Chapter 2

Tools from complex analysis

We discuss some topics from complex analysis that are used in this course and that are usually not treated in a basic complex analysis course. In exams, we will not ask questions on exams about the proofs in this chapter, but students are expected to know the results, and be able to apply them. In some theorems there are assumptions on measurability of some occurring functions. If you are willing to take for granted that all functions in this course are measurable, it is not necessary to know what this means. The Prerequisites contain more information on measurable functions.

2.1 A quick review

We quickly recall a few basic facts. For definitions and more details we refer to Section 0.7 of the Prerequisites. Recall that a complex function f, defined on a non-empty open subset U of $\mathbb{C}$ is said to be *analytic* or *holomorphic* on U if for every $z_0 \in U$, the limit $f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

1) Let U be an open subset of $\mathbb{C}$, $f : U \to \mathbb{C}$ an analytic function, $z_0 \in U$, and R the largest number such that the open disk with center z_0 and radius R, $D(z_0, R)$ is contained in U. Then on $D(z_0, R)$ we have a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. We say that z_0 is a zero of order k of f if $a_0 = \cdots = a_{k-1} = 0$ but $a_k \neq 0$.

A consequence of this is, that the derivatives f', $f^{(2)}$, $f^{(3)}$,... exist and are all analytic on U. Further, we have

$$a_n = f^{(n)}(z_0)/n! = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}} \cdot dz \quad \text{for all } n \in \mathbb{Z}_{>0}, \tag{2.1}$$

where γ is any contour contained in $D(z_0, R)$ with z_0 in its interior.

2) Let U be an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ an analytic function and suppose that for some $R > 0$, the punctured disk $D^0(z_0, R) = \{z \in \mathbb{C} : |z - z_0| < R, z \neq z_0\}$ with center z_0 and radius R is contained in U. Then on $D^0(z_0, R)$ we can express $f(z)$ as a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$. If γ is a contour contained in $D^0(z_0, R)$ with z_0 in its interior, we have

$$a_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}} \cdot dz \quad \text{for all } n \in \mathbb{Z}. \tag{2.2}$$

3) We keep the notation from 2). We call z_0 an *essential singularity* of f if $a_n \neq 0$ for infinitely many $n < 0$, a *pole of f of order k* if $a_{-k} \neq 0$ for some $k > 0$ but $a_n = 0$ for $n < -k$, and a *removable singularity of f* if $a_n = 0$ for $n < 0$. In that case, we can make f analytic on $U \cup \{z_0\}$ by defining $f(z_0) := a_0$.

The coefficient a_{-1} is called the *residue of f at z_0*, notation $\operatorname{res}(z_0, f)$. For instance, if z_0 is a simple pole of f (i.e., of order 1), then

$$\operatorname{res}(z_0, f) = \lim_{z \to z_0} (z - z_0) f(z).$$

In case that z_0 is a removable singularity of f we have $\operatorname{res}(z_0, f) = 0$.

4) We say that a complex function f is *meromorphic around z_0* if it is defined and can be expressed as a Laurent series $\sum_{n=n_0}^{\infty} a_n(z - z_0)^n$ on $D^0(z_0, r)$ for some $r > 0$. This means that z_0 is a pole or removable singularity of f. If f is not identically 0, and if we take $a_{n_0} \neq 0$, we define $\operatorname{ord}_{z_0}(f) := n_0$. If f is identically 0, we put $\operatorname{ord}_{z_0}(f) := \infty$.

One can verify that if f, g are meromorphic around z_0 then so are $f \pm g$, $f \cdot g$ and f/g (if g is not identically 0) and in that case $\operatorname{ord}_{z_0}(f \cdot g) = \operatorname{ord}_{z_0}(f) + \operatorname{ord}_{z_0}(g)$, $\operatorname{ord}_{z_0}(f/g) = \operatorname{ord}_{z_0}(f) - \operatorname{ord}_{z_0}(g)$, and $\operatorname{ord}_{z_0}(f \pm g) \geqslant \min(\operatorname{ord}_{z_0}(f), \operatorname{ord}_{z_0}(g))$.

Further, if f is meromorphic around z_0 and not identically 0, then z_0 is a simple pole of the logarithmic derivative f'/f and $\operatorname{ord}_{z_0}(f) = \operatorname{res}(z_0, f'/f)$.

5) A very powerful tool is the so-called *Residue Theorem*, which in fact implies (2.1) and (2.2):

Let U be a non-empty, open subset of $\mathbb{C}$, γ a contour such that both γ and its interior are contained in U and $z_1, \ldots, z_q$ in the interior of γ. Let $f : U \setminus \{z_1, \ldots, z_q\} \to \mathbb{C}$ be analytic. Then

$$\frac{1}{2\pi i} \oint_\gamma f(z) \cdot dz = \sum_{i=1}^{q} \operatorname{res}(z_i, f).$$

2.2 Unicity of analytic functions

In this section we show that two analytic functions f, g defined on a connected open set U are equal on the whole set U, if they are equal on a sufficiently large subset of U.

We start with the following result.

Theorem 2.1. *Let U be a non-empty, open, connected subset of $\mathbb{C}$, and $f : U \to \mathbb{C}$ an analytic function. Assume that $f = 0$ on an infinite subset of U having a limit point in U. Then $f = 0$ on U.*

Proof. Our assumption that U is connected means, that any non-empty subset S of U that is both open and closed in U, must be equal to U.

Let Z be the set of $z \in U$ with $f(z) = 0$. Let S be the set of $z \in U$ such that z is a limit point of Z. By assumption, S is non-empty. Since f is continuous, we have $S \subseteq Z$. Any limit point in U of S is therefore a limit point of Z and so it belongs to S. Hence S is closed in U. We show that S is also open; then it follows that $S = U$ and we are done.

