Chapter 9

Hua's lemma and a first expedition into the major arcs

We continue the proof of Theorem 8.1 regarding $R(n)$, which denotes the number of representations of a large positive integer n as a sum of 9 positive integer cubes. In Chapter 8 we used the circle method to show that $R(n)$ equals the sum of an integral over the major arcs $\mathfrak{M}$ and an integral over the minor arcs $\mathfrak{m}$. In addition, we started proving (8.10) which states that the integral over the minor arcs has a rather small value. In this chapter we finish the proof of (8.10) by using Hua's lemma and begin the proof of (8.9), which evaluates precisely the value of the integral over the major arcs.

9.1 Hua's lemma

Let again $N := [n^{1/3}]$ and $f(\alpha) = \sum_{m=1}^{N} \mathrm{e}(\alpha m^3)$. To prove (8.10) we observe that

$$\Big| \int_{\mathfrak{m}} f(\alpha)^9 \mathrm{e}(-\alpha n) \mathrm{d}\alpha \Big| \leqslant \int_{\mathfrak{m}} |f(\alpha)|^9 \mathrm{d}\alpha \leqslant \sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \int_{\mathfrak{m}} |f(\alpha)|^8 \mathrm{d}\alpha. \tag{9.1}$$

To bound the supremum above we will employ Weyl's inequality (Theorem 8.3). This requires that each $\alpha \in \mathfrak{m}$ can be approximated by a rational $\frac{a}{q}$ with denominator not too large and not too small. Namely we need to find an integer in the range

$n^{\frac{1}{300}} \leqslant q \leqslant n^{1-\frac{1}{300}}$ and an integer a such that

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{qn^{1-\frac{1}{300}}}. \tag{9.2}$$

To see this, let $m := [n^{1-\frac{1}{300}}]$ and observe that the m real numbers

$$\beta_q = \alpha q - [\alpha q], (q = 1, 2, \ldots, m),$$

are contained in the interval $[0, 1)$. We split $[0, 1)$ into $m + 1$ subintervals of equal size,

$$B_r := \Big[\frac{r-1}{m+1}, \frac{r}{m+1}\Big), (r = 1, \ldots, m+1),$$

and now there are 2 cases to consider. Firstly, if there exists some β_q in B_1 or B_{m+1} then (9.2) holds with $a = [\alpha q]$ and $a = 1 + [\alpha q]$ respectively. If there is no $\beta_q \in B_1 \cup B_{m+1}$ then, by the pigeonhole principle, there must be at least one box B_r that has 2 (or more) of the numbers β_q; denote them as β_{q_1} and β_{q_2}, where $q_1 < q_2$. We then obtain

$$\frac{1}{m+1} \geqslant \Big|\beta_{q_1} - \beta_{q_2}\Big| = \Big|\alpha(q_2 - q_1) - ([\alpha q_2] - [\alpha q_1])\Big|,$$

hence (9.2) holds with $q = q_2 - q_1$ and $a = [\alpha q_2] - [\alpha q_1]$. We have therefore proved (9.2); note that by writing $\frac{a}{q}$ in lowest terms the value of q cannot increase, hence we can solve (9.2) with $1 \leqslant q \leqslant n^{1-\frac{1}{300}}$ and a coprime to q.

In order to apply Theorem 8.3 we need to know that $q \geqslant n^{\frac{1}{300}}$; we will show by contradiction that this is a consequence of the fact that $\alpha \in \mathfrak{m}$. Taking $q, a = 1$ in (8.5) shows that

$$\Big[1 - \frac{1}{n^{1-\frac{1}{300}}}, 1 + \frac{1}{n^{1-\frac{1}{300}}}\Big] \subset \mathfrak{M}, \text{ hence } \mathfrak{m} \subset \left(\frac{1}{n^{1-\frac{1}{300}}}, 1 - \frac{1}{n^{\frac{1}{300}}}\right).$$

Therefore α must be in the last interval, hence (9.2) reveals that $a \leqslant q$.

