Chapter 10

The singular integral

Our aim in this chapter is to replace the functions $\mathfrak{S}^*(n)$ and $J^*(n)$ by more convenient expressions; these will be called the singular series $\mathfrak{S}(n)$ and the singular integral $J(n)$. This will be done in section 10.1. We shall show that the order of magnitude of the singular integral is n^2 in section 10.2.

10.1 Introducing $\mathfrak{S}(n)$ and $J(n)$

Define

$$S(q) := \sum_{\substack{1\leqslant a\leqslant q\\ \gcd(a,q)=1}} S(q,a)^9 \mathrm{e}(-an/q),$$

where

$$S(q,a) = \sum_{m=1}^{q} \mathrm{e}(am^3/q).$$

Thus,

$$\mathfrak{S}^*(n) = \sum_{q\leqslant n^{1/300}} \frac{S(q)}{q^9}.$$

Lemma 10.1. *Let a, q be coprime integers and ϵ any positive real. Then*

$$|S(a,q)| \ll_\epsilon q^{\frac{3}{4}+\epsilon}.$$

Proof. The argument is an analogue of the differencing process (see pages 151-153 of Chapter 8). We have

$$|S(q,a)|^2 = \sum_{m_2 (\text{mod } q)} \sum_{m_1 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}(m_1^3 - m_2^3)\right).$$

The transformation $m_1 \mapsto h_1$ given by $m_1 \equiv h_1 + m_2 (\text{mod } q)$ shows that

$$|S(q,a)|^2 = \sum_{h_1 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}h_1^3\right) \sum_{m_2 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}(3h_1(h_1 m_2 + m_2^2))\right),$$

hence the triangle inequality gives

$$|S(q,a)|^2 \leqslant \sum_{h_1 (\text{mod } q)} \Big| \sum_{m_2 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}(3h_1(h_1 m_2 + m_2^2))\right) \Big|.$$

Now Cauchy's inequality reveals that

$$|S(q,a)|^4 \leqslant q \sum_{h_1 (\text{mod } q)} \Big| \sum_{m_2 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}(3h_1(h_1 m_2 + m_2^2))\right) \Big|^2.$$

The inner term is

$$\Big| \sum_{m_2 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}(3h_1(h_1 m_2 + m_2^2))\right) \Big|^2 = \sum_{m_2, m_3 (\text{mod } q)} \mathrm{e}\left(\frac{a}{q}(3h_1(h_1(m_2 - m_3) + (m_2^2 - m_3^3)))\right)$$

and the substitution $m_2 \mapsto h_2$ given by $m_2 \equiv hm_3 + h_2 (\text{mod } q)$ leads to

$$\sum_{h_2 (\text{mod } q)} \mathrm{e}\left(3\frac{a}{q}(h_1^2 h_2 + h_1 h_2^2)\right) \sum_{m_3 (\text{mod } q)} \mathrm{e}\left(6\frac{a}{q}h_1 h_2 m_3\right).$$

Note that the coprimality of a, q shows that the sum over m_3 equals q when q divides $6h_1h_2$ and vanishes otherwise. We obtain that

$$|S(q,a)|^4 \leqslant q^2 \# \{1 \leqslant h_1, h_2 \leqslant q : q | 6h_1 h_2\}.$$

The integers $6h_1h_2$ lie in the range $[1, 6q^2]$ and are divisible by q. Hence there exists $i \in [1, 6q]$ such that $6h_1h_2 = iq$. Therefore

$$|S(q,a)|^4 \leqslant q^2 \sum_{i=1}^{6q} \# \{1 \leqslant h_1, h_2 \leqslant q : 6h_1 h_2 = iq\}.$$