Pick $z_0 \in S$. We have to show that there is $\delta > 0$ such that $D(z_0, \delta) \subset S$. There is $\delta > 0$ such that f has a Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converging on $D(z_0, \delta)$. Assume that f is not identically 0 on $D(z_0, \delta)$. Then not all coefficients a_n are 0. Assume that $a_m \neq 0$ and $a_n = 0$ for $n < m$, say. Then $f(z) = (z - z_0)^m h(z)$ with $h(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m}$. Since $h(z_0) = a_m \neq 0$ and

h is continuous, there is $\delta_1 > 0$ such that $h(z) \neq 0$ for all $z \in D(z_0, \delta_1)$. But then $f(z) \neq 0$ for all z with $0 < |z - z_0| < \delta_1$, contradicting that $z_0 \in S$.

Hence f is identically 0 on $D(z_0, \delta)$. Clearly, every point of $D(z_0, \delta)$ is a limit point of $D(z_0, \delta)$, hence of Z. So $D(z_0, \delta) \subset S$. This shows that indeed, S is open. $\square$

Corollary 2.2. *Let U be a non-empty, open, connected subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be an analytic function that is not identically 0 on U. Then the set of zeros of f in U is discrete in U, i.e., every compact subset of U contains only finitely many zeros of f.*

Proof. Suppose that some compact subset of U contains infinitely many zeros of f. Then by the Bolzano-Weierstrass Theorem, the set of these zeros would have a limit point in this compact set, implying that $f = 0$ on U. $\square$

Corollary 2.3. *Let U be a non-empty, open, connected subset of $\mathbb{C}$, and f, $g : U \to \mathbb{C}$ two analytic functions. Assume that $f = g$ on an infinite subset of U having a limit point in U. Then $f = g$ on U.*

Proof. Apply Theorem 2.1 to $f - g$. $\square$

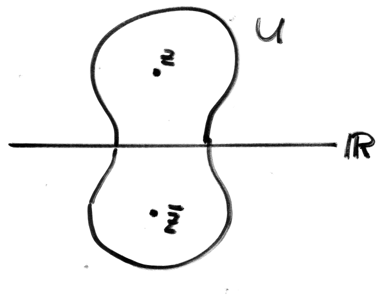
Let U, U' be open subsets of $\mathbb{C}$ with $U \subset U'$. Let $f : U \to \mathbb{C}$ be an analytic function. An *analytic continuation* of f to U' is an analytic function $g : U' \to \mathbb{C}$ such that $g(z) = f(z)$ for $z \in U$. It is often a difficult problem to figure out whether such an analytic continuation exists. The next Corollary shows that if it exists, it must be unique.

Corollary 2.4. *Let U, U' be non-empty, open subsets of $\mathbb{C}$, such that $U \subset U'$ and U' is connected. Let $f : U \to \mathbb{C}$ be an analytic function. Then f has at most one analytic continuation to U'.*

Proof. . Let g_1, g_2 be two analytic continuations of f to U'. Then $g_1(z) = g_2(z) = f(z)$ for $z \in U$. Since U has a limit point in itself, hence in U', it follows that $g_1(z) = g_2(z)$ for $z \in U'$. $\square$

Another consequence of Theorem 2.1 is the so-called Schwarz' reflection priciple, which states that analytic functions assuming real values on the real line have nice symmetric properties.

Corollary 2.5 (Schwarz' reflection principle)**.**



Let U be an open, connected subset of $\mathbb{C}$, such that $U \cap \mathbb{R} \neq \emptyset$ and such that U is symmetric about $\mathbb{R}$, i.e., $\overline{z} \in U$ for every $z \in U$. Further, let $f : U \to \mathbb{C}$ be a non-identically zero analytic function with the property that

$$\{z \in U \cap \mathbb{R} : \, f(z) \in \mathbb{R}\}$$

has a limit point in U.

Then f has the following properties:

(i) $f(z) \in \mathbb{R}$ for $z \in U \cap \mathbb{R}$;
(ii) $\overline{f(\overline{z})} = f(z)$ for $z \in U$;
(iii) If z_0 and $r > 0$ are such that $D^0(z_0, r) \subset U$, then $\mathrm{ord}_{\overline{z_0}}(f) = \mathrm{ord}_{z_0}(f)$.

Proof. We first show that the function $z \mapsto \overline{f(\overline{z})}$ is analytic on U. Indeed, for $z_0 \in U$, the limit

$$\lim_{z \to z_0} \frac{\overline{f(\overline{z})} - \overline{f(\overline{z_0})}}{z - z_0} = \lim_{z \to z_0} \overline{\left(\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}} \right)} = \overline{f'(\overline{z_0})}$$

exists.

Notice that for every $z \in U \cap \mathbb{R}$ with $f(z) \in \mathbb{R}$, we have $\overline{f(\overline{z})} = f(z)$. So by our assumption on f, the set of $z \in U$ with $\overline{f(\overline{z})} = f(z)$ has a limit point in U. Now Corollary 2.3 implies that $\overline{f(\overline{z})} = f(z)$ for $z \in U$. This implies (i) and (ii).

We finish with proving (iii). Our assumption implies that f has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

converging on $D^0(z_0, r)$. Then for $z \in D^0(\overline{z_0}, r)$ we have $\overline{z} \in D^0(z_0, r)$ and

$$f(z) = \overline{f(\overline{z})} = \overline{\left(\sum_{n=-\infty}^{\infty} a_n (\overline{z} - z_0)^n \right)} = \sum_{n=-\infty}^{\infty} \overline{a_n} (z - \overline{z_0})^n,$$

which clearly implies (iii). $\square$

2.3 Analytic functions defined by integrals

In analytic number theory, quite often one has to deal with complex functions that are defined by infinite series, infinite products, infinite integrals, or even worse, infinite integrals of infinite series. In this section we have collected some useful results that allow us to verify in a not too difficult manner that such complicated functions are analytic. We could not find a convenient reference for these results, therefore we have included the not too exciting proofs.

We start with a general theorem on analytic functions defined by an integral.

