Analytic Number Theory Fall 2016, Assignment 1

Deadline: Monday October 17

• Don't forget to write your name and student number on your homework. To simplify the grading, it is preferable that you submit your homework in latex.
• You may either submit your homework at the course, or to Marc Paul Noordman, or send him an electronic version of it by email.
• The number of points for each exercise is indicated in the left margin.
The total number of points is 70. Grade=(number of points)/7.

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5 **1.** *a)* Let k be a positive integer. Prove that

$$\int_2^x \frac{dt}{(\log t)^k} = O\Big(\frac{x}{(\log x)^k}\Big) \quad \text{as } x \to \infty.$$

Hint. Split the integral into $\int_2^{f(x)} + \int_{f(x)}^x$ for a well-chosen function $f(x)$ with $2 \le f(x) < x$ and estimate both parts from above.

5 *b)* Using integration by parts, prove that for every integer $n > 0$,

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t} = \sum_{i=1}^{n} (i-1)! \frac{x}{(\log x)^i} + O\Big(\frac{x}{(\log x)^{n+1}}\Big) \quad \text{as } x \to \infty.$$

Remark. The error term will increase with n. So the finite sum cannot be expanded into an infinite series.

2. Euclid's proof that there are infinitely many primes runs as follows. Suppose there are only finitely many primes, $p_1, p_2, \ldots, p_n$, say. Consider the number $P := p_1 p_2 \cdots p_n + 1$. Then either P itself is a prime or P is divisible by a prime but in both cases, this prime must be different from $p_1, \ldots, p_n$. Thus we arrive at a contradiction.
In certain cases, it is possible to give a similar proof for the fact that there are infinitely many primes p with $p \equiv a \pmod q$. Assume there are only finitely many such primes, $p_1, \ldots, p_n$, say. Construct a function $P(p_1, \ldots, p_n)$ which is divisible by a prime which is congruent to a modulo q but which is different from $p_1, \ldots, p_n$.

3 *a)* Let p be a prime with $p \equiv 3 \pmod 4$. Show that there is no integer x with $x^2 \equiv -1 \pmod p$.
Hint. Suppose there does exist such an integer x. Consider the order of $x \pmod p$ in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ of non-zero residue classes modulo p.

3 *b)* Show that there are infinitely many primes p with $p \equiv 1 \pmod 4$.
Hint. Take $P(p_1, \dots, p_n) = 4(p_1 p_2 \cdots p_n)^2 + 1$.

4 *c)* Show that there are infinitely many primes p with $p \equiv 3 \pmod 4$.
(You have to find yourself a suitable expression $P(p_1, \dots, p_n)$.)

5 *d)* Let p, q be distinct prime numbers with $q \geq 3$, $p \not\equiv 1 \pmod q$. Prove that there is no integer x with $1 + x + x^2 + \cdots + x^{q-1} \equiv 0 \pmod p$.

5 *e)* Let q be a prime number ≥ 3. Prove that there are infinitely many primes p with $p \equiv 1 \pmod q$.

3. In this exercise you are asked to prove Bertrand's postulate: for every positive integer n there is a prime number p with $n < p \leq 2n$. You have to use the theorems and lemmas proved in Chapter 1 of the lecture notes.

4 *a)* Prove that for every real $x \geq 2$ we have $\prod_{p \leq x} p \leq 4^x$ (product taken over all prime numbers $\leq x$).
Hint. Let $m := [x]$, and proceed by induction on m. If m is even, you can immediately apply the induction hypothesis. Assume that $m = 2k+1$ is odd and consider $\prod_{k+1 < p \leq 2k+1} p$.

It suffices to prove Bertrand's postulate for $n \geq 1000$ since the remaining cases can be verified by straightforward computation. In b),c),d) below let n be an integer ≥ 1000, and assume that there is no prime p with $n < p \leq 2n$.

5 *b)* Prove that the binomial coefficient $\binom{2n}{n}$ is not divisible by any prime p with $\frac{2}{3}n < p \leq n$.
Hint. Compute $\mathrm{ord}_p\left(\binom{2n}{n}\right)$.

4 *c)* Prove that $\binom{2n}{n} \leq (2n)^{\pi(\sqrt{2n})} \cdot 4^{2n/3}$.
Hint. Write $\binom{2n}{n} = p_1^{k_1} \cdots p_t^{k_t}$ with p_i distinct primes and $k_i > 0$ and split into primes p_i with $p_i \leq \sqrt{2n}$ and $p_i > \sqrt{2n}$; for the latter, $k_i = 1$.

2 *d)* Derive a contradiction.

