Chapter 3

Characters and Gauss sums

3.1 Characters on finite abelian groups

In what follows, abelian groups are multiplicatively written, and the unit element of an abelian group A is denoted by 1. We denote the order (number of elements) of A by $|A|$.

Let A be a finite abelian group. A *character* on A is a group homomorphism $\chi: A \to \mathbb{C}^*$ (i.e., $\mathbb{C} \setminus \{0\}$ with multiplication).

If $|A| = n$ then $a^n = 1$, hence $\chi(a)^n = 1$ for each $a \in A$ and each character χ on A. Therefore, a character on A maps A to the roots of unity.

The product $\chi_1\chi_2$ of two characters χ_1, χ_2 on A is defined by $(\chi_1\chi_2)(a) := \chi_1(a)\chi_2(a)$ for $a \in A$. With this product, the characters on A form an abelian group, the so-called *character group of* A, which we denote by $\widehat{A}$ (or $\mathrm{Hom}(A, \mathbb{C}^*)$). The unit element of $\widehat{A}$ is the trivial character $\chi_0^{(A)}$ that maps A to 1. Since any character on A maps A to the roots of unity, the inverse $\chi^{-1} : a \mapsto \chi(a)^{-1}$ of a character χ is equal to its complex conjugate $\overline{\chi} : a \mapsto \overline{\chi(a)}$.

We first construct an isomorphism from A to $\widehat{A}$. This will not be canonical, since it will depend on a choice of generators for A.

Lemma 3.1. *Let A be a cyclic group of order n. Then $\widehat{A}$ is also a cyclic group of order n.*

Proof. Let $A = \langle g \rangle$. Let ρ_1 be a primitive n-th root of unity. Since g has order n, there is a character χ_1 on A with $\chi_1(g) = \rho_1$. Clearly, χ_1 has order n. Let $\chi \in \widehat{A}$. Then $\chi(g)^n = 1$, so $\chi(g) = \rho_1^k$ for some integer k, and hence $\chi = \chi_1^k$ since a character on A is determined by its value in g. So $\widehat{A} = \langle \chi_1 \rangle$ is a cyclic group of order n. $\square$

Lemma 3.2. *Let $A = A_1 \times \cdots \times A_r$ be the direct product of finite abelian groups $A_1, \ldots, A_r$. Then $\widehat{A}$ is isomorphic to $\widehat{A_1} \times \cdots \times \widehat{A_r}$.*

Proof. Define a map

$$\begin{aligned} \varphi : \quad & \widehat{A_1} \times \cdots \times \widehat{A_r} \to \widehat{A} : \ (\chi_1, \ldots, \chi_r) \mapsto \chi_1 \cdots \chi_r, \\ & \chi_1 \cdots \chi_r((g_1, \ldots, g_r)) := \chi_1(g_1) \cdots \chi_r(g_r) \text{ for } g_i \in A_i, \ i = 1, \ldots, r. \end{aligned}$$

It is easy to see that φ is a group homomorphism. Substituting $g_j = 1_{A_j}$ for $j \neq i$, we see that χ_i is uniquely determined by $\chi_1 \cdots \chi_r$, for $i = 1, \ldots, r$. Hence φ is injective. Conversely, let $\chi \in \widehat{A}$, and for $i = 1, \ldots, r$ define $\chi_i \in \widehat{A_i}$ by

$$\chi_i(g_i) := \chi(\ldots, g_i, \ldots) \ \text{ for } g_i \in A_i,$$

with on the j-th place the unit element of A_i, for $j \neq i$. Then one easily verifies that $\chi = \chi_1 \cdots \chi_r$. Hence φ is also surjective. $\square$

Proposition 3.3. *Every finite abelian group is isomorphic to a direct product of cyclic groups.*

Proof. See S. Lang, Algebra, Chap.1, §10. $\square$

Theorem 3.4. *Let A be a finite abelian group. Then there exists an isomorphism from A to $\widehat{A}$. So in particular, $|\widehat{A}| = |A|$.*

Proof. By Proposition 3.3, A is isomorphic to a direct product $C_1 \times \cdots \times C_r$ of finite cyclic groups. By Lemmas 3.1, 3.2, $\widehat{C_i}$ is a cyclic group of the same order as C_i, for $i = 1, \ldots, r$, and $\widehat{A}$ is isomorphic to $\widehat{C_1} \times \cdots \times \widehat{C_r}$. Now the isomorphism from A to $\widehat{A}$ can be established by mapping a generator of C_i to one of $\widehat{C_i}$, for $i = 1, \ldots, r$. $\square$

Remark. The isomorphism constructed above depends on choices for generators of C_i, $\widehat{C_i}$, for $i = 1, \ldots, r$. So it is not canonical.

Corollary 3.5. *Let A be a finite abelian group, and $g \in A$ with $g \neq 1$. Then there is a character χ on A with $\chi(g) \neq 1$.*

Proof. First assume that $A = \langle g_1 \rangle$ is a cyclic group of order n. Then $g = g_1^k$ with $1 \leqslant k < n$. Let χ_1 be a generator of $\widehat{A}$ as constructed in the proof of Lemma 3.1. Then clearly, $\chi_1(g) \neq 1$.

Now let A be an arbitrary finite abelian group. We may assume that $A = C_1 \times \cdots \times C_r$, where $C_1, \ldots, C_r$ are finite cyclic groups, and $g = (g_1, \ldots, g_r)$ with $g_i \in C_i$ for $i = 1, \ldots, r$ and, say, $g_1 \neq 1_{C_1}$. Choose $\chi_1 \in \widehat{C_1}$ with $\chi_1(g_1) \neq 1$, let $\chi_2, \ldots, \chi_r$ be the principal characters on $C_2, \ldots, C_r$, and put $\chi := \chi_1 \cdots \chi_r$. Then clearly, $\chi(g) = \chi_1(g_1) \neq 1$. $\square$

For a finite abelian group A, let $\widehat{\widehat{A}}$ denote the character group of $\widehat{A}$. We construct a canonical isomorphism from A to $\widehat{\widehat{A}}$. Notice that each element $a \in A$ gives rise to a character $\widehat{a}$ on $\widehat{A}$, given by $\widehat{a}(\chi) := \chi(a)$.

Theorem 3.6 (Duality)**.** *Let A be a finite abelian group. Then the map $a \mapsto \widehat{a}$ defines an isomorphism from A to $\widehat{\widehat{A}}$.*

Proof. The map $\varphi : a \mapsto \widehat{a}$ obviously defines a group homomorphism from A to $\widehat{\widehat{A}}$. By Corollary 3.5 we have $\mathrm{Ker}(\varphi) = \{a \in A : \widehat{a}(\chi) = 1 \,\forall\, \chi \in \widehat{A}\} = \{1\}$; hence φ is injective. By Theorem 3.4 we have $|\widehat{\widehat{A}}| = |\widehat{A}| = |A|$. Hence φ is also surjective. $\square$

Theorem 3.7 (Orthogonality relations for characters)**.** *Let A be a finite abelian group.*

(i) For any two characters χ_1, χ_2 on A we have

$$\sum_{a \in A} \chi_1(a)\overline{\chi_2(a)} = \begin{cases} |A| & \text{if } \chi_1 = \chi_2, \\ 0 & \text{if } \chi_1 \neq \chi_2. \end{cases}$$

(ii) For any two elements a, b of A we have

$$\sum_{\chi \in \widehat{A}} \chi(a)\overline{\chi(b)} = \begin{cases} |A| & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

Proof. Part (ii) follows by applying part (i) with $\widehat{A}$ instead of A, and using Theorem 3.6 and $|\widehat{A}| = |A|$. So we prove only (i). Let $\chi_1, \chi_2 \in \widehat{A}$ and put $S := \sum_{a \in A} \chi_1(a)\overline{\chi_2(a)}$. Let $\chi := \chi_1\overline{\chi_2} = \chi_1\chi_2^{-1}$. Then $S = \sum_{a \in A} \chi(a)$. Clearly, if