In order to have $6h_1h_2 = iq$ both integers h_1, h_2 must divide iq and there are only

$$\tau(iq)^2 \ll_\epsilon (iq)^{\epsilon/2} \ll q^\epsilon$$

such pairs, where ϵ is any positive real. This concludes our proof. $\square$

The last lemma shows that

$$\frac{|S(q)|}{q^9} \ll_\epsilon \frac{1}{q^{1+\frac{1}{4}+\epsilon}},$$

therefore the following series, usually referred to as the *singular series*,

$$\mathfrak{S}(n) := \sum_{q=1}^{\infty} \frac{S(q)}{q^9} \tag{10.1}$$

converges absolutely and satisfies

$$\mathfrak{S}(n) - \mathfrak{S}^*(n) \ll_\epsilon \sum_{q>n^{1/300}} \frac{1}{q^{1+\frac{1}{4}+\epsilon}} \ll \int_{n^{1/300}}^{\infty} \frac{\mathrm{d}t}{t^{1+\frac{1}{4}+\epsilon}} \ll_\epsilon n^{-\frac{1}{1200}+\epsilon}.$$

This shows that

$$\mathfrak{S}(n) \ll 1 \tag{10.2}$$

and by (9.6) we obtain

$$R^*(n) = (\mathfrak{S}(n) + O_\epsilon(n^{-\frac{1}{1200}+\epsilon}))J^*(n). \tag{10.3}$$

We next replace $J^*(n)$ by a more suitable integral. For this we shall need to need the behaviour of $v(\beta) = \frac{1}{3}\sum_{m=1}^{n} \mathrm{e}(\beta m)/m^{2/3}$ in the range $|\beta| \leqslant \frac{1}{2}$.

Lemma 10.2. *Let $\beta \in \mathbb{R}$ with $|\beta| \leqslant \frac{1}{2}$. Then*

$$|v(\beta)| \ll \min\{n^{1/3}, |\beta|^{-1/3}\}.$$

Proof. If β is close to 0 then the terms $\mathrm{e}(\beta m)$ in the definition of $v(\beta)$ remain close to 1. Hence using the triangle inequality one does not loose much information,

$$|v(\beta)| \leqslant \frac{1}{3} \sum_{1\leqslant m\leqslant n} \frac{1}{m^{\frac{2}{3}}} \leqslant \frac{1}{3}\int_1^{n-1} \frac{\mathrm{d}t}{t^{\frac{2}{3}}} + O(1) \ll n^{1/3}.$$

If $|\beta| \leqslant 1/n$ then $|\beta|^{-1/3} > n^{1/3}$, hence the claim of our lemma is evident.

In the remaining case $|\beta| > 1/n$ we see that

$$\Big| \sum_{m \leqslant 1/|\beta|} \frac{\mathrm{e}(\beta m)}{m^{2/3}} \Big| \leqslant \sum_{m \leqslant 1/|\beta|} \frac{1}{m^{2/3}} \ll (1/|\beta|)^{1/3},$$

which is acceptable. We use partial summation to estimate the remaining sum

$$\sum_{1/|\beta| < m \leqslant n} \frac{\mathrm{e}(\beta m)}{m^{2/3}}.$$

For this purpose we define for $t \in \mathbb{R}$,

$$A(t) := \sum_{1 \leqslant m \leqslant t} \mathrm{e}(\beta m) = \mathrm{e}(\beta m) \frac{\mathrm{e}(\beta [t]) - 1}{\mathrm{e}(\beta) - 1}$$

and observe that the inequality $|\mathrm{e}(\beta) - 1| \gg |\beta|$, valid for $|\beta| < 1/2$, yields

$$A(t) \ll \frac{1}{|\beta|},$$

with an implied constant that is independent of t. Partial summation now gives

$$\sum_{1/|\beta| < m \leqslant n} \frac{\mathrm{e}(\beta m)}{m^{2/3}} = \frac{A(n)}{n^{2/3}} - \frac{A(1/|\beta|)}{|\beta|^{-2/3}} + \int_{1/|\beta|}^{n} A(t) \frac{\mathrm{d}t}{t^{5/3}},$$

which is

$$\ll \frac{1/|\beta|}{n^{2/3}} + \frac{|\beta|^{2/3}}{|\beta|} + \frac{1}{|\beta|} \frac{1}{|\beta|^{-2/3}} \ll |\beta|^{-1/3}.$$