Now if $q \leqslant n^{\frac{1}{300}}$ holds, then owing to the definition (8.6) (which states that $\mathfrak{M}(a, q)$ is a subset of $\mathfrak{M}$ whenever $q \leqslant n^{\frac{1}{300}}$) we deduce that α must be in $\mathfrak{M}$. This is a contradiction because we assume that $\alpha \in \mathfrak{m}$, which was defined by (8.7) as the complement of $\mathfrak{M}$. So we can apply Theorem 8.3 and obtain

$$\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll n^{\frac{1}{3} - \frac{1}{2000}}.$$

Combining this with (9.1) and the next lemma provides the bound

$$\Big| \int_{\mathfrak{m}} f(\alpha)^9 \mathrm{e}(-\alpha n) \mathrm{d}\alpha \Big| \ll n^{2-\frac{1}{4000}},$$

which proves (8.10).

Lemma 9.1 (Hua's lemma)**.** *We have*

$$\int_0^1 |f(\alpha)|^8 \mathrm{d}\alpha \ll n^{\frac{5}{3}}.$$

Proof. Recall that we have defined $N := [n^{\frac{1}{3}}]$ and $f(\alpha) = \sum_{m=1}^{N} \mathrm{e}(\alpha m^3)$. First let us observe that

$$\int_0^1 |f(\alpha)|^2 \mathrm{d}\alpha = \sum_{1 \leqslant m_1, m_2 \leqslant N} \int_0^1 \mathrm{e}(\alpha(m_1^3 - m_2^3)) \mathrm{d}\alpha,$$

which equals N since the inner integral vanishes whenever $m_1 \neq m_2$. In the notation of §8.2 we also have

$$|f(\alpha)|^2 = \sum_{|h|<N} \sum_{m_2 \in I_h} \mathrm{e}(\alpha(h(h^2 + 3hm_2 + 3m_2^2))), \tag{9.3}$$

where

$$I_h := \mathbb{N} \cap [1, N] \cap [1-h, N-h].$$

The contribution of the term with $h = 0$ is bounded by the cardinality of I_h, which is $\leqslant N$, therefore

$$|f(\alpha)|^2 \leqslant N + \sum_{0<|h|<N} \sum_{m_2 \in I_h} \mathrm{e}(\alpha(h(h^2 + 3hm_2 + 3m_2^2))).$$

Multiplying this with $|f(\alpha)|^2$ and integrating over the interval $[0, 1)$ we obtain

$$\int_0^1 |f(\alpha)|^4 \mathrm{d}\alpha \leqslant N^2 + \sum_{1 \leqslant x_1, x_2 \leqslant N} \sum_{0<|h|<N} \sum_{m_2 \in I_h} \int_0^1 \mathrm{e}(\alpha(x_1^3 - x_2^3 + h(h^2 + 3hm_2 + 3m_2^2))) \mathrm{d}\alpha.$$

We see that the last integral equals the number of solutions of

$$x_1^3 - x_2^3 + h(h^2 + 3hm_2 + 3m_2^2) = 0$$

with $x_1, x_2 \in \mathbb{N} \cap [1, N]$ and $0 < |h| < N, m_2 \in I_h$. The number of possibilities for x_1, x_2 is $\leqslant N^2$. If $x_1 = x_2$ then for each h there are at most 2 solutions for m_2 in $h^2 + 3hm_2 + 3m_2^2 > 0$; this contribution is $\ll N^2$. In any other case, h must divide $x_1^3 - x_2^3$, for which there are

$$\leqslant \tau(x_1^3 - x_2^3) \ll_\epsilon |x_1^3 - x_2^3|^\epsilon \ll_\epsilon N^\epsilon$$

possibilities for h, where ϵ is any positive real. For each such value for x_1, x_2, h there are at most 2 values of m_2 satisfying the equation above. This shows that for any $\epsilon > 0$ we have

$$\int_0^1 |f(\alpha)|^4 \mathrm{d}\alpha \ll_\epsilon N^{2+\epsilon}.$$

The rest of the proof proceeds in a similar manner; one proves bounds for $\int_0^1 |f(\alpha)|^8 \mathrm{d}\alpha$ from the last bound for $\int_0^1 |f(\alpha)|^4 \mathrm{d}\alpha$ in the same way that we obtained the bound for $\int_0^1 |f(\alpha)|^4 \mathrm{d}\alpha$ from the one regarding $\int_0^1 |f(\alpha)|^2 \mathrm{d}\alpha$. This is explained in detail in pages 12 and 13 of Davenport's book, *Analytic methods for Diophantine equations and Diophantine inequalities*, Cambridge Mathematical Library, 2005. □