$\chi_1 = \chi_2$ then $\chi = \chi_0^{(A)}$, hence $S = |A|$. Let $\chi_1 \neq \chi_2$. Then $\chi \neq \chi_0^{(A)}$, hence there is $g \in A$ with $\chi(g) \neq 1$. Further,

$$\chi(g)S = \sum_{a \in A} \chi(ga) = S,$$

since ga runs through the elements of A. Hence $S = 0$. $\square$

3.2 Dirichlet characters

Let $q \in \mathbb{Z}_{\geqslant 2}$. Denote the residue class of a mod q by $\overline{a}$. Recall that the prime residue classes mod q, $(\mathbb{Z}/q\mathbb{Z})^* = \{\overline{a} : \gcd(a, q) = 1\}$ form a group of order $\varphi(q)$ under multiplication of residue classes. We can lift any character $\widetilde{\chi}$ on $(\mathbb{Z}/q\mathbb{Z})^*$ to a map $\chi : \mathbb{Z} \to \mathbb{C}$ by setting

$$\chi(a) := \begin{cases} \widetilde{\chi}(\overline{a}) & \text{if } \gcd(a, q) = 1; \\ 0 & \text{if } \gcd(a, q) > 1. \end{cases}$$

Notice that χ has the following properties:

(i) $\chi(1) = 1$;
(ii) $\chi(ab) = \chi(a)\chi(b)$ for $a, b \in \mathbb{Z}$;
(iii) $\chi(a) = \chi(b)$ if $a \equiv b \pmod{q}$;
(iv) $\chi(a) = 0$ if $\gcd(a, q) > 1$.

Any map $\chi : \mathbb{Z} \to \mathbb{C}$ with properties (i)–(iv) is called a *(Dirichlet) character modulo* q. Conversely, from a character χ mod q one easily obtains a character $\widetilde{\chi}$ on $(\mathbb{Z}/q\mathbb{Z})^*$ by setting $\widetilde{\chi}(\overline{a}) := \chi(a)$ for $a \in \mathbb{Z}$ with $\gcd(a, q) = 1$.

Let $G(q)$ be the set of characters modulo q. We define the product $\chi_1\chi_2$ of $\chi_1, \chi_2 \in G(q)$ by $(\chi_1\chi_2)(a) = \chi_1(a)\chi_2(a)$ for $a \in \mathbb{Z}$. With this operation, $G(q)$ becomes a group, with unit element the *principal character modulo* q given by

$$\chi_0^{(q)}(a) = \begin{cases} 1 & \text{if } \gcd(a, q) = 1; \\ 0 & \text{if } \gcd(a, q) > 1. \end{cases}$$

The inverse of $\chi \in G(q)$ is its complex conjugate

$$\overline{\chi} : a \mapsto \overline{\chi(a)}.$$

It is clear, that this makes $G(q)$ into a group, and that $\chi \mapsto \widetilde{\chi}$ defines an isomorphism from $G(q)$ to the character group of $(\mathbb{Z}/q\mathbb{Z})^*$.

One of the advantages of viewing characters as maps from $\mathbb{Z}$ to $\mathbb{C}$ is that this allows to multiply characters of different moduli: if χ_1 is a character mod q_1 and χ_2 a character mod q_2, then their product $\chi_1\chi_2$ is a character mod $\operatorname{lcm}(q_1, q_2)$.

We can easily translate the orthogonality relations for characters of $(\mathbb{Z}/q\mathbb{Z})^*$ into orthogonality relations for Dirichlet characters modulo q. Recall that a *complete residue system modulo* q is a set, consisting of precisely one integer from every residue class modulo q, e.g., $\{3, 5, 11, 22, 104\}$ is a complete residue system modulo 5.

Theorem 3.8. *Let $q \in \mathbb{Z}_{\geqslant 2}$, and let S_q be a complete residue system modulo q.*
(i) Let $\chi_1, \chi_2 \in G(q)$. Then

$$\sum_{a \in S_q} \chi_1(a)\overline{\chi_2(a)} = \begin{cases} \varphi(q) & \text{if } \chi_1 = \chi_2; \\ 0 & \text{if } \chi_1 \neq \chi_2. \end{cases}$$

(ii) Let $a, b \in \mathbb{Z}$. Then

$$\sum_{\chi \in G(q)} \chi(a)\overline{\chi(b)} = \begin{cases} \varphi(q) & \text{if } \gcd(ab, q) = 1,\ a \equiv b \pmod{q}; \\ 0 & \text{if } \gcd(ab, q) = 1,\ a \not\equiv b \pmod{q}; \\ 0 & \text{if } \gcd(ab, q) > 1. \end{cases}$$

Proof. Easy exercise. □

Let χ be a character mod q and d a positive divisor of q.

We say that q is *induced* by a character χ' mod d if $\chi(a) = \chi'(a)$ for every $a \in \mathbb{Z}$ with $\gcd(a, q) = 1$. Here we define the principal character mod 1 by $\chi_0^{(1)}(a) = 1$ for $a \in \mathbb{Z}$. For instance, $\chi_0^{(q)}$ is induced by $\chi_0^{(1)}$. Notice that if $\gcd(a, d) = 1$ and $\gcd(a, q) > 1$, then $\chi'(a) \neq 0$ but $\chi(a) = 0$.

An alternative formulation of χ being induced by χ' is that $\chi = \chi' \cdot \chi_0^{(q)}$.

The *conductor* of χ is the smallest positive divisor d of q such that χ is induced by a character mod d.

We define the principal character mod 1 by $\chi_0^{(1)}(n) = 1$ for all $n \in \mathbb{Z}$. Clearly, if q is an integer $\geqslant 2$ then $\chi_0^{(q)}$ is induced by $\chi_0^{(1)}$, so $\chi_0^{(q)}$ has conductor 1.

A character χ is called *primitive* if there is no divisor $d < q$ of q such that χ is induced by a character mod d, in other words, if χ has conductor q.

Theorem 3.9. *Let $q \in \mathbb{Z}_{\geqslant 2}$, χ a character mod q. Denote by f the conductor of χ.*
(i) There is a unique character χ^* mod f that induces χ, and this is necessarily primitive.
(ii) Let d be a divisor of q and χ' a character mod d that induces χ. Then f is a divisor of d and χ^* induces χ'.

We need some lemmas.

Lemma 3.10. *Let d be a divisor of q and a an integer with $\gcd(a, d) = 1$. Then there is $b \in \mathbb{Z}$ with $b \equiv a \pmod{d}$, $\gcd(b, q) = 1$.*

Proof. Write $q = q_1 q_2$, where q_1 is composed of the primes occurring in the factorization of d, and where q_2 is composed of primes not dividing d. Thus, d and q_2 are coprime. By the Chinese Remainder Theorem, there is $b \in \mathbb{Z}$ with

$$b \equiv a \pmod{d}, \quad b \equiv 1 \pmod{q_2}.$$

This integer b is coprime with d, hence with q_1, and also coprime with q_2, so it is coprime with q. $\square$

Lemma 3.11. *Let χ be a character mod q, and d a divisor of q. Then there is at most one character mod d that induces χ.*

Proof. Suppose χ is induced by a character χ_1 mod d. Let $a \in \mathbb{Z}$ with $\gcd(a, d) = 1$. Choose $b \in \mathbb{Z}$ with $b \equiv a \pmod{d}$ and $\gcd(b, q) = 1$. Then $\chi_1(a) = \chi_1(b) = \chi(b)$. Hence χ_1 is uniquely determined by χ. $\square$

The next lemma gives a method to verify if a character χ is induced by a character mod d.

Lemma 3.12. *Let χ be a character mod q, and d a divisor of q. Then the following assertions are equivalent:*
(i) χ is induced by a character mod d;
(ii) $\chi(a) = \chi(b)$ for all $a, b \in \mathbb{Z}$ with $a \equiv b \pmod{d}$ and $\gcd(ab, q) = 1$;
(iii) $\chi(a) = 1$ for all $a \in \mathbb{Z}$ with $a \equiv 1 \pmod{d}$ and $\gcd(a, q) = 1$.

