Chapter 4

The Riemann zeta function and L-functions

4.1 Basic facts

We prove some results that will be used in the proof of the Prime Number Theorem (for arithmetic progressions). The L-function of a Dirichlet character χ modulo q is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

We view $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ as the L-function of the principal character modulo 1, more precisely, $\zeta(s) = L(s, \chi_0^{(1)})$, where $\chi_0^{(1)}(n) = 1$ for all $n \in \mathbb{Z}$.

We first prove that $\zeta(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \operatorname{Re} s > 0\} \setminus \{1\}$. We use an important summation formula, due to Euler.

Lemma 4.1.1 (Euler's summation formula)**.** *Let a, b be integers with $a < b$ and $f : [a,b] \to \mathbb{C}$ a continuously differentiable function. Then*

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + f(a) + \int_a^b (x - [x]) f'(x) dx.$$

Remark. This result often occurs in the more symmetric form

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \tfrac{1}{2}(f(a)+f(b)) + \int_a^b (x-[x]-\tfrac{1}{2})f'(x)dx.$$

Proof. Let $n \in \{a, a+1, \ldots, b-1\}$. Then

$$\begin{aligned}\int_n^{n+1} \big(x-[x]\big)f'(x)dx &= \int_n^{n+1} (x-n)f'(x)dx \\ &= \Big[(x-n)f(x)\Big]_n^{n+1} - \int_n^{n+1} f(x)dx = f(n+1) - \int_n^{n+1} f(x)dx.\end{aligned}$$

By summing over n we get

$$\int_a^b (x-[x])f'(x)dx = \sum_{n=a+1}^{b} f(n) - \int_a^b f(x)dx,$$

which implies at once Lemma 4.1.1. □

We also need a result from the Prerequisites on analytic functions defined by integrals, which we recall here.

Theorem 0.6.25. *Let D be a measurable subset of $\mathbb{R}^m$, U an open subset of $\mathbb{C}$ and $f : D \times U \to \mathbb{C}$ a function with the following properties:*
(i) f is measurable on $D \times U$ (with U viewed as subset of $\mathbb{R}^2$);
(ii) for every fixed $x \in D$, the function $z \mapsto f(x,z)$ is analytic on U;
(iii) for every compact subset K of U there is a measurable function $M_K : D \to \mathbb{R}$ such that

$$|f(x,z)| \leqslant M_K(x) \text{ for } x \in D,\ z \in K, \qquad \int_D M_K(x)dx < \infty.$$

Then the function F given by

$$F(z) := \int_D f(x,z)dx$$

is analytic on U, and for every $k \geqslant 1$,

$$F^{(k)}(z) = \int_D f^{(k)}(x,z)dx,$$

where $f^{(k)}(x,z)$ *denotes the* k*-th derivative with respect to* z *of the analytic function* $z \mapsto f(x,z)$.

We now state and prove our result on the Riemann zeta function.

Theorem 4.1.2. $\zeta(s)$ *has a unique analytic continuation to the set* $\{s \in \mathbb{C} : \operatorname{Re} s > 0,\ s \neq 1\}$, *with a simple pole with residue* 1 *at* $s = 1$.

Proof. By Corollary 0.6.22 we know that an analytic continuation of $\zeta(s)$, if such exists, is unique.

For the moment, let $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. Then by Lemma 4.1.1, with $f(x) = x^{-s}$,

$$\begin{aligned}\sum_{n=1}^{N} n^{-s} &= \int_1^N x^{-s}dx + 1 + \int_1^N (x-[x])(-sx^{-1-s})dx \\ &= \frac{1-N^{1-s}}{s-1} + 1 - s\int_1^N (x-[x])x^{-1-s}dx.\end{aligned}$$

If we let $N \to \infty$ then the left-hand side converges, and also the first term on the right-hand side, since $|N^{-1-s}| = N^{-1-\operatorname{Re} s} \to 0$. Hence the integral on the right-hand side must converge as well. Thus, letting $N \to \infty$, we get for $\operatorname{Re} s > 1$,

$$\zeta(s) = \frac{1}{s-1} + 1 - s\int_1^\infty (x-[x])x^{-1-s}dx. \tag{4.1.1}$$

We now show that the integral on the right-hand side defines an analytic function on $U := \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, by means of Theorem 0.6.25.

