Analytic Number Theory Fall 2016, Assignment 2

Deadline: Monday November 14

The total number of points is 60. Grade=(number of points)/6.

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5 **6.** *a)* Let $f : \mathbb{Z}_{>0} \to \mathbb{C}$ be a strongly multiplicative function, and $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ the associated Dirichlet series. Prove that for $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma_a(f)$ (abscissa of absolute convergence) we have

$$L_f(s)^{-1} = \sum_{n=1}^{\infty} f(n)\mu(n)n^{-s}, \quad \frac{L_f'(s)}{L_f(s)} = -\sum_{n=1}^{\infty} f(n)\Lambda(n)n^{-s}.$$

5 *b)* For any positive integer n put $\omega(1) := 0$ and $\omega(n) := t$ if t is the number of distinct primes dividing n (i.e., if $n = p_1^{k_1} \cdots p_t^{k_t}$ where $p_1, \dots, p_t$ are distinct primes and $k_i > 0$ for $i = 1, \dots, t$). Prove that for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ we have

$$\frac{\zeta(s)^2}{\zeta(2s)} = \sum_{n=1}^{\infty} 2^{\omega(n)} n^{-s}.$$

7. Let $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be a Dirichlet series.

5 *a)* Suppose that there are constants C, σ such that $|\sum_{n=1}^{N} f(n)| \leq CN^{\sigma}$ for every $N \geq 1$. Prove that $\sigma_0(f) \leq \sigma$, i.e., $L_f(s)$ converges for every $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma$. **Hint.** Use partial summation.

5 *b)* Prove that for every $\sigma > \max(0, \sigma_0(f))$ there is a constant $C > 0$ such that $|\sum_{n=1}^{N} f(n)| \leq CN^{\sigma}$ for every $N \geq 1$.

Does this result remain true if $\sigma_0(f) < 0$ and $\sigma_0(f) < \sigma < 0$?

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10 **8.** For an integer $q \geq 1$, denote by $F(q)$ the number of primitive characters modulo q. Prove that F is a multiplicative arithmetic function, and compute $F(p^k)$ for every prime power p^k.
Hint. Use Theorem 4.10 from the lecture notes.

9. In this exercise you have to use the identity

$$\zeta(s) = \frac{1}{s-1} + 1 - s\int_1^\infty (x - [x])x^{-1-s}dx \quad (\mathrm{Re}\, s > 0).$$

4 *a)* Prove that $\zeta(s) < 0$ for $s \in \mathbb{R}$ with $0 < s < 1$.

6 *b)* Around $s = 1$, $\zeta(s)$ has a Laurent series expansion

$$\frac{1}{s-1} + \sum_{n=0}^{\infty} a_n(s-1)^n.$$

Prove that $a_0 = \gamma$, where γ is the Euler-Mascheroni constant,

$$\gamma = \lim_{N\to\infty} \Big(\sum_{n=1}^{N} \frac{1}{n} - \log N\Big).$$

Hint. Explain that $a_0 = 1 - \int_1^\infty (x - [x])dx/x^2$ and compute the integral.

10. In this exercise you are asked to prove the following theorem: $\zeta(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$, with a pole with residue 1 at $s = 1$.

3 *a)* For $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$ and $k \in \mathbb{Z}_{>0}$, define $F_k(s) := \int_1^\infty (x - [x])^k x^{-s-k}dx$. Prove that

$$F_k(s) = \frac{1}{k+1}\Big(\zeta(s+k) - 1 + (s+k)F_{k+1}(s)\Big).$$

Hint. First express $F_{n,k}(s) := \int_n^{n+1} (x-n)^k x^{-s-k}dx$ in terms of $F_{n,k+1}(s)$ and then sum over n.

2 *b)* Let $r \in \mathbb{Z}_{\geq 1}$. Prove that for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$ we have

$$\zeta(s) = \frac{1}{s-1} + 1 - \sum_{k=1}^{r} \frac{s(s+1)\cdots(s+k-1)\cdot\big(\zeta(s+k)-1\big)}{(k+1)!} - \frac{s(s+1)\cdots(s+r)}{(r+1)!}F_{r+1}(s).$$

5 *c)* Prove for every $r \in \mathbb{Z}_{>0}$ that $\zeta(s)$ has an analytic continuation to the set $U_r := \{s \in \mathbb{C} : \operatorname{Re} s > -r\} \setminus \{1\}$, with a pole with residue 1 at $s = 1$. Then deduce the theorem.

Hint. Assume that it has been proved that $\zeta(s)$ has an analytic continuation to U_{r-1}, with a simple pole with residue 1 at $s = 1$. Then prove that the right-hand side of b) defines an analytic function on U_r.

11. Let q be an integer ≥ 2, and a an integer with $1 \leq a \leq q/2$.

5 *a)* Show that $\sum_{m=-\infty}^{\infty}(qm+a)^{-2}$ converges and prove that

$$\sum_{m=-\infty}^{\infty}(qm+a)^{-2} = \frac{(\pi/q)^2}{(\sin \pi a/q)^2}.$$

Use the method of proof of Exercise 4 with $f(z) = (qz+a)^{-2}$ (it is not necessary that f is even).

5 *b)* Let χ be a non-principal character modulo q with $\chi(-1) = 1$. Express $L(2, \chi)$ as a finite sum.