Proof. The implications (i)⇒(ii)⇒(iii) are trivial.

(iii) ⇒ (ii). Let $a, b \in \mathbb{Z}$ with $a \equiv b \pmod{d}$ and $\gcd(ab, q) = 1$. There is $c \in \mathbb{Z}$ with $\gcd(c, q) = 1$ such that $a \equiv bc \pmod{q}$. For this c we have $c \equiv 1 \pmod{d}$. Now by (iii) we have $\chi(a) = \chi(b)\chi(c) = \chi(b)$.

(ii) ⇒ (i). We define a character χ' mod d as follows. For $a \in \mathbb{Z}$ with $\gcd(a, d) > 1$ put $\chi'(a) := 0$. For $a \in \mathbb{Z}$ with $\gcd(a, d) = 1$, choose $b \in \mathbb{Z}$ such that $b \equiv a \pmod{d}$ and $\gcd(b, q) = 1$ (which is possible by Lemma 3.10), and put $\chi'(a) := \chi(b)$. By (ii) this gives a well-defined character mod d that clearly induces χ. □

Remark. Notice that this lemma provides a method to compute the conductor of a character χ mod q: check for every divisor d of q whether $\chi(a) = 1$ for all integers a with $1 \leqslant a < q$, $a \equiv 1 \pmod{d}$ and $\gcd(a, q) = 1$. The smallest divisor d of q for which this holds is the conductor of χ.

Lemma 3.13. *Let χ be a character mod q. Let d_1, d_2 be divisors of q. Assume that χ is induced by characters χ_1 mod d_1, χ_2 mod d_2. Then there is a character χ_3 mod* $\gcd(d_1, d_2)$ *that induces χ, χ_1 and χ_2.*

Proof. Let $d := \gcd(d_1, d_2)$, $d_0 := \mathrm{lcm}(d_1, d_2)$. We first show that χ_1 is induced by a character mod d. We apply criterion (iii) of the previous lemma. That is, we have to show that if a is an integer with $\gcd(a, d_1) = 1$ and $a \equiv 1 \pmod{d}$, then $\chi_1(a) = 1$.

Take such a. Then $a = 1 + td$ with $t \in \mathbb{Z}$. There are $x, y \in \mathbb{Z}$ with $xd_1 + yd_2 = d$. Hence $a = 1 + txd_1 + tyd_2$. The number $c := 1 + tyd_2 = a - txd_1$ is clearly coprime with d_2, and it is also coprime with d_1 since a is coprime with d_1. Hence c is coprime with d_0. By Lemma 3.10, there is b with $b \equiv c \pmod{d_0}$ and $\gcd(b, q) = 1$. We have $b \equiv a \pmod{d_1}$, $b \equiv 1 \pmod{d_2}$. So by Lemma 3.12 applied with d_1 and d_2, $\chi_1(a) = \chi(b) = \chi_2(1) = 1$.

It follows that χ_1 is induced by a character, say χ_3 mod d. Similarly, χ_2 is induced by a character χ_3' mod d. Both χ_3, χ_3' induce χ. So by Lemma 3.11, $\chi_3 = \chi_3'$. □

Proof of Theorem 3.9. (i) By Lemma 3.11 there is a unique character χ^* mod f inducing χ. If χ^* were induced by a character χ' modulo a divisor $d < f$ of f, then χ were induced by χ', contradicting the definition of the conductor. So χ^* is primitive.

(ii) By Lemma 3.13 there is a character χ'' mod $\gcd(d, f)$ inducing χ, χ^* and χ'. Since χ^* is primitive we must have $f|d$ and $\chi'' = \chi^*$. So χ^* induces χ'. □

3.3 Computation of $G(q)$

We give a method to compute the character group modulo q. We first make a reduction to prime powers.

Theorem 3.14. *Let $q = p_1^{k_1} \cdots q_t^{k_t}$, where $p_1, \ldots, p_t$ are distinct primes and $k_1, \ldots, k_t$ positive integers. Then the map*

$$G(p_1^{k_1}) \times \cdots \times G(p_t^{k_t}) \to G(q) : \ (\chi_1, \ldots, \chi_t) \mapsto \chi_1 \cdots \chi_t$$

is a group isomorphism.

Proof. Let ρ denote the map under consideration. Then ρ is a homomorphism. Since $G(p_1^{k_1}) \times \cdots \times G(p_t^{k_t})$ and $G(q)$ have the same order $\varphi(q)$, it suffices to show that ρ is injective. That is, we have to show that if $\chi_i \in G(p_i^{k_i})$ $(i = 1, \ldots, t)$ are such that $\chi_1 \cdots \chi_t = \chi_0^{(q)}$, then $\chi_i = \chi_0^{(p_i^{k_i})}$ for $i = 1, \ldots, t$.

To prove this, let $i \in \{1, \ldots, t\}$ and $a \in \mathbb{Z}$ with $\gcd(a, p_i) = 1$. By the Chinese Remainder Theorem, there is $b \in \mathbb{Z}$ such that

$$b \equiv a \,(\mathrm{mod}\, p_i^{k_i}), \quad b \equiv 1 \,(\mathrm{mod}\, p_j^{k_j}) \text{ for } j \neq i,$$

and using this b we infer $\chi_i(a) = \chi_1(b) \cdots \chi_t(b) = \chi_0^{(q)}(b) = 1$. Hence $\chi_i = \chi_0^{(p_i^{k_i})}$. □

To compute $G(p^k)$ for a prime power p^k, we need some information about the structure of $(\mathbb{Z}/p^k\mathbb{Z})^*$. This is provided by the following theorem.

Theorem 3.15. *(i) Let p be a prime $\geqslant 3$. Then the group $(\mathbb{Z}/p^k\mathbb{Z})^*$ is cyclic of order $p^{k-1}(p-1)$.*
(ii) $(\mathbb{Z}/4\mathbb{Z})^*$ is cyclic of order 2.
Further, if $k \geqslant 3$ then $(\mathbb{Z}/2^k\mathbb{Z})^* = \langle \overline{-1} \rangle \times \langle \overline{5} \rangle$ is isomorphic to the direct product of a cyclic group of order 2 and a cyclic group of order 2^{k-2}.

We skip the proof of $k = 1$ of (i), which belongs to a basic algebra course. For the proof of the remaining parts, we need a lemma.

For a prime number p, and for $a \in \mathbb{Z} \setminus \{0\}$, we denote by $\mathrm{ord}_p(a)$ the largest integer k such that p^k divides a.

Lemma 3.16. *Let p be a prime number and a an integer such that* $\mathrm{ord}_p(a-1) \geqslant 1$ *if $p \geqslant 3$ and* $\mathrm{ord}_p(a-1) \geqslant 2$ *if $p = 2$. Then*

$$\mathrm{ord}_p(a^{p^k} - 1) = \mathrm{ord}_p(a-1) + k.$$

Proof. We prove the assertion only for $k = 1$; then the general statement follows easily by induction on k. Our assumption on a implies that $a = 1 + p^t b$, where $t \geqslant 1$ if $p \geqslant 3$ and $t \geqslant 2$ if $p = 2$, and where b is an integer not divisible by p. By the binomial formula,

$$a^p - 1 = p^{t+1}b + \binom{p}{2}p^{2t}b^{2t} + \cdots + \binom{p}{p-1}p^{(p-1)t}b^{(p-1)t} + p^{pt}b^{pt} \equiv p^{t+1}b \pmod{p^{t+2}}$$

since $\binom{p}{2}, \ldots, \binom{p}{p-1}$ are all divisible by p and $pt \geqslant t+2$ in both the cases $p \geqslant 3$, $p = 2$. So $\mathrm{ord}_p(a^p - 1) = t + 1$. □

Lemma 3.17. *Let $p \geqslant 3$ be a prime number. Then there is an integer g such that $g \pmod p$ is a generator of* $(\mathbb{Z}/p\mathbb{Z})^*$ *and* $\mathrm{ord}_p(g^{p-1} - 1) = 1$.