The function $F(x,s) := (x-[x])x^{-1-s}$ is measurable on $[1,\infty) \times U$ (for its set of discontinuities has Lebesgue measure 0 since it is the countable union of the sets $\{n\} \times U$ with $n \in \mathbb{Z}_{\geqslant 2}$, which all have Lebesgue measure 0) and for every fixed x it is analytic in s. So conditions (i) and (ii) of Theorem 0.6.25 are satisfied. We verify (iii). Let K be a compact subset of U. Then there is $\sigma > 0$ such that $\operatorname{Re} s \geqslant \sigma$ for all $s \in K$. Now for $x \geqslant 1$ and $s \in K$ we have

$$|(x-[x])x^{-1-s}| \leqslant x^{-1-\sigma}$$

and $\int_1^\infty x^{-1-\sigma}dx < \infty$. So (iii) is also satisfied, and we may indeed conclude that the integral on the right-hand side of (4.1.1) defines an analytic function on U.

Consequently, the right-hand side of (4.1.1) is analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\} \setminus \{1\}$ and since it is of the shape $\frac{1}{s-1}$+something analytic around $s = 1$ it has a simple pole at $s = 1$ with residue 1.

We may take this as our analytic continuation of $\zeta(s)$. □

Theorem 4.1.3. *Let $q \in \mathbb{Z}_{\geqslant 2}$, and let χ be a Dirichlet character mod q.*
(i) $L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}$ for $s \in \mathbb{C}$, $\operatorname{Re} s > 1$.
(ii) If $\chi \neq \chi_0^{(q)}$, then $L(s, \chi)$ converges, and is analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$.
(iii) $L(s, \chi_0^{(q)})$ can be continued to an analytic function on $\{s \in \mathbb{C} : \operatorname{Re} s > 0,\ s \neq 1\}$, and for s in this set we have

$$L(s, \chi_0^{(q)}) = \zeta(s) \cdot \prod_{p|q} (1 - p^{-s}).$$

Hence $L(s, \chi_0^{(q)})$ has a simple pole at $s = 1$.

Proof. (i) χ is a strongly multiplicative function, and $L(s, \chi)$ converges absolutely for $\operatorname{Re} s > 1$. Apply Corollary 2.3.3.

(ii) Let N be any positive integer. Then $N = tq + r$ for certain integers t, r with $t \geqslant 0$ and $0 \leqslant r < q$. By one of the orthogonality relations for characters (see Theorem 3.2.1), we have $\sum_{m=1}^{q} \chi(m) = 0$, $\sum_{m=q+1}^{2q} \chi(m) = 0$, etc. Hence

$$\left| \sum_{n=1}^{N} \chi(n) \right| = \Big| \chi(tq + 1) + \cdots + \chi(tq + r) \Big| \leqslant r < q.$$

This last upper bound is independent of N. Now Theorem 2.1.2 implies that the L-series $L(s, \chi)$ converges and is analytic on $\operatorname{Re} s > 0$.

(iii) By (i) we have for $\operatorname{Re} s > 1$,

$$L(s, \chi_0^{(q)}) = \prod_{p \nmid q} (1 - p^{-s})^{-1} = \zeta(s) \prod_{p|q} (1 - p^{-s}).$$

The right-hand side is defined and analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0,\ s \neq 1\}$, and so it can be taken as an analytic continuation of $L(s, \chi_0^{(q)})$ on this set. □

Corollary 4.1.4. *Both $\zeta(s)$ and $L(s,\chi)$ for any character χ modulo an integer $q \geqslant 2$ are $\neq 0$ on $\{s \in \mathbb{C} : \operatorname{Re} s > 1\}$.*

Proof. Use part (i) of the above theorem, together with Corollary 2.3.3. □

4.2 Non-vanishing on the line $\operatorname{Re} s = 1$

We prove that $\zeta(s) \neq 0$ if $\operatorname{Re} s = 1$ and $s \neq 1$, and $L(s,\chi) \neq 0$ for any $s \in \mathbb{C}$ with $\operatorname{Re} s = 1$ and any non-principal character χ modulo an integer $q \geqslant 2$. We have to distinguish two cases, which are treated quite differently. We interpret $\zeta(s)$ as $L(s, \chi_0^{(1)})$. Recall that a character χ mod q is called *real* if $\chi(a) \in \mathbb{R}$ for every $a \in \mathbb{Z}$. Clearly, real characters modulo q assume only the values $-1, 1$ on the integers coprime with q and so, χ is real if and only if $\chi^2 = \chi_0^{(q)}$.