Theorem 2.6. *Let D be a measurable subset of $\mathbb{R}^m$, U an open subset of $\mathbb{C}$ and $f : D \times U \to \mathbb{C}$ a function with the following properties:*
(i) f is measurable on $D \times U$ (with U viewed as subset of $\mathbb{R}^2$);
(ii) for every fixed $x \in D$, the function $z \mapsto f(x, z)$ is analytic on U;
(iii) for every compact subset K of U there is a measurable function $M_K : D \to \mathbb{R}$ such that

$$|f(x,z)| \leqslant M_K(x) \text{ for } x \in D,\ z \in K, \qquad \int_D M_K(x)dx < \infty.$$

Then the function F given by

$$F(z) := \int_D f(x,z)dx$$

is analytic on U, and for every $k \geqslant 1$,

$$F^{(k)}(z) = \int_D f^{(k)}(x,z)dx,$$

where $f^{(k)}(x, z)$ denotes the k-th derivative with respect to z of the analytic function $z \mapsto f(x, z)$.

Proof. Fix $z \in U$. Choose $r > 0$ such that $\overline{D}(z, r) \subset U$, and let $0 < \delta < \frac{1}{2}r$. We show that for $w \in D(z, \delta)$, $F(w)$ can be expanded into a Taylor series around z; then it follows that F is analytic on $D(z, \delta)$ and so in particular in z. Let $M(x) : D \to \mathbb{R}$ be a measurable function such that $|f(x, w)| \leqslant M(x)$ for $x \in D$, $w \in \overline{D}(z, r)$ and $\int_D M(x)dx < \infty$.

For $w \in D(z,\delta)$ we have by (2.1),

$$F(w) = \int_D f(x,w)dx = \int_D \left\{ \frac{1}{2\pi i} \oint_{\gamma_{z,2\delta}} \frac{f(x,\zeta)}{\zeta - w} \cdot d\zeta \right\} dx.$$

By inserting

$$\begin{aligned} \frac{f(x,\zeta)}{\zeta - w} &= \frac{f(x,\zeta)}{(\zeta - z) - (w - z)} = \frac{f(x,\zeta)}{\zeta - z}\left(1 - \frac{w-z}{\zeta - z}\right)^{-1} \\ &= \sum_{n=0}^{\infty} \frac{f(x,\zeta)}{(\zeta - z)^{n+1}} \cdot (w-z)^n \end{aligned}$$

we obtain

$$\begin{aligned} F(w) &= \int_D \left\{ \frac{1}{2\pi i} \oint_{\gamma_{z,2\delta}} \left(\sum_{n=0}^{\infty} \frac{f(x,\zeta)}{(\zeta - z)^{n+1}} (w-z)^n \right) d\zeta \right\} dx \\ &= \int_D \left\{ \int_0^1 \left(\sum_{n=0}^{\infty} \frac{f(x, z + 2\delta e^{2\pi i t})}{(2\delta e^{2\pi i t})^n} (w-z)^n \right) dt \right\} dx. \end{aligned}$$

We apply the Fubini-Tonelli Theorem (see the Prerequisites). Note that since $|w - z| < \delta$,

$$\begin{aligned} &\int_D \left\{ \int_0^1 \left(\sum_{n=0}^{\infty} \left| \frac{f(x, z + 2\delta e^{2\pi i t})}{(2\delta e^{2\pi i t})^n} (w-z)^n \right| \right) dt \right\} dx \\ &\qquad \leqslant \int_D \left\{ \int_0^1 \left(\sum_{n=0}^{\infty} M(x) 2^{-n} \right) dt \right\} dx \leqslant \int_D 2M(x)dx < \infty. \end{aligned}$$

So the conditions of the Fubini-Tonelli Theorem are satisfied, and in the expression for $F(w)$ derived above we can interchange the integrations and the summation. Performing this interchange, we obtain

$$\begin{aligned} F(w) &= \sum_{n=0}^{\infty} (w-z)^n \left(\int_D \left\{ \int_0^1 \frac{f(x, z + 2\delta e^{2\pi i t})}{(2\delta e^{2\pi i t})^n} dt \right\} dx \right) \\ &= \sum_{n=0}^{\infty} (w-z)^n \left(\int_D \left\{ \frac{1}{2\pi i} \oint_{\gamma_{z,2\delta}} \frac{f(x,\zeta)}{(\zeta - z)^{n+1}} \cdot d\zeta \right\} dx \right) \\ &= \sum_{n=0}^{\infty} (w-z)^n \left(\int_D \frac{f^{(n)}(x,z)}{n!} \cdot dx \right), \end{aligned}$$

where in the last step we applied (2.1). This shows that indeed, $F(w)$ has a Taylor expansion around z converging on $D(z,\delta)$. So in particular, F is analytic in z. Further, $F^{(k)}(z)$ is equal to $k!$ times the coefficient of $(w-z)^k$, that is, $\int_D f^{(k)}(x,z)dx$. This proves our Theorem. $\square$

We deduce a result, which states that under certain conditions, the pointwise limit of a sequence of analytic functions is again analytic.

Theorem 2.7. *Let $U \subset \mathbb{C}$ be a non-empty open set, and $\{f_n : U \to \mathbb{C}\}_{n=0}^{\infty}$ a sequence of analytic functions, converging pointwise to a function f on U. Assume that for every compact subset K of U there is a constant $C_K < \infty$ such that*

$$|f_n(z)| \leqslant C_K \quad \text{for all } z \in K,\ n \geqslant 0.$$

Then f is analytic on U, and $f_n^{(k)} \to f^{(k)}$ pointwise on U for all $k \geqslant 1$.

Proof. The set U can be covered by disks $D(z_0,\delta)$ with $z_0 \in U$, $\delta > 0$, such that the closed disk with center z_0 and radius 2δ, $\overline{D}(z_0,2\delta)$ is contained in U. We fix such a disk $D(z_0,\delta)$ and prove that f is analytic on $D(z_0,\delta)$ and $f_n^{(k)} \to f^{(k)}$ pointwise on $D(z_0,\delta)$ for $k \geqslant 1$. This clearly suffices.