Proof. We take for granted that $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic of order $p - 1$; then there is an integer h such that $h \pmod p$ is a generator of $(\mathbb{Z}/p\mathbb{Z})^*$. So $\mathrm{ord}_p(h^{p-1} - 1) \geqslant 1$. Put $g := h$ if $\mathrm{ord}_p(h^{p-1} - 1) = 1$ and $g := h + p$ if $\mathrm{ord}_p(h^{p-1} - 1) \geqslant 2$. In the latter case, we have

$$g^{p-1} - 1 = h^{p-1} - 1 + (p-1)h^{p-2}p + \binom{p-1}{2}h^{p-3}p^2 + \cdots + p^{p-1} \equiv -h^{p-2}p \pmod{p^2},$$

hence $\mathrm{ord}_p(g^{p-1} - 1) = 1$. □

Proof of Theorem 3.15. (i). Let $p \geqslant 3$ and $k \geqslant 2$. Take g as in Lemma 3.17. We show that $\overline{g} := g \pmod{p^k}$ generates $(\mathbb{Z}/p^k\mathbb{Z})^*$ or equivalently, that the order n of $\overline{g}$ in $(\mathbb{Z}/p^k\mathbb{Z})^*$ equals the order of $(\mathbb{Z}/p^k\mathbb{Z})^*$, which is $p^{k-1}(p-1)$. In any case, n divides $p^{k-1}(p-1)$. Further, $g^n \equiv 1 \pmod p$, hence $p - 1$ divides n. So $n = p^s(p-1)$ with $s \leqslant k - 1$. By Lemma 3.16 we have

$$\mathrm{ord}_p(g^n - 1) = \mathrm{ord}_p(g^{p-1} - 1) + s = s + 1.$$

This has to be at least k, so $s = k - 1$. Hence indeed $n = p^{k-1}(p-1)$.

(ii). Assume that $k \geqslant 3$. Define the subgroup

$$H := \{\overline{a} \in (\mathbb{Z}/2^k\mathbb{Z})^* : a \equiv 1 \pmod 4\}.$$

Note that $\overline{a} \in (\overline{-1})H$ if $a \equiv 3 \pmod 4$. So

$$(\mathbb{Z}/2^k\mathbb{Z})^* = H \cup (\overline{-1})H = \langle\overline{-1}\rangle \times H.$$

Similarly as above, one shows that H is cyclic of order 2^{k-2}, and that $H = \langle\overline{5}\rangle$. $\square$

We can now give an explicit description for the groups $G(p^k)$, following the proofs of Lemmas 3.1, 3.2.

If $p > 2$, choose $g \in \mathbb{Z}$ such that $g \pmod{p^k}$ generates $(\mathbb{Z}/p^k\mathbb{Z})^*$, and choose a primitive $p^{k-1}(p-1)$-th root of unity ρ. Then $G(p^k) = \langle\chi_1\rangle$ where χ_1 is the Dirichlet character determined by $\chi_1(g) = \rho$, and $G(p^k)$ is cyclic of order $p^{k-1}(p-1)$.

Clearly, $G(2) = \{\chi_0^{(2)}\}$ and $G(4) = \{\chi_0^{(4)}, \chi_4\}$, where $\chi_4(a) = 1$ if $a \equiv 1 \pmod 4$, $\chi_4(a) = -1$ if $a \equiv 3 \pmod 4$, $\chi_4(a) = 0$ if a is even.

As for 2^k with $k \geqslant 3$, choose a primitive 2^{k-2}-th root of unity ρ. Then $G(2^k) = \langle\chi_1\rangle \times \langle\chi_2\rangle$, where χ_1, χ_2 are given by

$$\chi_1(-1) = -1,\ \chi_1(5) = 1; \quad \chi_2(-1) = 1,\ \chi_2(5) = \rho,$$

χ_1 has order 2, and χ_2 has order 2^{k-2}.

3.4 Gauss sums

Let $q \in \mathbb{Z}_{\geqslant 2}$. For a character χ mod q and for $b \in \mathbb{Z}$, we define the Gauss sum

$$\tau(b, \chi) := \sum_{a \in S_q} \chi(a) e^{2\pi i b a/q},$$

where S_q is a full system of representatives modulo q. This does not depend on the choice of S_q. The Gauss sum $\tau(1, \chi)$ occurs for instance in the functional equation for the L-function $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ (later).

We prove some basic properties of Gauss sums.

Theorem 3.18. *Let $q \in \mathbb{Z}_{\geqslant 2}$ and let χ be a character mod q. Further, let $b \in \mathbb{Z}$.*
(i) If $\gcd(b,q) = 1$, then $\tau(b,\chi) = \overline{\chi(b)} \cdot \tau(1,\chi)$.
(ii) If $\gcd(b,q) > 1$ and χ is primitive, then $\tau(b,\chi) = \overline{\chi(b)} \cdot \tau(1,\chi) = 0$.

Proof. (i) Suppose $\gcd(b,q) = 1$. If a runs through a complete residue system S_q mod q, then ba runs through another complete residue system S'_q mod q. Write $y = ba$. Then $\chi(y) = \chi(b)\chi(a)$, hence $\chi(a) = \overline{\chi(b)}\chi(y)$. Therefore,

$$\begin{aligned}\tau(b,\chi) &= \sum_{a \in S_q} \chi(a) e^{2\pi i ba/q} = \sum_{y \in S'_q} \overline{\chi(b)} \chi(y) e^{2\pi i y/q} \\ &= \overline{\chi(b)} \cdot \tau(1,\chi).\end{aligned}$$

(ii) Let $\gcd(b,q) =: d > 1$ and put $b_1 := b/d$, $q_1 := q/d$. Then χ is not induced by a character mod q_1, so by Lemma 3.12 there is $c \in \mathbb{Z}$ such that $c \equiv 1 \pmod{q_1}$, $\gcd(c,q) = 1$, and $\chi(c) \neq 1$. With this c we have

$$\chi(c)\tau(b,\chi) = \sum_{a \in S_q} \chi(ca) e^{2\pi i ba/q}.$$

If a runs through a complete residue system S_q mod q, then $y := ca$ runs through another complete residue system S'_q mod q. Further, since $c \equiv 1 \pmod{q_1}$ we have

$$e^{2\pi i ab/q} = e^{2\pi i ab_1/q_1} = e^{2\pi i cab_1/q_1} = e^{2\pi i yb/q}.$$

Hence

$$\chi(c)\tau(b,\chi) = \sum_{y \in S'_q} \chi(y) e^{2\pi i by/q} = \tau(b,\chi).$$

Since $\chi(c) \neq 1$ this implies that $\tau(b,\chi) = 0$. $\square$

Theorem 3.19. *Let $q \in \mathbb{Z}_{\geqslant 2}$ and let χ be a primitive character mod q. Then*

$$|\tau(1,\chi)| = \sqrt{q}.$$

Proof. We have by Theorem 3.18,

$$\begin{aligned}
|\tau(1,\chi)|^2 &= \overline{\tau(1,\chi)}\cdot\tau(1,\chi) = \sum_{a=0}^{q-1}\overline{\chi(a)}e^{-2\pi ia/q}\tau(1,\chi) \\
&= \sum_{a=0}^{q-1} e^{-2\pi ia/q}\tau(a,\chi) = \sum_{a=0}^{q-1} e^{-2\pi ia/q}\left(\sum_{b=0}^{q-1}\chi(b)e^{2\pi iab/q}\right) \\
&= \sum_{a=0}^{q-1}\left(\sum_{b=0}^{q-1}\chi(b)e^{2\pi ia(b-1)/q}\right) \\
&= \sum_{b=0}^{q-1}\chi(b)\left(\sum_{a=0}^{q-1}e^{2\pi ia(b-1)/q}\right) = \sum_{b=0}^{q-1}\chi(b)S(b), \text{ say.}
\end{aligned}$$

If $b = 1$, then $S(b) = \sum_{a=0}^{q-1} 1 = q$, while if $b \neq 1$, then by the sum formula for geometric sequences,

$$S(b) = \frac{e^{2\pi i(b-1)} - 1}{e^{2\pi i(b-1)/q} - 1} = 0.$$

Hence $|\tau(1,\chi)|^2 = \chi(1)q = q$. □

Remark. Theorem 3.19 implies that $\varepsilon_\chi := \tau(1,\chi)/\sqrt{q}$ lies on the unit circle. Gauss gave an easy explicit expression for ε_χ in the case that χ is a primitive real character mod q, i.e., χ assumes its values in $\mathbb{R}$, so in $\{0,\pm1\}$. There is no general efficient method known to compute ε_χ for non-real characters χ modulo large values of q.