9.2 Major arcs

We begin by establishing a lemma that relates the value of $f(\alpha)$ to that of $f(a/q)$ when α belongs to a major arc with centre a/q. For this we need to define

$$S(q, a) := \sum_{m=1}^{q} \mathrm{e}(am^3/q), \quad a, q \in \mathbb{N},$$

and

$$v(\beta) := \frac{1}{3} \sum_{m=1}^{n} \frac{\mathrm{e}(\beta m)}{m^{\frac{2}{3}}}, \quad \beta \in \mathbb{R}.$$

Lemma 9.2. *Let $a, q \in \mathbb{N}$ be coprime with $1 \leqslant a \leqslant q \leqslant n^{\frac{1}{300}}$. If $\alpha \in \mathfrak{M}(a, q)$ then*

$$f(\alpha) = \frac{S(q, a)}{q} v(\alpha - a/q) + O(n^{\frac{2}{300}}).$$

Proof. For fixed a, q the function of m given by $\mathrm{e}(am^3/q)$ is periodic with period q. Therefore for all $t \geqslant 1$ we have

$$\sum_{1 \leqslant m \leqslant t} \mathrm{e}(am^3/q) = \sum_{r=1}^{q} \mathrm{e}(ar^3/q) \sum_{\substack{1 \leqslant m \leqslant t \\ m \equiv r (\mathrm{mod}\ q)}} 1.$$

Splitting the interval $[1, t]$ into the subintervals $[1, q], [q+1, 2q], \ldots$, we see that the inner sum equals $t/q + O(1)$. Hence the sum over m is

$$\sum_{1 \leqslant m \leqslant t} \mathrm{e}(am^3/q) = \frac{t}{q} S(q, a) + O(q).$$

The sum on the left-hand side can be rewritten as

$$\sum_{\substack{1 \leqslant k \leqslant t^3 \\ \exists m \in \mathbb{Z} : k = m^3}} \mathrm{e}(ak/q),$$

therefore writing $\mathbf{1}(k)$ for the characteristic function of the positive integer cubes (i.e., $\mathbf{1}(k) := 1$ if k is an integer cube and 0 otherwise) we have shown that

$$\sum_{1 \leqslant k \leqslant t^3} \mathbf{1}(k) \mathrm{e}(ak/q) = \frac{t}{q} S(q, a) + O(q). \tag{9.4}$$

Observe that

$$\sum_{1 \leqslant k \leqslant t^3} \frac{1}{k^{\frac{2}{3}}} = \int_1^{t^3} \frac{\mathrm{d}t}{t^{\frac{2}{3}}} + O(1) = 3t + O(1),$$

therefore

$$\sum_{1 \leqslant k \leqslant t^3} \frac{1}{3k^{\frac{2}{3}}} = t + O(1).$$

By (9.4) we obtain that

$$\sum_{1 \leqslant k \leqslant t^3} \left(\mathbf{1}(k) \mathrm{e}(ak/q) - \frac{1}{3k^{\frac{2}{3}}} \frac{S(q, a)}{q} \right) = O(q).$$

Define $c_k := \mathrm{e}(ak/q) - S(q, a)/(3qk^{2/3})$ when k is an integer cube and otherwise let $c_k := -S(q, a)/(3qk^{2/3})$. Then the last inequality becomes

$$\sum_{1 \leqslant k \leqslant t^3} c_k = O(q).$$

Note that this inequality is valid for all $t \geqslant 1$, hence the same inequality holds if t^3 is replaced by t. Now define for $t \in \mathbb{R}_{\geqslant 1}$,

$$F(t) := \sum_{1 \leqslant m \leqslant t} c_m,$$

a function which satisfies $F(t) = O(q)$. Now assume that $\beta \in \mathbb{R}$ is such that $|\beta| \leqslant 1/(n^{1-\frac{1}{300}})$. Then by partial summation we get

$$\sum_{1 \leqslant m \leqslant n} c_m \mathrm{e}(\beta m) = F(n)\mathrm{e}(\beta n) - 2\pi i \beta \int_1^n F(t)\mathrm{e}(\beta t)\mathrm{d}t,$$

therefore

$$\Big| \sum_{1 \leqslant m \leqslant n} c_m \mathrm{e}(\beta m) \Big| \ll q + |\beta| q \int_1^n 1 \mathrm{d}t \ll (1 + |\beta| n) q \ll n^{\frac{2}{300}}. \tag{9.5}$$