□

Define the following integral (which is usually called *singular integral*),

$$J(n) := \int_{-1/2}^{1/2} v(\beta)^9 \mathrm{e}(-\beta n) \mathrm{d}\beta \tag{10.4}$$

and observe that Lemma 10.2 shows that

$$J(n) \ll \int_0^{1/n} n^{9/3} \mathrm{d}\beta + \int_{1/n}^{1/2} \frac{\mathrm{d}\beta}{|\beta|^3},$$

hence

$$J(n) \ll n^2. \tag{10.5}$$

Now recall the definition of $J^*(n)$ in (9.8). We have

$$J(n) - J^*(n) = \int_{n^{-1+1/300} \leqslant |\beta| \leqslant 1/2} v(\beta)^9 \mathrm{e}(-\beta n) \mathrm{d}\beta,$$

which according to Lemma 10.2 is

$$\ll \int_{n^{1/300}/n}^{1/2} \beta^{-3} \mathrm{d}\beta \ll n^{2-\frac{1}{150}}.$$

Using (10.2), (10.3) and (10.5) we find an absolute constant $\delta > 0$ such that

$$R^*(n) = \mathfrak{S}(n) J(n) + O(n^{2-\delta}),$$

which when combined with (8.8), (8.10) and (9.5) yields the following theorem.

Theorem 10.3. *We have*

$$\lim_{n \to +\infty} \left| \frac{R(n)}{n^2} - \frac{\mathfrak{S}(n) J(n)}{n^2} \right| = 0.$$

10.2 The singular integral

The *Beta function* is defined as

$$B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} \mathrm{d}t, \quad \text{for } x, y > 0.$$

Before relating the singular integral $J(n)$ to the Beta function we need some information on sums of monotone arithmetic functions. Let $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{R}$ be any monotonic function. Comparing the sum $\sum_{y < n \leqslant x} f(n)$ with the integral $\int_y^x f(t) \mathrm{d}t$ we see that

$$\sum_{y < n \leqslant x} f(n) = \int_y^x f(t) \mathrm{d}t + O(1 + |f(y)| + |f(x)|).$$

Therefore if $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{R}$ is monotonic in each interval

$$(0, x_1), (x_1, x_2), \ldots, (x_k, x)$$

then

$$\sum_{0\leqslant m\leqslant n} f(m) = \int_0^n f(t)\mathrm{d}t + O(1+|f(0)|+|f(n)|+\sum_{i=1}^k |f(x_i)|).$$

The function $f(x) := x^{\beta-1}(n-x)^{\alpha-1}$, defined for

$$\beta\in(0,1],\ \alpha\geqslant\beta,$$

has derivative $x^{\beta-2}(n-x)^{\alpha-2}((\beta-1)n-x(\alpha+\beta-2))$, which vanishes at 0 and n and $X=\frac{\beta-1}{\alpha+\beta-2}$. If $X\in(0,n)$ then

$$\sum_{0\leqslant m\leqslant n} f(m) = \int_0^n f(t)\mathrm{d}t + O(1+|f(X)|)$$

and if $X\notin(0,n)$ then the error term is $O(1)$. The substitution $t\mapsto y$ given by $t=ny$ shows that

$$\int_0^n f(t)\mathrm{d}t = n^{\alpha+\beta-1}B(\beta,\alpha)$$

and the error term is $\ll n^{\alpha-1}$, therefore

$$\sum_{m=1}^{n-1} m^{\beta-1}(n-m)^{\alpha-1} = n^{\alpha+\beta-1}\left(B(\beta,\alpha)+O(n^{-\beta})\right). \tag{10.6}$$

Before proceeding we need to recall a few standard facts about the *Gamma function*. It is defined as

$$\Gamma(t) := \int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t \quad \text{for } x>0$$

and satisfies

$$\Gamma(1)=1, \tag{10.7}$$

$$\Gamma(t+1)=t\Gamma(t) \quad \text{for } t>0, \tag{10.8}$$

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{for } x,y>0. \tag{10.9}$$

Observe that $\Gamma(x)=(x-1)!$ for every positive integer x. So $B(x,y)=\dfrac{x+y}{xy}\cdot\dbinom{x+y}{x}^{-1}$ for all positive integers x,y.