3.5 Character sums

For many purposes one needs good estimates for expressions $|\sum_{a=M+1}^{M+N}\chi(a)|$, where χ is a non-principal character modulo an integer $q \geqslant 2$. We prove the following classic result, which, apart from the constant 3 in front of $\sqrt{q}\log q$, was obtained independently by Polyá and I.N. Vinogradov in 1918.

Theorem 3.20. *Let q be an integer $\geqslant 2$, χ a non-principal character modulo q, and M, N integers with $N \geqslant 1$. Then*

$$\left|\sum_{a=M+1}^{M+N}\chi(a)\right| \leqslant 3\sqrt{q}\log q.$$

Of course, the left-hand side is at most N. So this estimate is non-trivial only if $N > 3\sqrt{q}\log q$.

We need the following simple exponential sum estimate.

Lemma 3.21. *Let* $0 < x < 1$. *Then*

$$\left|\sum_{a=M+1}^{M+N} e^{2\pi i a x}\right| \leqslant \frac{1}{2}\cdot\max\Big(\frac{1}{x},\ \frac{1}{1-x}\Big).$$

Proof. By the sum formula for geometric series,

$$\begin{aligned}(3.1)\quad \sum_{a=M+1}^{M+N} e^{2\pi i a x} &= e^{2(M+1)\pi i x}\cdot\frac{e^{2N\pi i x}-1}{e^{2\pi i x}-1} = e^{(2M+N+1)\pi i x}\cdot\frac{e^{N\pi i x}-e^{-N\pi i x}}{e^{\pi i x}-e^{-\pi i x}}\\ &= e^{(2M+N+1)\pi i x}\cdot\frac{\sin(\pi N x)}{\sin(\pi x)}.\end{aligned}$$

The lemma easily follows by taking absolute values, using $|e^{\pi i y}| = 1$ and $|\sin \pi y| \leqslant 1$ for every $y \in \mathbb{R}$, and $\sin \pi y \geqslant 2\min(y, 1-y)$ for every y with $0 \leqslant y \leqslant 1$ (check the graph of sin). $\square$

Proof of Theorem 3.20. We give an elementary proof, due to Schur (1918). We first assume that χ is a primitive character modulo q. Then by Theorem 3.18,

$$\begin{aligned}\sum_{a=M+1}^{M+N}\overline{\chi(a)} &= \tau(1,\chi)^{-1}\sum_{a=M+1}^{M+N}\tau(a,\chi)\\ &= \tau(1,\chi)^{-1}\sum_{a=M+1}^{M+N}\Big(\sum_{n=1}^{q-1}\chi(n)e^{2\pi i a n/q}\Big)\\ &= \tau(1,\chi)^{-1}\sum_{n=1}^{q-1}\chi(n)\Big(\sum_{a=M+1}^{M+N} e^{2\pi i a n/q}\Big).\end{aligned}$$

Now from Theorem 3.19, $|\chi(n)| \leqslant 1$ for all n and Lemma 3.21, we infer

$$\begin{aligned}\left|\sum_{a=M+1}^{M+N}\chi(a)\right| &\leqslant \sqrt{q}^{-1}\sum_{n=1}^{q-1}\cdot\frac{1}{2}\cdot\max\Big(\frac{1}{n/q},\frac{1}{1-(n/q)}\Big)\\ &\leqslant \sqrt{q}\sum_{n=1}^{[q/2]}\frac{1}{n} \leqslant \sqrt{q}\Big(1+\int_1^{q/2}\frac{dx}{x}\Big) = \sqrt{q}\Big(1+\log(q/2)\Big),\end{aligned}$$

(clear from the graph of $1/x$) and thus, using $1+\log(x/2) \leqslant \frac{3}{2}\log x$ for $x \geqslant 2$,

$$\left|\sum_{a=M+1}^{M+N} \chi(a)\right| \leqslant \frac{3}{2}\sqrt{q}\log q. \tag{3.2}$$

This proves our theorem for primitive characters χ modulo q. Now let χ be a non-primitive, non-principal character modulo q, and let f be the conductor of χ. Then χ is induced by a primitive character χ^* modulo f. We write $q = f \cdot q'$. If $\gcd(a, q') = 1$ then $\gcd(a, f) = \gcd(a, q)$, hence $\chi(a) = \chi^*(a)$. If $\gcd(a, q') > 1$, then $\chi(a) = 0$. Thus,

$$\sum_{a=M+1}^{M+N} \chi(a) = \sum_{\substack{a=M+1 \\ \gcd(a,q')=1}}^{M+N} \chi^*(a).$$

The following trick is used quite often. Recall the property of the Möbius function

$$\sum_{d|q',\, d|a} \mu(d) = \sum_{d|\gcd(a,q')} \mu(d) = \begin{cases} 1 & \text{if } \gcd(a,q') = 1, \\ 0 & \text{if } \gcd(a,q') = 0. \end{cases}$$

By inserting this into the above identity and interchanging the summations, we obtain

$$\begin{aligned}
\sum_{a=M+1}^{M+N} \chi(a) &= \sum_{a=M+1}^{M+N} \Big(\sum_{d|q',\, d|a} \mu(d)\Big)\chi^*(a) \\
&= \sum_{d|q'} \mu(d)\Big(\sum_{\substack{a=M+1 \\ a\equiv 0 \,(\mathrm{mod}\, d)}}^{M+N} \chi^*(a)\Big) \\
&= \sum_{d|q'} \mu(d)\chi^*(d)\Big(\sum_{(M+1)/d \leqslant b \leqslant (M+N)/d} \chi^*(b)\Big),
\end{aligned}$$

where we have written $a = db$ and used the multiplicativity of χ^*. The inner sum has absolute value at most $\frac{3}{2}\sqrt{f}\log f$ by (3.2) with χ^*, f instead of χ, q, the quantities $\mu(d)$ and $\chi^*(d)$ have absolute value at most 1 and the number of summands d is precisely the number of divisors $\tau(q')$ of q'. Hence

$$\left|\sum_{a=M+1}^{M+N} \chi(a)\right| \leqslant \frac{3}{2}\tau(q')\sqrt{f}\log f.$$

Note that for each divisor d of q' with $\sqrt{q'} \leqslant d \leqslant q$ there is a divisor $q'/d \leqslant \sqrt{q'}$. Hence $\tau(q') \leqslant 2\sqrt{q'}$ (of course there are much better estimates). Since also $f \leqslant q$, we arrive at

$$\left|\sum_{a=M+1}^{M+N} \chi(a)\right| \leqslant 3\sqrt{q'}\sqrt{f}\log f \leqslant 3\sqrt{q}\log q.$$

□

We mention that the estimate in Theorem 3.20 can not be improved very much, since by a result of Schur, for every primitive character χ modulo an integer $q \geqslant 2$ one has

$$\max_N \left|\sum_{a=1}^{N} \chi(a)\right| > \frac{\sqrt{q}}{2\pi}.$$

As mentioned above, Theorem 3.20 improves the trivial bound N only if $N > 3\sqrt{q}\log q$. It would be important to have non-trivial estimates also for smaller values of N. Burgess proved in 1962 that for every $\varepsilon > 0$ there is a number $C(\varepsilon) > 0$ such that for every integer $q \geqslant 2$, every primitive character χ modulo q, and every pair of integers M, N with $N > 0$,

$$\left|\sum_{a=M+1}^{M+N} \chi(a)\right| \leqslant C(\varepsilon) N^{1/2} q^{(3/16)+\varepsilon}.$$

This upper bound is non-trivial (smaller than N) if $N \gg q^{(3/8)+2\varepsilon}$.