Theorem 4.2.1. *Let $q \in \mathbb{Z}_{\geqslant 1}$, χ a character mod q, and t a real. Assume that either $t \neq 0$ and χ is arbitrary, or $t = 0$ but χ is not real. Then $L(1+it,\chi) \neq 0$.*

Proof. We use a famous idea, due to Hadamard. It is based on the inequality

$$3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2 \geqslant 0 \quad \text{for } \theta \in \mathbb{R}. \tag{4.2.1}$$

Fix a real t such that $t \neq 0$ if χ is real. Suppose that $L(1+it,\chi) = 0$. Consider the function in s,

$$F(s) := L(s, \chi_0^{(q)})^3 \cdot L(s+it,\chi)^4 \cdot L(s+2it, \chi^2).$$

By the conditions imposed on χ and t, the function $L(s+2it,\chi^2)$ is analytic around $s = 1$. Further, $L(s,\chi_0^{(q)})$ has a simple pole at $s = 1$, while $L(s+it,\chi)$ has by assumption a zero at $s = 1$. Recall that for a function f meromorphic around $s = s_0$, $\operatorname{ord}_{s=s_0}(f)$ is the integer n_0 such that $f(s)$ has a Laurent series expansion $\sum_{n=n_0}^{\infty} a_n (s-s_0)^n$ around $s = s_0$ with $a_{n_0} \neq 0$. This $\operatorname{ord}_{s=s_0}$-function is additive on products. Thus,

$$\begin{aligned}\operatorname{ord}_{s=1}(F) &= 3\cdot\operatorname{ord}_{s=1}\big(L(s,\chi_0^{(q)})\big) + 4\cdot\operatorname{ord}_{s=1}\big(L(s+it,\chi)\big) + \operatorname{ord}_{s=1}\big(L(s+2it,\chi^2)\big)\\ &\geqslant 3\cdot(-1) + 4 + 0 = 1.\end{aligned}$$

This shows that F is analytic around $s = 1$, and has a zero at $s = 1$. We now prove that $|F(\sigma)| \geqslant 1$ (or rather, $\log|F(\sigma)| \geqslant 0$) for $\sigma > 1$. This gives a contradiction since by continuity, $\lim_{\sigma\downarrow 1}|F(\sigma)|$ should be 0. So our assumption that $L(1+it,\chi)=0$ must be false.

From the definition of the function F we obtain that for $\sigma>1$ we have

$$\log|F(\sigma)| = \log\prod_p\left(\left|\frac{1}{1-\chi_0^{(q)}(p)p^{-\sigma}}\right|^3\cdot\left|\frac{1}{1-\chi(p)p^{-\sigma-it}}\right|^4\cdot\left|\frac{1}{1-\chi(p)^2p^{-\sigma-2it}}\right|\right)$$

$$=\sum_{p\nmid q}\left(3\log\left|\frac{1}{1-p^{-\sigma}}\right|+4\log\left|\frac{1}{1-\chi(p)p^{-\sigma-it}}\right|+\log\left|\frac{1}{1-\chi(p)^2p^{-\sigma-2it}}\right|\right).$$

Note that if $p\nmid q$ then $\chi(p)$ is a root of unity. Hence $|\chi(p)p^{-it}|=|\chi(p)e^{-it\log p}|=1$. So we have $\chi(p)p^{-it}=e^{i\varphi_p}$ with $\varphi_p\in\mathbb{R}$. Hence

$$\log|F(\sigma)|=\sum_{p\nmid q}\left(3\log\left|\frac{1}{1-p^{-\sigma}}\right|+4\log\left|\frac{1}{1-p^{-\sigma}e^{i\varphi_p}}\right|+\log\left|\frac{1}{1-p^{-\sigma}e^{2i\varphi_p}}\right|\right).$$

Recall that

$$\log\frac{1}{1-z}=\sum_{n=1}^{\infty}\frac{z^n}{n},\quad \log\left|\frac{1}{1-z}\right|=\mathrm{Re}\,\log\frac{1}{1-z}\quad\text{for } z\in\mathbb{C}\text{ with } |z|<1.$$