Let $z \in D(z_0,\delta)$, $k \geqslant 0$. Then by (2.1), we have

$$\begin{aligned} f_n^{(k)}(z) &= \frac{k!}{2\pi i}\oint_{\gamma_{z_0,2\delta}} \frac{f_n(\zeta)}{(\zeta-z)^{k+1}}\cdot d\zeta \\ &= \int_0^1 k!\cdot\frac{f_n(z_0+2\delta e^{2\pi it})2\delta e^{2\pi it}}{(z_0+2\delta e^{2\pi it}-z)^{k+1}}\cdot dt = \int_0^1 g_{n,k}(t,z)dt, \end{aligned}$$

say. By assumption, there is $C < \infty$ such that $|f_n(w)| \leqslant C$ for $w \in \overline{D}(z_0,2\delta)$, $n \geqslant 0$. Further, for $t \in [0,1]$ we have $|z_0+2\delta e^{2\pi it}-z| > \delta$. Hence

$$(2.3) \qquad |g_{n,k}(t,z)| \leqslant C\cdot k!\cdot 2\delta/\delta^{k+1} = 2C\cdot k!\delta^{-k} \quad \text{for } n,k \geqslant 0.$$

Notice that for $k \geqslant 0$, $t \in [0,1]$, $z \in D(z_0,\delta)$ we have

$$g_{n,k}(t,z) \to k!\cdot\frac{f(z_0+2\delta e^{2\pi it})2\delta e^{2\pi it}}{(z_0+2\delta e^{2\pi it}-z)^{k+1}} = g^{(k)}(t,z),$$

where

$$g(t,z) := \frac{f(z_0+2\delta e^{2\pi it})2\delta e^{2\pi it}}{z_0+2\delta e^{2\pi it}-z}$$

and $g^{(k)}(t,z)$ is the k-th derivative of the analytic function in z, $z \mapsto g(t,z)$.

Thanks to (2.3) we can apply the dominated convergence theorem, and obtain

$$f_n^{(k)}(z) \to \int_0^1 g^{(k)}(t,z)dt \ \text{ for } z \in D(z_0,\delta),\ k \geqslant 0.$$

Applying this with $k=0$ and using $f_n \to f$ pointwise, we obtain

$$f(z) = \int_0^1 g(t,z)dt \ \text{ for } z \in D(z_0,\delta).$$

It follows from Theorem 2.6 that the right-hand side, and hence f, is analytic on $D(z_0,\delta)$, and moreover,

$$f^{(k)}(z) = \int_0^1 g^{(k)}(t,z)dt \ \text{ for } z \in D(z_0,\delta),\ k \geqslant 1.$$

Indeed, $g(t,z)$ is measurable on $[0,1] \times D(z_0,\delta)$ and for every fixed t, the function $z \mapsto g(t,z)$ is analytic on $D(z_0,\delta)$. Further, by (2.3) and since $g_{n,0}(t,z) \to g(t,z)$, we have $|g(t,z)| \leqslant 2C$ for $t \in [0,1]$, $z \in D(z_0,\delta)$. So all conditions of Theorem 2.6 are satisfied.

Now it follows that

$$\lim_{n\to\infty} f_n^{(k)}(z) = \int_0^1 g^{(k)}(t,z)dt = f^{(k)}(z) \ \text{ for } z \in D(z_0,\delta),\ k \geqslant 1,$$

which is what we wanted to prove. □

Corollary 2.8. *Let $U \subset \mathbb{C}$ be a non-empty open set, and $\{f_n : U \to \mathbb{C}\}_{n=0}^{\infty}$ a sequence of analytic functions, converging to a function f pointwise on U, and uniformly on every compact subset of U.*
Then f is analytic on U and $f_n^{(k)} \to f^{(k)}$ pointwise on U for every $k \geqslant 1$.

Proof. Take a compact subset K of U. Let $\varepsilon > 0$. Then there is N such that $|f_n(z) - f_m(z)| < \varepsilon$ for all $z \in K$, $m,n \geqslant N$. Choose $m \geqslant N$. Then there is $C > 0$ such that $|f_m(z)| \leqslant C$ for $z \in K$ since f_m is continuous. Hence $|f_n(z)| \leqslant C + \varepsilon$ for $z \in K$, $n \geqslant N$. Now our Corollary follows at once from Theorem 2.7. □

Corollary 2.9. *let $U \subset \mathbb{C}$ be a non-empty open set, and $\{f_n : U \to \mathbb{C}\}_{n=0}^{\infty}$ a sequence of analytic functions, converging to a function f pointwise on U and uniformly on every compact subset of U. Then*

$$\lim_{n\to\infty} \frac{f_n'(z)}{f_n(z)} = \frac{f'(z)}{f(z)}$$

for all $z \in U$ with $f(z) \neq 0$, where the limit is taken over those n for which $f_n(z) \neq 0$.

Proof. Obvious. □

Corollary 2.10. *Let $U \subset \mathbb{C}$ be a non-empty open set and $\{f_n : U \to \mathbb{C}\}_{n=0}^{\infty}$ a sequence of analytic functions. Assume that for every compact subset K of U there are reals $M_{n,K}$ such that $|f_n(z)| \leqslant M_{n,K}$ for $z \in K$ and $\sum_{n=0}^{\infty} M_{n,K}$ converges. Then*

(i) $\sum_{n=0}^{\infty} f_n$ is analytic on U, and $\left(\sum_{n=0}^{\infty} f_n\right)^{(k)} = \sum_{n=0}^{\infty} f_n^{(k)}$ for $k \geqslant 0$,

(ii) $\prod_{n=0}^{\infty}(1+f_n)$ is analytic on U.

Proof. Our assumption on the functions f_n implies that both the series $\sum_{n=0}^{\infty} f_n$ and the infinite product $\prod_{n=0}^{\infty}(1+f_n)$ converge uniformly on every compact subset of U (see the Prerequisites). Now apply Corollary 2.8. □

Corollary 2.11. *Let U, $\{f_n\}_{n=0}^{\infty}$ be as in Corollary 2.10 and assume in addition that $f_n \neq -1$ on U for every $n \geqslant 0$. Then for the function $F = \prod_{n=0}^{\infty}(1+f_n)$ we have*

$$\frac{F'}{F} = \sum_{n=0}^{\infty} \frac{f_n'}{1+f_n}.$$

Proof. Let $F_m := \prod_{n=0}^{m}(1+f_n)$. Then $F_m \to F$ uniformly on every compact subset of U. Hence by Corollary 2.9,

$$\frac{F'}{F} = \lim_{m\to\infty} \frac{F_m'}{F_m} = \lim_{m\to\infty} \sum_{n=0}^{m} \frac{f_n'}{1+f_n}$$

which clearly implies Corollary 2.11. □

2.4 Euler's Gamma function

The Gamma function plays an important role in the functional equation for the Riemann zeta function. We have collected here some properties of this function.