3.6 Quadratic reciprocity

We give an analytic proof of Gauss' Quadratic Reciprocity Theorem, by computing certain special Gauss sums.

Let $p > 2$ be a prime number. An integer a is called a *quadratic residue modulo* p if $x^2 \equiv a \,(\mathrm{mod}\, p)$ is solvable in $x \in \mathbb{Z}$ and $p \nmid a$, and a *quadratic non-residue modulo* p if $x^2 \equiv a \,(\mathrm{mod}\, p)$ is not solvable in $x \in \mathbb{Z}$. Further, a quadratic (non-)residue class modulo p is a residue class modulo p represented by a quadratic (non-)residue.

We define the *Legendre symbol*

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p; \\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } p; \\ 0 & \text{if } p|a. \end{cases}$$

Lemma 3.22. *Let p be a prime > 2.*
(i) $\left(\frac{\cdot}{p}\right)$ is a primitive character mod p.
(ii) There are precisely $\frac{1}{2}(p-1)$ quadratic residue classes, and precisely $\frac{1}{2}(p-1)$ quadratic non-residue classes modulo p.
(iii) $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod p$ for $a \in \mathbb{Z}$.

Proof. (i) The group $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic of order $p-1$. Let $g(\text{mod } p)$ be a generator of this group. Take $a \in \mathbb{Z}$ with $\gcd(a,p) = 1$. Then there is $t \in \mathbb{Z}$ such that $a \equiv g^t \pmod p$. Now clearly, $x^2 \equiv a \pmod p$ is solvable in $x \in \mathbb{Z}$ if and only if t is even. Hence $\left(\frac{a}{p}\right) = (-1)^t$. This shows that $\left(\frac{\cdot}{p}\right)$ is a character mod p. It is not the principal character mod p, since $\left(\frac{g}{p}\right) = -1$. Since p is a prime, it must be primitive.

(ii) The group $(\mathbb{Z}/p\mathbb{Z})^*$ consists of $g^t \pmod p$ $(t = 0, \dots, p-1)$. As we have seen, the quadratic residue classes are those with t even, and the quadratic non-residue classes those with t odd. This implies (ii).

(iii) The assertion is clearly true if $p|a$. Assume that $p \nmid a$. Then there is $t \in \mathbb{Z}$ with $a \equiv g^t \pmod p$. Note that $(g^{(p-1)/2})^2 \equiv 1 \pmod p$, hence $g^{(p-1)/2} \equiv \pm 1 \pmod p$. But $g^{(p-1)/2} \not\equiv 1 \pmod p$ since $g \pmod p$ is a generator of $(\mathbb{Z}/p\mathbb{Z})^*$. Hence $g^{(p-1)/2} \equiv -1 \pmod p$. As a consequence,

$$a^{(p-1)/2} \equiv (-1)^t \equiv \left(\frac{a}{p}\right) \pmod p.$$

□

The following is immediate:

Corollary 3.23. *Let p be a prime > 2. Then*

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = \begin{cases} 1 & \textit{if } p \equiv 1 \pmod 4, \\ -1 & \textit{if } p \equiv 3 \pmod 4. \end{cases}$$

We now come to the formulation of Gauss' Quadratic Reciprocity Theorem:

Theorem 3.24. *Let p, q be distinct primes > 2. Then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4} = \begin{cases} -1 & \text{if } p \equiv q \equiv 3 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore, as a supplement we have:

Theorem 3.25. *Let p be a prime > 2. Then*

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod 8, \\ -1 & \text{if } p \equiv \pm 3 \pmod 8). \end{cases}$$

Example. Check if $x^2 \equiv 33 \pmod{97}$ is solvable.

$$\begin{aligned} \left(\frac{33}{97}\right) &= \left(\frac{3}{97}\right)\cdot\left(\frac{11}{97}\right) = \left(\frac{97}{3}\right)\cdot\left(\frac{97}{11}\right) \\ &= \left(\frac{1}{3}\right)\cdot\left(\frac{-2}{11}\right) = \left(\frac{1}{3}\right)\cdot\left(\frac{-1}{11}\right)\cdot\left(\frac{2}{11}\right) = 1\cdot(-1)\cdot(-1) = 1. \end{aligned}$$

We prove only Theorem 3.24 and leave Theorem 3.25 as an exercise. We give an analytic proof, based on exponential sums $S(q) := \sum_{x=0}^{q-1} e^{2\pi i x^2/q}$, which are closely connected to certain Gauss sums.

We start with a simple result from Fourier analysis, which will be used also elsewhere.

We define the Fourier coefficients of an integrable function $f : [0,1] \to \mathbb{C}$ by

$$c_n(f) := \int_0^1 f(t)e^{-2\pi i n t}dt \quad \text{for } n \in \mathbb{Z}.$$

Theorem 3.26. *Let f be a complex analytic function, defined on an open subset of $\mathbb{C}$ containing the real interval $[0,1]$. Then*

$$\lim_{N\to\infty} \sum_{n=-N}^{N} c_n(f) = \tfrac{1}{2}\big(f(0)+f(1)\big).$$

Remarks.

1. Theorem 3.26 holds in fact for measurable functions $f : [0,1] \to \mathbb{C}$ for which

$\int_0^1 |f(t)|dt < \infty$ and f has bounded variation. The version we state and prove with a much more restrictive condition on f is amply sufficient for our purposes.

2. It may be that $\lim_{N\to\infty}\sum_{n=-N}^{N} a_n$ converges, whereas the doubly infinite series $\sum_{n=-\infty}^{\infty} a_n = \lim_{M,N\to\infty}\sum_{n=-M}^{N} a_n$ (with $M,N\to\infty$ independently of each other) diverges. For instance, if $a_{-n} = -a_n$ for $n\in\mathbb{Z}\setminus\{0\}$, then $\lim_{N\to\infty}\sum_{n=-N}^{N} a_n = a_0$, but $\sum_{n=-\infty}^{\infty} a_n$ may be horribly divergent.

Proof. We first consider some special cases. For the constant function $f(z)=1$ we have $c_0(f)=1$, while $c_n(f)=0$ for $n\neq 0$, and so in this case, $\sum_{n=-N}^{N} c_n(f) = 1 = \frac{1}{2}(f(0)+f(1))$ for all N. For the function $f(z)=z$ we have $c_0(f)=\frac{1}{2}$, while $c_n(f) = -\frac{1}{2\pi i n}$ for $n\neq 0$. So also in this case, $\sum_{n=-N}^{N} c_n(f) = \frac{1}{2} = \frac{1}{2}(f(0)+f(1))$ for all N.

We now take an arbitrary function f as in the statement of the theorem, say analytic on an open subset U of $\mathbb{C}$ containing $[0,1]$. Define the function $f^*(z) := f(z) - f(0) + (f(0)-f(1))z$. Then f^* is analytic on U and $f^*(0)=f^*(1)=0$. We prove that $\lim_{N\to\infty}\sum_{n=-N}^{N} c_n(f^*) = 0$. Together with the special cases just considered and the linearity of $c_n(\cdot)$ over $\mathbb{C}$ this implies $\lim_{N\to\infty}\sum_{n=-N}^{N} c_n(f) = \frac{1}{2}(f(0)+f(1))$.