Hence for $r,\varphi\in\mathbb{R}$ with $0<r<1$,

$$\begin{aligned}\log\left|\frac{1}{1-re^{i\varphi}}\right| &= \mathrm{Re}\left(\log\frac{1}{1-re^{i\varphi}}\right)=\mathrm{Re}\left(\sum_{n=1}^{\infty}\frac{(re^{i\varphi})^n}{n}\right)\\ &= \sum_{n=1}^{\infty}\frac{r^n}{n}\mathrm{Re}\,(e^{in\varphi})=\sum_{n=1}^{\infty}\frac{r^n}{n}\cdot\cos n\varphi.\end{aligned}$$

This leads to

$$\begin{aligned}\log|F(\sigma)| &= \sum_{p\nmid q}\left(3\sum_{n=1}^{\infty}\frac{p^{-n\sigma}}{n}+4\sum_{n=1}^{\infty}\frac{p^{-n\sigma}}{n}\cdot\cos n\varphi_p+\sum_{n=1}^{\infty}\frac{p^{-n\sigma}}{n}\cos 2n\varphi_p\right)\\ &= \sum_{p\nmid q}\sum_{n=1}^{\infty}\frac{p^{-n\sigma}}{n}(3+4\cos n\varphi_p+\cos 2n\varphi_p)\geqslant 0,\end{aligned}$$

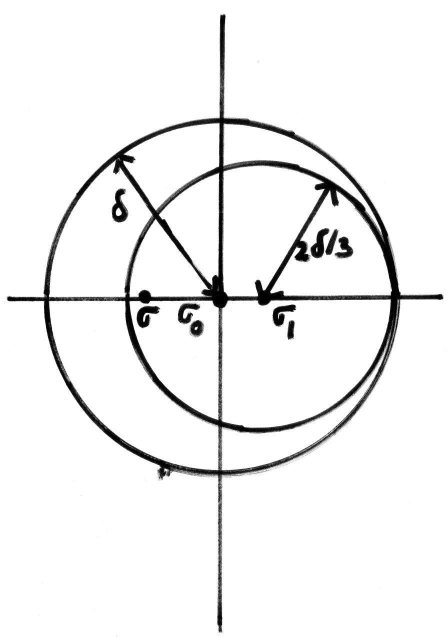
using (4.2.1). This shows that indeed, $|F(\sigma)|\geqslant 1$ for $\sigma>1$, giving us the contradiction we want. □

It remains to prove that $L(1,\chi) \neq 0$ for any real character χ mod q that is not principal. Dirichlet needed this fact already in his proof that for every pair of integers q, a with $q \geqslant 3$ and $\gcd(a, q) = 1$ there are infinitely many primes p with $p \equiv a \pmod{q}$. Dirichlet had a rather complicated proof that $L(1,\chi) \neq 0$, based on Dirichlet series associated with quadratic forms (in modern language: Dedekind zeta functions for orders in quadratic number fields) and class number formulas.

Landau found a much more direct proof, which we give here, based on a simple result for Dirichlet series, which more or less asserts that a Dirichlet series with non-negative real coefficients can not be continued analytically beyond the boundary of its half plane of convergence.

Lemma 4.2.2 (Landau). *Let $f : \mathbb{Z}_{>0} \to \mathbb{R}$ be an arithmetic function with $f(n) \geqslant 0$ for all n. Suppose that $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ has abscissa of convergence σ_0. Then $L_f(s)$ cannot be continued analytically to any open set containing $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma_0\} \cup \{\sigma_0\}$.*

Proof.



Suppose $L_f(s)$ can be continued to an analytic function $g(s)$ on an open set containing $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma_0\} \cup \{\sigma_0\}$. Then there is $\delta > 0$ such that $g(s)$ is analytic on the open disk $D(\sigma_0, \delta)$ with center σ_0 and radius δ. Let $\sigma_1 := \sigma_0 + \delta/3$. Then $D(\sigma_1, 2\delta/3) \subset D(\sigma_0, \delta)$, so $g(s)$ is analytic and has a Taylor series expansion around σ_1 converging on $D(\sigma_1, 2\delta/3)$. Now take σ with $\sigma_0 - \delta/3 < \sigma < \sigma_0$, so that $\sigma \in D(\sigma_1, 2\delta/3)$. Using the Taylor series expansion of $g(s)$ around σ_1, we get

$$g(\sigma) = \sum_{k=0}^{\infty} \frac{g^{(k)}(\sigma_1)}{k!} \cdot (\sigma - \sigma_1)^k.$$