For $t \in \mathbb{R}_{>0}$, $z \in \mathbb{C}$, define $t^z := e^{z \log t}$, where $\log t$ is the ordinary real logarithm. Euler's Gamma function is defined by the integral

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt \quad (z \in \mathbb{C},\ \operatorname{Re} z > 0).$$

Lemma 2.12. *$\Gamma(z)$ defines an analytic function on $\{z \in \mathbb{C} :\ \operatorname{Re} z > 0\}$.*

Proof. This is standard using Theorem 2.6. Let $U := \{z \in \mathbb{C} :\ \operatorname{Re} z > 0\}$. First, the function $F(t,z) := e^{-t}t^{z-1}$ is continuous, hence measurable on $\mathbb{R}_{>0} \times U$. Second, for each fixed $t > 0$, $z \mapsto e^{-t}t^{z-1}$ is analytic on U. Third, let K be a compact subset of U. Then there exist $\delta, R > 0$ such that $\delta \leqslant \operatorname{Re} z \leqslant R$ for $z \in K$. This implies that for $z \in K$, $t > 0$,

$$|e^{-t}t^{z-1}| \leqslant M(t) := \begin{cases} t^{\delta-1} & \text{for } 0 \leqslant t \leqslant 1, \\ e^{-t}t^{R-1} \leqslant Ce^{-t/2} & \text{for } t > 1, \end{cases}$$

where C is some constant. Now we have

$$\int_0^\infty M(t)dt = \int_0^1 t^{\delta-1}dt + C\int_1^\infty e^{-t/2}dt = \delta^{-1} + 2C < \infty.$$

Hence all conditions of Theorem 2.6 are satisfied, and thus, $\Gamma(z)$ is analytic on U. $\square$

Using integration by parts, one easily shows that for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$,

$$\begin{aligned} \Gamma(z) &= z^{-1}\int_0^\infty e^{-t}dt^z \\ &= z^{-1}\left([e^{-t}t^z|_{t=0}^{t=\infty} + \int_0^\infty e^{-t}t^z dt \right) = z^{-1}\Gamma(z+1), \end{aligned}$$

that is,

$$\Gamma(z+1) = z\Gamma(z) \quad \text{if } \operatorname{Re} z > 0. \tag{2.4}$$

One easily shows that $\Gamma(1) = 1$ and then by induction, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}_{>0}$.

We now show that Γ has a meromorphic continuation to $\mathbb{C}$.

Theorem 2.13. *There exists a unique meromorphic function* Γ *on* $\mathbb{C}$ *with the following properties:*
(i) $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ *for* $z \in \mathbb{C}$, $\mathrm{Re}\, z > 0$*;*
(ii) the function Γ *is analytic on* $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$*;*
(iii) Γ *has a simple pole with residue* $(-1)^n/n!$ *at* $z = -n$ *for* $n = 0, 1, 2, \ldots$*;*
(iv) $\Gamma(z + 1) = z\Gamma(z)$ *for* $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

Proof. The function Γ has already been defined for $\mathrm{Re}\, z > 0$ by $\int_0^\infty e^{-t}t^{z-1}dt$. By Corollary 2.4, Γ has at most one analytic continuation to any larger connected open set, hence there is at most one function Γ with properties (i)–(iv). We proceed to construct such a function.

Let $z \in \mathbb{C}$ with $\mathrm{Re}\, z > 0$. By repeatedly applying (2.4) we get

$$\Gamma(z) = \frac{1}{z(z+1)\cdots(z+n-1)} \cdot \Gamma(z+n) \quad \text{for } \mathrm{Re}\, z > 0, \quad n = 1, 2, \ldots. \tag{2.5}$$

We continue Γ to $B := \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ as follows. For $z \in B$, choose $n \in \mathbb{Z}_{>0}$ such that $\mathrm{Re}\, z + n > 0$ and define $\Gamma(z)$ by the right-hand side of (2.5). This does not depend on the choice of n. For if m, n are any two integers with $m > n > -\mathrm{Re}\, z$, then by (2.5) with $z + n$, $m - n$ instead of z, n we have

$$\Gamma(z+n) = \frac{1}{z+n)\cdots(z+m-1)} \cdot \Gamma(z+m),$$

and so

$$\frac{1}{z(z+1)\cdots(z+n-1)} \cdot \Gamma(z+n) = \frac{1}{z(z+1)\cdots(z+m-1)} \cdot \Gamma(z+m).$$

Hence Γ is well-defined on B, and it is analytic on B since the right-hand side of (2.5) is analytic if $\mathrm{Re}\, z + n > 0$. This proves (ii).

We prove (iii). By (2.5) we have

$$\begin{aligned}
\lim_{z \to -n} (z+n)\Gamma(z) &= \lim_{z \to -n} (z+n) \frac{1}{z(z+1)\cdots(z+n)} \Gamma(z+n+1) \\
&= \frac{1}{(-n)(-n+1)\cdots(-1)} \Gamma(1) = \frac{(-1)^n}{n!}.
\end{aligned}$$

Hence Γ has a simple pole at $z = -n$ of residue $(-1)^n/n!$.

We prove (iv). Both functions $\Gamma(z+1)$ and $z\Gamma(z)$ are analytic on B, and by (2.4), they are equal on the set $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ which has limit points in B. So by Corollary 2.3, $\Gamma(z+1) = z\Gamma(z)$ for $z \in B$. □

Theorem 2.14. *We have* $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$ *for* $z \in \mathbb{C} \setminus \mathbb{Z}$.