From the identity

$$\begin{aligned}\sum_{n=-N}^{N} e^{-2\pi i n t} &= e^{2\pi i N t}\sum_{n=0}^{2N} e^{-2\pi i n t} = e^{2\pi i N t}\cdot\frac{e^{-2\pi i(2N+1)t}-1}{e^{-2\pi i t}-1}\\ &= \frac{e^{-\pi i(2N+1)t}-e^{\pi i(2N+1)t}}{e^{-\pi i t}-e^{\pi i t}} = \frac{\sin((2N+1)\pi t)}{\sin \pi t}\end{aligned}$$

we obtain

$$\sum_{n=-N}^{N} c_n(f^*) = \int_0^1 \frac{f^*(t)}{\sin \pi t}\cdot\sin((2N+1)\pi t)\cdot dt = \int_0^1 g(t)\cdot\sin(h_N(t))dt,$$

where

$$g(z) := \frac{f^*(z)}{\sin \pi z},\quad h_N(z) := (2N+1)\pi z.$$

Assume that U is small enough, so that it does not contain any integers other than $0,1$. Then g is analytic on U. Indeed, $\sin \pi z\neq 0$ on U except at $z=0$, $z=1$ where it has simple zeros, but these are cancelled by the zeros of f^* at $z=0$, $z=1$. Now using integration by parts, we obtain

$$\begin{aligned}
\left|\sum_{n=-N}^{N} c_n(f^*)\right| &= \left|\int_0^1 g(t)\sin(h_N(t))dt\right| = \tfrac{1}{(2N+1)\pi}\left|\int_0^1 g(t)d\cos(h_N(t))\right| \\
&= \tfrac{1}{(2N+1)\pi}\left|-g(1)-g(0)-\int_0^1 g'(t)\cos(h_N(t))dt\right| \\
&\leqslant \tfrac{1}{(2N+1)\pi}\Big(|g(1)|+|g(0)|+\int_0^1 |g'(t)|dt\Big) \to 0 \ \text{ as } N\to\infty.
\end{aligned}$$

Here we used that g' is analytic on U, hence $t \mapsto |g'(t)|$ is continuous and bounded on $[0,1]$. This completes our proof. $\square$

Corollary 3.27 (Poisson's summation formula for finite sums)**.** *Let a, b be integers with $a<b$ and let f be a complex analytic function, defined on an open subset of $\mathbb{C}$ containing the interval $[a,b]$. Then*

$$\begin{aligned}
\sum_{m=a}^{b} f(m) &= \tfrac{1}{2}\big(f(a)+f(b)\big) + \lim_{N\to\infty}\sum_{n=-N}^{N}\int_a^b f(t)e^{-2\pi i n t}dt \\
&= \tfrac{1}{2}\big(f(a)+f(b))\big) + \int_a^b f(t)dt + 2\sum_{n=1}^{\infty}\int_a^b f(t)\cos 2\pi n t\cdot dt.
\end{aligned}$$

Proof. Pick $m \in \{a,\dots,b-1\}$. Then by Theorem 3.26, applied to $z\mapsto f(z+m)$, using $e^{2\pi i m}=1$,

$$\begin{aligned}
\tfrac{1}{2}\big(f(m)+f(m+1)\big) &= \lim_{N\to\infty}\sum_{n=-N}^{N}\int_0^1 f(t+m)e^{-2\pi i n t}dt \\
&= \lim_{N\to\infty}\sum_{n=-N}^{N}\int_m^{m+1} f(t)e^{-2\pi i n t}dt \\
&= \int_m^{m+1} f(t)dt + \lim_{N\to\infty}\sum_{n=1}^{N}\int_m^{m+1} f(t)\left(e^{2\pi i n t}+e^{-2\pi i n t}\right)dt \\
&= \int_m^{m+1} f(t)dt + 2\sum_{n=1}^{\infty}\int_m^{m+1} f(t)\cos 2\pi n t\cdot dt.
\end{aligned}$$

Now take the sum over $m = a, a+1, \ldots, b-1$. $\square$

Let q be any integer $\geqslant 1$, and b any integer coprime with q. Define the exponential sums

$$S(b,q) := \sum_{a=0}^{q-1} e^{2\pi i b a^2/q}, \quad S(q) := S(1,q).$$

Lemma 3.28. *Let q be an odd prime and b an integer coprime with q. Then $S(b,q) = \tau(b, \left(\frac{\cdot}{q}\right)) = \left(\frac{b}{q}\right) S(q)$.*

Proof. Let $Q := \sum^{(1)} e^{2\pi i b a/q}$, $N := \sum^{(2)} e^{2\pi i b a/q}$, where $\sum^{(1)}$ denotes the summation over the quadratic residues $a \in \{0, \ldots, q-1\}$ and $\sum^{(2)}$ that over the quadratic non-residues $a \in \{0, \ldots, q-1\}$. Then

$$1 + Q + N = \sum_{a=0}^{q-1} e^{2\pi i b a/q} = \frac{e^{2\pi i b} - 1}{e^{2\pi i b/q} - 1} = 0.$$

If a runs through $1, \ldots, q-1$, then $a^2 \pmod{q}$ runs twice through the quadratic residue classes mod q (note that a^2 and $(q-a)^2$ give the same quadratic residue). So

$$S(b,q) = 1 + 2Q = Q - N = \sum_{a=0}^{q-1} \left(\frac{a}{q}\right) e^{2\pi b a/q} = \tau(b, \left(\frac{\cdot}{q}\right)).$$

The second equality in the statement follows from Theorem 3.18. $\square$

Lemma 3.29. *Let p, q be two distinct odd primes. Then*

$$S(pq) = S(q,p) S(p,q) = \left(\frac{q}{p}\right)\left(\frac{p}{q}\right) S(p) S(q).$$

Proof. If a runs through $0, \ldots, p-1$ and b through $0, \ldots, q-1$, then $qa + pb$ runs through a complete system of residues mod pq. Thus,

$$\begin{aligned} S(pq) &= \sum_{a=0}^{p-1} \sum_{b=0}^{q-1} e^{2\pi i (qa+pb)^2/pq} = \sum_{a=0}^{p-1} \sum_{b=0}^{q-1} e^{2\pi i ((qa^2/p) + pb^2/q) + 2ab)} \\ &= \sum_{a=0}^{p-1} \sum_{b=0}^{q-1} e^{2\pi i q a^2/p} \cdot e^{2\pi i p b^2/q} = \sum_{a=0}^{p-1} e^{2\pi i q a^2/p} \sum_{b=0}^{q-1} e^{2\pi i p b^2/q} = S(q,p) S(p,q). \end{aligned}$$

By Lemma 3.28, the latter is $\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) S(p) S(q)$. $\square$

Lemma 3.30. *Let q be a positive integer. Then*

$$S(q) = \begin{cases} (1+i)\sqrt{q} & \text{if } q \equiv 0 \pmod 4, \\ \sqrt{q} & \text{if } q \equiv 1 \pmod 4, \\ 0 & \text{if } q \equiv 2 \pmod 4, \\ i\sqrt{q} & \text{if } q \equiv 3 \pmod 4. \end{cases}.$$

Proof. By Corollary 3.27 we have

$$\begin{aligned} S(q) &= -1 + \sum_{a=0}^{q} e^{2\pi i a^2/q} = \lim_{N\to\infty} \sum_{n=-N}^{N} \int_0^q e^{(2\pi i t^2/q) - 2\pi i n t} dt \\ &= \sqrt{q} \cdot \lim_{N\to\infty} \sum_{n=-N}^{N} \int_0^{\sqrt{q}} e^{2\pi i u^2 - 2\pi i n\sqrt{q}} du \quad (\text{substituting } u = t/\sqrt{q}) \\ &= \sqrt{q} \cdot \lim_{N\to\infty} \sum_{n=-N}^{N} \int_0^{\sqrt{q}} e^{2\pi i((u - n\sqrt{q}/2)^2 - n^2 q/4)} du \\ &= \sqrt{q} \cdot \lim_{N\to\infty} \sum_{n=-N}^{N} e^{-\pi i n^2 q/2} \int_0^{\sqrt{q}} e^{2\pi i (u - n\sqrt{q})^2} du. \end{aligned}$$