Letting $\beta = \alpha - a/q$ we see that $f(\alpha) = \sum_{1 \leqslant k \leqslant n^{\frac{1}{3}}} \mathrm{e}(\alpha k^3)$ equals

$$\sum_{1 \leqslant m \leqslant n} c_m \mathrm{e}\left(\beta m\right) + \frac{S(q,a)}{3q} v(\beta),$$

which concludes our proof. □

Define

$$V(\alpha, q, a) := \frac{S(q,a)}{q} v(\alpha - a/q).$$

The last lemma proves that for $\alpha \in \mathfrak{M}(a,q)$ and a, q as in its statement we have

$$V(\alpha, q, a) \ll |f(\alpha)| + n^{2/300} \ll n^{1/3}.$$

Therefore

$$f(\alpha)^9 - V(\alpha, q, a)^9 \ll n^{8/3} |f(\alpha) - V(\alpha, q, a)| \ll n^{8/3 + 2/300}.$$

Define

$$\begin{aligned} R^*(n) &:= \sum_{q \leqslant n^{1/300}} \sum_{\substack{1 \leqslant a \leqslant q \\ \gcd(a,q)=1}} \int_{\mathfrak{M}(a,q)} V(\alpha - a/q)^9 \mathrm{e}(-\alpha n) \mathrm{d}\alpha \\ &= \sum_{q \leqslant n^{1/300}} \frac{1}{q^9} \sum_{\substack{1 \leqslant a \leqslant q \\ \gcd(a,q)=1}} S(q,a)^9 \int_{\mathfrak{M}(a,q)} v(\alpha - a/q)^9 \mathrm{e}(-\alpha n) \mathrm{d}\alpha. \end{aligned}$$

Then

$$\int_{\mathfrak{M}} f(\alpha)^9 \mathrm{e}(-\alpha n)\mathrm{d}\alpha = R^*(n) + O(n^{2-\delta}), \tag{9.6}$$

for $\delta = \frac{19}{60}$. This is because by comparing the left-hand side of (9.6) with $R^*(n)$ we make an error

$$\begin{aligned}
&\sum_{q\leqslant n^{1/300}} \sum_{\substack{1\leqslant a\leqslant q\\ \gcd(a,q)=1}} \int_{\mathfrak{M}(a,q)} (f(\alpha)^9 - V(\alpha,q,a)^9)\mathrm{e}(-\alpha n)\mathrm{d}\alpha \\
&\quad \ll \sum_{1\leqslant q\leqslant n^{\frac{1}{300}}} \sum_{a=1}^{q} n^{8/3+2/300} |\mathfrak{M}(a,q)| \ll \sum_{1\leqslant q\leqslant n^{\frac{1}{300}}} \sum_{a=1}^{q} n^{8/3+2/300} \cdot n^{-1+1/300} \\
&\quad \ll n^{2/300} \cdot n^{5/3+3/300} = n^{2-19/60}.
\end{aligned}$$

The change of variables $\alpha \mapsto \beta$ given by $\beta = \alpha - a/q$ shows that

$$\int_{\mathfrak{M}(a,q)} v(\alpha - a/q)^9 \mathrm{e}(-\alpha n)\mathrm{d}\alpha = \mathrm{e}(-an/q) \int_{-n^{-1+\frac{1}{300}}}^{n^{-1+\frac{1}{300}}} v(\beta)^9 \mathrm{e}(-\beta n)\mathrm{d}\beta,$$

therefore we can write

$$R^*(n) = \mathfrak{S}^*(n) J^*(n), \tag{9.7}$$

where

$$\mathfrak{S}^*(n) := \sum_{q\leqslant n^{1/300}} \frac{1}{q^9} \sum_{\substack{1\leqslant a\leqslant q\\ \gcd(a,q)=1}} S(q,a)^9 \mathrm{e}(-an/q) \tag{9.8}$$

and

$$J^*(n) := \int_{-n^{-1+\frac{1}{300}}}^{n^{-1+\frac{1}{300}}} v(\beta)^9 \mathrm{e}(-\beta n)\mathrm{d}\beta. \tag{9.9}$$