Proof. We prove that $z\Gamma(z)\Gamma(1-z) = \pi z / \sin \pi z$, or equivalently,

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin \pi z} \quad \text{for } z \in A := (\mathbb{C} \setminus \mathbb{Z}) \cup \{0\}, \tag{2.6}$$

which implies Theorem 2.14. Notice that by Theorem 2.13 the left-hand side is analytic on A, while by $\lim_{z \to 0} \pi z / \sin \pi z = 1$ the right-hand side is also analytic on A. By Corollary 2.3, it suffices to prove that (2.6) holds for every z in an infinite subset of A having a limit point in A. For this infinite set we take $S := \{\frac{1}{2n} : n \in \mathbb{Z}_{>0}\}$; this set has limit point 0 in A. Thus, (2.6), and hence Theorem 2.14, follows once we have proved that

$$\Gamma(1 + \tfrac{1}{2n}) \cdot \Gamma(1 - \tfrac{1}{2n}) = \frac{\pi/2n}{\sin \pi/2n} \quad (n = 1, 2, \ldots). \tag{2.7}$$

Notice that

$$\begin{aligned} \Gamma(1 + \tfrac{1}{2n}) \cdot \Gamma(1 - \tfrac{1}{2n}) &= \int_0^\infty e^{-s} s^{1/2n} ds \cdot \int_0^\infty e^{-t} t^{-1/2n} dt \\ &= \int_0^\infty \int_0^\infty e^{-s-t} (s/t)^{1/2n} ds dt. \end{aligned}$$

Define new variables $u = s + t$, $v = s/t$. Then $s = uv/(v+1)$, $t = u/(v+1)$. The Jacobian of the substitution $(s,t) \mapsto (u,v)$ is

$$\begin{aligned} \frac{\partial(s,t)}{\partial(u,v)} &= \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{v}{v+1} & \frac{u}{(v+1)^2} \\ \frac{1}{v+1} & -\frac{u}{(v+1)^2} \end{vmatrix} \\ &= \frac{-uv - u}{(v+1)^3} = \frac{-u}{(v+1)^2}. \end{aligned}$$

It follows that

$$\Gamma(1 + \tfrac{1}{2n}) \cdot \Gamma(1 - \tfrac{1}{2n}) = \int_0^\infty \int_0^\infty e^{-u} v^{1/2n} \left| \frac{\partial(s,t)}{\partial(u,v)} \right| \cdot du dv$$

$$= \int_0^\infty \int_0^\infty e^{-u} v^{1/2n} \frac{u}{(v+1)^2} \cdot du dv = \int_0^\infty e^{-u} u du \cdot \int_0^\infty \frac{v^{1/2n}}{(v+1)^2} dv.$$

In the last product, the first integral is equal to 1, while for the second integral we have, by the exercise below,

$$\begin{aligned}\int_0^\infty \frac{v^{1/2n}}{(v+1)^2}dv &= -\int_0^\infty v^{1/2n} d\Big(\frac{1}{v+1}\Big)\\ &= -\left[\frac{v^{1/2n}}{v+1}\right]_0^\infty + \int_0^\infty \frac{1}{v+1}\cdot dv^{1/2n} = \int_0^\infty \frac{dw}{w^{2n}+1} = \frac{\pi/2n}{\sin \pi/2n}.\end{aligned}$$

This implies (2.7), hence Theorem 2.14. $\square$

Exercise 2.1. *Let n be a positive integer. Prove that*

$$\int_0^\infty \frac{dz}{z^{2n}+1} = -\frac{1}{2}\cdot 2\pi i\cdot \sum_{k=0}^{n-1}\frac{1}{2n}\cdot e^{(\pi i/2n)+(k\pi i/n)} = \frac{\pi}{2n}\cdot\frac{1}{\sin(\pi/2n)}.$$

Hint. *Let Γ_R be the contour consisting of the line segment from $-R$ to R, and the semi-circle in the upper half plane from R to $-R$ with center R. Compute $\oint_{\Gamma_R}\frac{dz}{z^{2n}+1}$ using the Residue Theorem, and show that the (absolute value of) the integral of $\frac{1}{z^{2n}+1}$ along the semi-circle converges to 0 as $R\to\infty$. Here, you have to use the general inequality*

$$\left|\int_\gamma g(z)dz\right| \leqslant L(\gamma)\cdot\sup_{z\in\gamma}|g(z)|,$$

where γ is a path in $\mathbb{C}$, $g:\gamma\to\mathbb{C}$ is a continuous function, and $L(\gamma)$ denotes the length of γ.

Corollary 2.15. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Proof. Substitute $z=\frac{1}{2}$ in Theorem 2.14, and use $\Gamma(\frac{1}{2})>0$. $\square$

Corollary 2.16. *(i) $\Gamma(z)\neq 0$ for $z\in\mathbb{C}\setminus\{0,-1,-2,\ldots\}$.*
(ii) $1/\Gamma$ is analytic on $\mathbb{C}$, and $1/\Gamma$ has simple zeros at $z=0,-1,-2,\ldots$.

Proof. (i) Recall that $\Gamma(n)=(n-1)!\neq 0$ for $n=1,2,\ldots$. Further, by Theorem 2.14 we have $\Gamma(z)\Gamma(1-z)\sin\pi z=\pi\neq 0$ for $z\in\mathbb{C}\setminus\mathbb{Z}$.

(ii) By (i), the function $1/\Gamma$ is analytic on $\mathbb{C}\setminus\{0,-1,-2,\ldots\}$. Further, at $z=0,-1,-2,\ldots$, Γ has a simple pole, hence $1/\Gamma$ is analytic and has a simple zero. $\square$

We give another expression for the Gamma function.