We have the following theorem.

Theorem 10.4. *We have*

$$J(n) = \Gamma\left(\frac{4}{3}\right)^9 \frac{n^2}{2}\left(1 + O(n^{-1/3})\right).$$

Proof. We begin by proving by induction that for every integer $s \geqslant 2$, that one has

$$\frac{1}{3^s} \sum_{\substack{1\leqslant m_1,\dots,m_s\leqslant n \\ m_1+\cdots+m_s=n}} \frac{1}{(m_1\cdots m_s)^{\frac{2}{3}}} = \Gamma\left(\frac{4}{3}\right)^s \Gamma\left(\frac{s}{3}\right)^{-1} n^{\frac{s}{3}-1}\left(1 + O(n^{-1/3})\right). \tag{10.10}$$

For $s = 2$ this is valid due to (10.6) with $\alpha = \beta = 1/3$, as well as (10.8) and (10.9). Assuming that (10.10) is valid for some integer $s \geqslant 2$ then

$$\frac{1}{3^{s+1}} \sum_{\substack{1\leqslant m_1,\dots,m_{s+1}\leqslant n \\ m_1+\cdots+m_{s+1}=n}} \frac{1}{(m_1\cdots m_{s+1})^{\frac{2}{3}}}$$

equals

$$\sum_{1\leqslant m_{s+1}\leqslant n-1} \frac{1}{3m_{s+1}^{\frac{2}{3}}}\left(\frac{1}{3^s} \sum_{\substack{1\leqslant m_1,\dots,m_s\leqslant n \\ m_1+\cdots+m_s=n-m_{s+1}}} \frac{1}{(m_1\cdots m_s)^{\frac{2}{3}}}\right),$$

which is

$$\Gamma\left(\frac{4}{3}\right)^s \Gamma\left(\frac{s}{3}\right)^{-1} \frac{1}{3} \sum_{1\leqslant m\leqslant n-1} \frac{m^{\frac{1}{3}}}{m}(n-m)^{\frac{s}{3}-1} + O\left(\sum_{1\leqslant m\leqslant n-1} m^{1/3-1}(n-m)^{\frac{(s-1)}{3}-1}\right)$$

due to the induction hypothesis. Using (10.6) with $\beta = \frac{1}{3}$ and $\alpha = \frac{s}{3}, \frac{(s-1)}{3} - 1$ respectively for the main and the error term, we conclude the proof of (10.10).

Combining (10.4) and the definition of v,

$$v(\beta) = \frac{1}{3} \sum_{1\leqslant m\leqslant n} \frac{\mathrm{e}(\beta m)}{m^{\frac{2}{3}}}$$

gives

$$J(n) = \frac{1}{3^9} \sum_{1\leqslant m_1,\dots,m_9\leqslant n} \frac{1}{(m_1\cdots m_9)^{\frac{2}{3}}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}(\beta(n - m_1 - \cdots - m_9))\mathrm{d}\beta.$$

The integral vanishes except when $n-m_1-\cdots-m_9=0$, thus obtaining

$$J(n)=\frac{1}{3^9}\sum_{\substack{1\leqslant m_1,\ldots,m_9\leqslant n\\ m_1+\cdots+m_9=n}}\frac{1}{(m_1\cdots m_9)^{\frac{2}{3}}}$$

and according to (10.10) our theorem is valid. $\square$