4. We describe a general method to compute series $\sum_{n=1}^{\infty} f(n)$, where f is an even meromorphic function on $\mathbb{C}$, i.e., $f(z) = f(-z)$ for $z \in \mathbb{C}$ minus the poles of f.
Let N be an integer ≥ 1 and let S_N be the square through the four points $\pm(N+\frac{1}{2}) \pm (N+\frac{1}{2})i$, traversed counterclockwise. Assume that f has only finitely many poles, and that none are lying at the non-zero integers.
1) Compute $\displaystyle\oint_{S_N} \frac{2\pi i f(z)}{e^{2\pi i z}-1} \cdot dz$, using the Residue Theorem.
2) Prove that $\displaystyle\lim_{N\to\infty} \oint_{S_N} \frac{2\pi i f(z)}{e^{2\pi i z}-1} \cdot dz = 0$. Here, you have to use the general inequality

$$\left| \int_\gamma g(z) dz \right| \leq L(\gamma) \cdot \sup_{z \in \gamma} |g(z)|,$$

where γ is a path in $\mathbb{C}$, $g : \gamma \to \mathbb{C}$ is a continuous function, and $L(\gamma)$ denotes the length of γ. Applying this estimate with $\gamma = S_N$, one has to show that the upper bounds converges to 0 as $N \to \infty$.

The following lemma, of which we have included a proof here, is crucial in 2).

Lemma. *There is a constant $c > 0$, independent of N, such that $|e^{2\pi i z} - 1| \geq c$ holds for all integers $N \geq 1$ and all $z \in S_N$.*

Proof. We consider the four edges of the square separately. First consider the edge from $(N+\frac{1}{2})(-1-i)$ to $(N+\frac{1}{2})(1-i)$. This can be parametrized by $(N+\frac{1}{2})(t-i)$ with $-1 \leq t \leq 1$. So for the points z on this edge we have

$$\begin{aligned} |e^{2\pi i z} - 1| &= |e^{2\pi i (N+\frac{1}{2})(t-i)} - 1| = |e^{2\pi i (N+\frac{1}{2})t} e^{2\pi (N+\frac{1}{2})} - 1| \\ &\geq e^{2\pi(N+\frac{1}{2})} - 1 \geq e^{3\pi} - 1. \end{aligned}$$

Next, consider the edge from $(N+\frac{1}{2})(1-i)$ to $(N+\frac{1}{2})(1+i)$. This can be parametrized by $(N+\frac{1}{2})(1+it)$ with $-1 \leq t \leq 1$. So for the points z on this edge we have

$$|e^{2\pi i z} - 1| = |e^{2\pi i (N+\frac{1}{2})(1+it)} - 1| = |-e^{-2\pi(N+\frac{1}{2})t} - 1| \geq 1.$$

Here we have used that $e^{2\pi i (N+\frac{1}{2})} = -1$. The other two edges can be treated in the same manner. □

3 *a)* Let f be a meromorphic function on $\mathbb{C}$ that has no poles or zeros at the non-zero integers. Prove that the function $\dfrac{2\pi i f(z)}{e^{2\pi i z}-1}$ has residue $f(k)$ at $z = k$ for every non-zero integer k.

7 *b)* The Bernouilli numbers B_n are given by $\dfrac{z}{e^z - 1} = \sum_{n=0}^{\infty} \dfrac{B_n}{n!} z^n$ $(z \in \mathbb{C},\ |z| < 2\pi)$.
Using the method sketched above, prove that

$$\zeta(2k) = (-1)^{k-1} 2^{2k-1} \frac{B_{2k}}{(2k)!} \cdot \pi^{2k} \quad \text{for } k = 1, 2, \ldots.$$

5. Consider the Dirichlet series

$$F(s) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots,$$
$$G(s) = 1^{-s} + 2^{-s} - 2 \times 3^{-s} + 4^{-s} + 5^{-s} - 2 \times 6^{-s} + \cdots$$

4 *a)* Prove that $F(s)$, $G(s)$ converge, and are analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$.

4 *b)* Prove that for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ we have

$$F(s) = (1 - 2^{1-s}) \sum_{n=1}^{\infty} n^{-s}, \quad G(s) = (1 - 3^{1-s}) \sum_{n=1}^{\infty} n^{-s}.$$

7 *c)* Use a) and b) to prove that $\sum_{n=1}^{\infty} n^{-s}$ can be continued to an analytic function $\zeta(s)$ on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\} \setminus \{1\}$, with a simple pole with residue 1 at $s = 1$, i.e., $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ if $\operatorname{Re} s > 1$, and $\lim_{s \to 1} (s-1)\zeta(s) = 1$.
Hint. Both functions $1 - 2^{1-s}$, $1 - 3^{1-s}$ have infinitely many zeros in $\mathbb{C}$. Which zeros do they have in common?