Theorem 2.17. *For $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ we have*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! \cdot n^z}{z(z+1)\cdots(z+n)}.$$

Proof. Define

$$F_n(z) := \frac{n! \cdot n^z}{z(z+1)\cdots(z+n)}.$$

We prove by induction that for every non-negative integer m we have $\Gamma(z) = \lim_{n\to\infty} F_n(z)$ for $z \in \mathbb{C}$ with $\operatorname{Re} z > -m$ and (if $m > 0$) $z \neq 0, -1, \ldots, 1-m$. For the moment, we assume that this assertion is true for $m = 0$, i.e., $\Gamma(z) = \lim_{n\to\infty} F_n(z)$ for $\operatorname{Re} z > 0$, and do the induction step. Assume our assertion holds for some integer $m \geqslant 0$. Let $z \in \mathbb{C}$ with $\operatorname{Re} z > -m-1$ and $z \neq 0, \ldots, m$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{F_n(z+1)}{zF_n(z)} &= \lim_{n\to\infty} \frac{n! \cdot n^{z+1}}{(z+1)\cdots(z+n+1)} \cdot \frac{(z+1)\cdots(z+n+1)}{(n+1)!(n+1)^z} \\
&= \lim_{n\to\infty} \Big(\frac{n}{n+1}\Big)^{z+1} = 1.
\end{aligned}$$

By the induction hypothesis we know that $\Gamma(z+1) = \lim_{n\to\infty} F_n(z+1)$, and so

$$\lim_{n\to\infty} F_n(z) = \lim_{n\to\infty} \frac{F_n(z+1)}{z} = \frac{\Gamma(z+1)}{z} = \Gamma(z).$$

This completes the induction step.

We now show that $\Gamma(z) = \lim_{n\to\infty} F_n(z)$ for $z \in \mathbb{C}$, $\operatorname{Re} z > 0$. For this, we need some lemmas.

Lemma 2.18. *Let $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Then*

$$F_n(z) = \int_0^n \big(1 - \frac{t}{n}\big)^n t^{z-1} dt.$$

Proof. By substituting $s = t/n$, the integral becomes

$$n^z \int_0^1 (1-s)^n s^{z-1} ds.$$

The rest is left as an exercise. $\square$

Lemma 2.19. *For every integer $n \geqslant 2$ and every real t with $0 \leqslant t \leqslant n$ we have*
$$0 \leqslant e^{-t} - \big(1 - \frac{t}{n}\big)^n \leqslant e^{-t} \cdot \frac{t^2}{n^2}.$$

Proof. This is equivalent to
$$1 - \frac{t^2}{n} \leqslant e^t \big(1 - \frac{t}{n}\big)^n \leqslant 1 \quad (0 \leqslant t \leqslant n,\ n \geqslant 2).$$
Recall that if f, g are continuously differentiable, real functions with $f(0) = g(0)$ and $f'(x) \leqslant g'(x)$ for $0 \leqslant x \leqslant A$, say, then $f(x) \leqslant g(x)$ for $0 \leqslant x \leqslant A$. From this observation, one easily deduces that
$$1 + x \leqslant e^x, \quad 1 - x \leqslant e^{-x}, \quad (1 - x)^r \geqslant 1 - rx \quad \text{for } 0 \leqslant x \leqslant 1,\ r \geqslant 0.$$
This implies on the one hand, for $n \geqslant 2$, $0 \leqslant t \leqslant n$,
$$e^t \big(1 - \frac{t}{n}\big)^n \leqslant e^t (e^{-t/n})^n \leqslant 1,$$
on the other hand
$$e^t \big(1 - \frac{t}{n}\big)^n \geqslant \big(1 + \frac{t}{n}\big)^n \cdot \big(1 - \frac{t}{n}\big)^n = \big(1 - \frac{t^2}{n^2}\big)^n \geqslant 1 - \frac{t^2}{n}.$$
□

Completion of the proof of Theorem 2.17. Let $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Put $\Gamma_n(z) := \int_0^n e^{-t} t^{z-1} dt$. Since $\lim_{n\to\infty} \Gamma_n(z) = \Gamma(z)$, it suffices to prove that $\Gamma_n(z) - F_n(z) \to 0$ as $n \to \infty$.

By Lemma 2.18 we have
$$\begin{aligned} \Gamma_n(z) - F_n(z) &= \int_0^n e^{-t} t^{z-1} dt - \int_0^n \big(1 - \frac{t}{n}\big)^n t^{z-1} dt \\ &= \int_0^n \Big(e^{-t} - \big(1 - \frac{t}{n}\big)^n\Big) t^{z-1} dt. \end{aligned}$$
Now using $|\int ...| \leqslant \int |...|$ and Lemma 2.19 we obtain
$$\begin{aligned} |\Gamma_n(z) - F_n(z)| &\leqslant \int_0^n e^{-t} \frac{t^2}{n} \cdot |t^{z-1}| dt \\ &\leqslant \frac{1}{n} \int_0^\infty e^{-t} t^{\operatorname{Re} z + 1} dt = \frac{\Gamma(\operatorname{Re} z + 2)}{n} \to 0 \ \text{ as } n \to \infty. \end{aligned}$$
□

We deduce some consequences. Recall that the *Euler-Mascheroni constant* γ is given by

$$\gamma := \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} \right) - \log N.$$

Corollary 2.20. *We have*

$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+z/n} \quad \textit{for } z \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}.$$

Proof. Let $z \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$. Then for $N \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned}
\Gamma(z) &= \lim_{N\to\infty} \frac{N^z \cdot N!}{z(z+1)\cdots(z+N)} = z^{-1} \lim_{N\to\infty} \frac{e^{z\log N}}{(1+z)(1+z/2)\cdots(1+z/N)} \\
&= z^{-1} \lim_{N\to\infty} e^{(\log N - 1 - \frac{1}{2} - \cdots - \frac{1}{N})z} \prod_{n=1}^{N} \frac{e^{z/n}}{1+z/n} \\
&= e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+z/n}.
\end{aligned}$$

□

As another consequence, we derive an infinite product expansion for $\sin \pi z$.