We split the summation into even n and odd n. Note that $e^{-\pi i \cdot n^2 q/2} = 1$ if n is even, and $e^{-\pi i q/2}$ if n is odd. So

$$\sum_{\substack{n=-N \\ n\equiv 0 \,(\mathrm{mod}\, 2)}}^{N} e^{-\pi i n^2 q/2} \int_0^{\sqrt{q}} e^{2\pi i (u - n\sqrt{q})^2} du = \sum_{\substack{n=-N \\ n\equiv 0 \,(\mathrm{mod}\, 2)}}^{N} \int_{-(n/2)\sqrt{q}}^{(1-n/2)\sqrt{q}} e^{2\pi i u^2} du = \int_{-N_1\sqrt{q}}^{N_2\sqrt{q}} e^{2\pi i u^2} du,$$

say, where we use that the intervals $[-\frac{1}{2}n\sqrt{q}, (1-\frac{1}{2}n)\sqrt{q}]$ ($n \in \{-N, \ldots, N\}$ even) apart from their begin points and end points do not overlap and paste together to a single interval $[-N_1\sqrt{q}, N_2\sqrt{q}]$ where $|N_i - \frac{1}{2}N| \leqslant 1$ for $i = 1, 2$. Likewise, the sum over the odd values of n in $\{-N, \ldots, N\}$ is

$$e^{-\pi i q/2} \int_{-N_3\sqrt{q}}^{N_4\sqrt{q}} e^{2\pi i u^2} du,$$

where $|N_i - \frac{1}{2}N| \leqslant 1$ for $i = 3, 4$. Taking for the moment for granted that the integral $C := \int_{-\infty}^{\infty} e^{2\pi i u^2} du$ converges, we get

$$S(q) = \sqrt{q} \lim_{N\to\infty} \Big(\int_{-N_1\sqrt{q}}^{N_2\sqrt{q}} e^{2\pi i u^2} du + e^{-\pi i q/2} \int_{-N_3\sqrt{q}}^{N_4\sqrt{q}} e^{2\pi i u^2} du \Big) = \sqrt{q}(1 + e^{-\pi i q/2})C.$$

Substituting $q = 1$ and using $S(1) = 1$ we read off $C = (1-i)^{-1}$. Thus we get $S(q) = \sqrt{q} \cdot (1 + e^{-\pi i q/2})/(1-i)$, which gives our lemma.

It remains to show that $\int_{-\infty}^{\infty} e^{2\pi i u^2} du$ converges. This integral is equal to $2\int_0^{\infty} e^{2\pi i u^2} du$, provided the latter converges. But this is indeed the case, since for any $B > A > 0$,

$$\begin{aligned} \left| \int_A^B e^{2\pi i u^2} du \right| &= \left| \int_A^B (4\pi i u)^{-1} de^{2\pi i u^2} \right| \\ &= \left| \frac{e^{2\pi i B^2}}{4\pi i B} - \frac{e^{2\pi i A^2}}{4\pi i A} + \frac{1}{4\pi i} \int_A^B u^{-2} e^{2\pi i u^2} du \right| \\ &\leqslant (4\pi)^{-1} \Big(B^{-1} + A^{-1} + \int_A^B u^{-2} du \Big) = (2\pi A)^{-1} \to 0 \text{ as } A, B \to \infty. \end{aligned}$$

This completes our proof. □

Proof of Theorem 3.24. Immediate from Lemmas 3.30 and 3.29. □

3.7 Exercises

Exercise 3.1. *Compute the characters modulo* 12 *and determine the conductor of each character.*

Exercise 3.2. *Recall that a character* χ *mod* q *is called real if* $\chi(a) \in \mathbb{R}$ *for every* $a \in \mathbb{Z}$*, i.e., if* $\chi(a) \in \{-1, 1\}$ *for every* $a \in \mathbb{Z}$ *with* $\gcd(a, q) = 1$.

a) *For a positive integer* q *denote by* $R(q)$ *the number of real characters mod* q*. Prove that* R *is a multiplicative arithmetic function, and compute* $R(p^k)$ *for every prime power* p^k*.*

b) *Determine those positive integers* q *such that every character mod* q *is real.*

Exercise 3.3. *For a positive integer* q*, denote by* $F(q)$ *the number of primitive characters mod* q*. Prove that* F *is a multiplicative arithmetic function, and compute* $F(p^k)$ *for every prime power* p^k*.*

Hint. *Prove that if* f *is a divisor of* q*, then* $F(f)$ *is precisely the number of characters mod* q *with conductor* f*. Use the results from the lecture notes.*

Exercise 3.4. *Let q be a positive integer. Prove that* $\tau(1, \chi_0^{(q)}) = \sum_{\substack{a=0 \\ \gcd(a,q)=1}}^{q-1} e^{2\pi i a/q} = \mu(q)$.

Exercise 3.5. *Prove Theorem 3.25.*
Hint. *Prove an analogue of Lemma 3.29 with $q = 8$.*

Exercise 3.6. *For an integer a and a positive odd integer b we define the* Jacobi-symbol

$$\left(\frac{a}{b}\right) := \prod_{i=1}^{t} \left(\frac{a}{p_i}\right)^{k_i},$$

where $b = p_1^{k_1} \cdots p_t^{k_t}$ is the unique prime factorization of b.

a) Let b be a positive odd integer. Prove that

$$\left(\frac{-1}{b}\right) = (-1)^{(b-1)/2}, \quad \left(\frac{2}{b}\right) = (-1)^{(b^2-1)/8}.$$

b) Let a, b be two odd, positive, coprime integers. Prove that

$$\left(\frac{a}{b}\right) \cdot \left(\frac{b}{a}\right) = (-1)^{(a-1)(b-1)/4}.$$

c) Let n be a positive odd, square-free integer which is not a prime. Prove that there are integers a such that $x^2 \equiv a \pmod{n}$ is not solvable, while $\left(\frac{a}{n}\right) = 1$.

Exercise 3.7. *Let p be a prime > 2 and m a divisor of $p - 1$ with $m \geqslant 2$. An integer a is called an m-th power residue modulo p if $p \nmid a$ and if there is an integer b with $a \equiv b^m \pmod{p}$. Let M, N be integers with $0 \leqslant M < M + N < p$. Denote by R_m the number of m-th power residues mod p in the interval $[M + 1, M + N]$. The purpose of this exercise is to show that*

$$\left|R_m - \frac{N}{m}\right| \leqslant 3(m-1)\sqrt{p}\log p.$$

In case that p is a large prime and N is much larger than $3m(m-1)\sqrt{p}\log p$ this implies that about a fraction of $1/m$ among the integers in $\{M + 1, \ldots, M + N\}$ is an m-th power residue modulo p. Perform the following steps:

a) Recall that $(\mathbb{Z}/p\mathbb{Z})^*$ *is a cyclic group of order* $p-1$*. Choose an integer* g *such that* g *mod* p *generates* $(\mathbb{Z}/p\mathbb{Z})^*$*. Choose a character* χ_1 *mod* p *such that* $\chi_1(g) = e^{2\pi i/(p-1)}$*; then* $G(p) = \langle \chi_1 \rangle$*. Let* $t := (p-1)/m$*. Prove that*

$$\sum_{j=0}^{m-1} \chi_1^{tj}(a) = \begin{cases} m & \text{if } a \text{ is an } m\text{-th power residue mod } p, \\ 0 & \text{otherwise.} \end{cases}$$

b) Compute $\sum_{j=0}^{m-1} \sum_{a=M+1}^{M+N} \chi_1^{tj}(a)$ *in two ways.*