Corollary 2.21. *We have*

$$\sin \pi z = \pi z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad \textit{for } z \in \mathbb{C}.$$

Proof. For $z \in \mathbb{C}$ we have by Theorem 2.14, Corollary 2.16 and Corollary 2.20,

$$\begin{aligned}
\sin \pi z &= \frac{\pi}{\Gamma(z)\Gamma(1-z)} = \frac{\pi}{\Gamma(z)(-z)\Gamma(-z)} \\
&= \pi(-z)^{-1} e^{\gamma z} z \prod_{n=1}^{\infty} \left(e^{-z/n}\left(1+\frac{z}{n}\right)\right) \cdot (-z)e^{-\gamma z} \prod_{n=1}^{\infty} \left(e^{-z/n}\left(1-\frac{z}{n}\right)\right) \\
&= \pi z \prod_{n=1}^{\infty} \left(1-\frac{z}{n}\right)\left(1+\frac{z}{n}\right) = \pi z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).
\end{aligned}$$

□

Recall that the Bernoulli numbers B_n ($n \geqslant 0$) are given by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} \cdot z^n \quad (|z| < 2\pi).$$

Corollary 2.22. *We have* $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_3 = B_5 = \cdots = 0$ *and*

$$\zeta(2n) = (-1)^{n-1} 2^{2n-1} \frac{B_{2n}}{(2n)!} \pi^{2n} \quad \textit{for } n = 1, 2, \ldots.$$

Proof. Let $z \in \mathbb{C}$ with $0 < |z| < 1$. Then $\sin \pi z \neq 0$ and so, by taking the logarithmic derivative of $\sin \pi z$,

$$\begin{aligned}
\frac{\sin' \pi z}{\sin \pi z} &= \frac{\pi \cos \pi z}{\sin \pi z} = \frac{\pi(e^{\pi i z} + e^{-\pi i z})/2}{(e^{\pi i z} - e^{-\pi i z})/2i} \\
&= \pi i + \frac{1}{z} \cdot \frac{2\pi i z}{e^{2\pi i z} - 1} \\
&= \pi i + \frac{1}{z} \sum_{n=0}^{\infty} \frac{B_n}{n!} \cdot (2\pi i)^n z^n. \qquad (2.8)
\end{aligned}$$

We obtain another expression for the logarithmic derivative of $\sin \pi z$ by applying Corollary 2.11 to the product identity from Corollary 2.21. Note that for $z \in \mathbb{C}$ with $|z| < 1$ we have $|z^2/n^2| < n^{-2}$ and that $\sum_{n=1}^{\infty} n^{-2}$ converges. Hence the logarithmic derivative of the infinite product is the infinite sum of the logarithmic derivatives of the factors, i.e.,

$$\begin{aligned}
\frac{\sin' \pi z}{\sin \pi z} &= \frac{(\pi z)'}{\pi z} + \sum_{n=1}^{\infty} \frac{(1 - z^2/n^2)'}{1 - z^2/n^2} \\
&= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-2z/n^2}{1 - z^2/n^2} = \frac{1}{z} - 2 \sum_{n=1}^{\infty} \frac{z}{n^2} \left(\sum_{k=0}^{\infty} \left(\frac{z^2}{n^2} \right)^k \right) \\
&= \frac{1}{z} - 2 \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{z^{2k+1}}{n^{2k+2}} \quad \text{(by absolute convergence)} \\
&= \frac{1}{z} - 2 \sum_{k=0}^{\infty} \zeta(2k+2) z^{2k+1}. \qquad (2.9)
\end{aligned}$$

Now Corollary 2.22 easily follows by comparing the coefficients of the Laurent series in (2.8) and (2.9). $\square$

We finish with another important consequence of Theorem 2.17, the so-called *duplication formula.*

Corollary 2.23. *We have*

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \cdot \Gamma(z)\Gamma(z+\tfrac{1}{2}) \quad \textit{for } z \in \mathbb{C},\ \ z \neq 0, -\tfrac{1}{2}, -1, -\tfrac{3}{2}, -2, \ldots .$$

Proof. Let A be the set of z indicated in the lemma. We show that the function $F(z) := 2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})/\Gamma(2z)$ is constant on A. Substituting $z = \frac{1}{2}$ gives that the constant is $2\sqrt{\pi}$, and then Corollary 2.23 follows.

Let $z \in A$. To get nice cancellations in the numerator and denominator, we use the expressions

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \frac{n!\cdot n^z}{z(z+1)\cdots(z+n)} = \lim_{n\to\infty} \frac{2^{n+1}\cdot n!\cdot n^z}{2z(2z+2)\cdots(2z+2n)}, \\
\Gamma(z+\tfrac{1}{2}) &= \lim_{n\to\infty} \frac{n!\cdot n^{z+1/2}}{(z+1/2)(z+3/2)\cdots(z+n+1/2)} \\
&\qquad\qquad = \lim_{n\to\infty} \frac{2^{n+1}\cdot n!\cdot n^{z+1/2}}{(2z+1)(2z+3)\cdots(2z+2n+1)}, \\
\Gamma(2z) &= \lim_{n\to\infty} \frac{(2n+1)!\cdot(2n+1)^{2z}}{2z(2z+1)\cdots(2z+2n+1)}
\end{aligned}$$

(i.e., in Theorem 2.17 we substitute $2z$ for z and take the limit over the odd integers). Thus,

$$\begin{aligned}
F(z) &= \frac{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)} \\
&= 2^{2z} \lim_{n\to\infty} \left\{ \frac{2^{2n+2}(n!)^2 n^{2z+1/2}}{2z(2z+1)\cdots(2z+2n+1)} \cdot \frac{2z(2z+1)\cdots(2z+2n+1)}{(2n+1)!\cdot(2n+1)^{2z}} \right\} \\
&= \lim_{n\to\infty} \left\{ \frac{2^{2n+2}(n!)^2\sqrt{n}}{(2n+1)!} \right\}
\end{aligned}$$

since

$$\lim_{n\to\infty} \frac{2^{2z}\cdot n^{2z}}{(2n+1)^{2z}} = \lim_{n\to\infty} e^{2z\log(2n/(2n+1))} = 1.$$

This shows that indeed $F(z)$ is constant. $\square$

Remark. By an argument similar to the proof of Corollary 2.23 (exercise), one can derive the *multiplication formula of Legendre-Gauss*,

$$\Gamma(nz) = (2\pi)^{-(n-1)/2} n^{nz-1/2} \Gamma(z)\Gamma(z+\tfrac{1}{n})\cdots\Gamma(z+\tfrac{n-1}{n}) \quad \text{for } n \in \mathbb{Z}_{\geqslant 2}.$$