Since σ_1 is larger than the abscissa of convergence σ_0 of $L_f(s)$, we have

$$g^{(k)}(\sigma_1) = L_f^{(k)}(\sigma_1) = \sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-\sigma_1} \quad \text{for } k \geqslant 0.$$

Hence

$$\begin{aligned} g(\sigma) &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-\sigma_1} \right) (\sigma - \sigma_1)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=1}^{\infty} f(n)(\log n)^k n^{-\sigma_1} \right) (\sigma_1 - \sigma)^k. \end{aligned}$$

Now all terms are non-negative, hence it is allowed to interchange the summations. Thus,

$$\begin{aligned} g(\sigma) &= \sum_{n=1}^{\infty} f(n) n^{-\sigma_1} \left(\sum_{k=0}^{\infty} \frac{1}{k!} (\log n)^k (\sigma_1 - \sigma)^k \right) \\ &= \sum_{n=1}^{\infty} f(n) n^{-\sigma_1} e^{(\log n)(\sigma_1 - \sigma)} \qquad \left(\text{using } e^z = \sum_{k=0}^{\infty} z^k/k! \right) \\ &= \sum_{n=1}^{\infty} f(n) n^{-\sigma_1} n^{\sigma_1 - \sigma} = \sum_{n=1}^{\infty} f(n) n^{-\sigma}. \end{aligned}$$

We see that $L_f(s)$ converges for $s = \sigma$. But this is impossible, since σ is smaller than the abscissa of convergence σ_0 of $L_f(s)$. So our initial assumption that $L_f(s)$ has an analytic continuation to an open set containing $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma_0\} \cup \{\sigma_0\}$ is false. □

Remark. Lemma 4.2.2 becomes false if we drop the condition that $f(n) \geqslant 0$ for all n. For instance, if χ is a non-principal character mod q, then $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ diverges if $\operatorname{Re} s < 0$, but one can show that $L(s, \chi)$ has an analytic continuation to the whole of $\mathbb{C}$.

Theorem 4.2.3. *Let $q \in \mathbb{Z}_{\geqslant 2}$, and let χ be a real, non-principal character mod q. Then $L(1, \chi) \neq 0$.*

Proof. Assume that $L(1, \chi) = 0$. Consider the function

$$F(s) := L(s, \chi)\zeta(s).$$

By Theorems 4.1.2, 4.1.3, this function is analytic at least on $\{s \in \mathbb{C} : \operatorname{Re} s > 0,\ s \neq 1\}$. Further, $\zeta(s)$ has a simple pole at $s = 1$, and since $L(s, \chi)$ has by assumption a zero at $s = 1$, we get

$$\operatorname{ord}_{s=1}(F) \geqslant 1 + (-1) = 0,$$

i.e., F is analytic at $s = 1$ as well. Hence F is analytic for all s with $\operatorname{Re} s > 0$. We show that for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$, $F(s)$ is expressable as a Dirichlet series with non-negative coefficients. By Lemma 4.2.2, this Dirichlet series should have abscissa of convergence $\leqslant 0$. But we show that the abscissa of convergence of this series is $\geqslant \frac{1}{2}$ and derive a contradiction.

The series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and $\sum_{n=1}^{\infty} \chi(n) n^{-s}$ converge absolutely if $\operatorname{Re} s > 1$. So by Theorem 2.3.1,

$$F(s) = L_f(s) = \sum_{n=1}^{\infty} f(n) n^{-s} \quad \text{for } s \in \mathbb{C},\ \operatorname{Re} s > 1,$$

where $f = E * \chi$, i.e.,

$$f(n) = \sum_{d|n} \chi(d) \ \text{ for } n \in \mathbb{Z}_{>0}.$$

Hence f is a multiplicative function. We compute f in the prime powers. We have $\chi(n) = \pm 1$ for all $n \in \mathbb{Z}$ with $\gcd(n, q) = 1$, while $\chi(n) = 0$ if $\gcd(n, q) > 1$. Hence, if p is a prime and k a non-negative integer, we have

$$f(p^k) = \sum_{j=0}^{k} \chi(p)^j = \begin{cases} 1 & \text{if } p|q, \\ k+1 & \text{if } p \nmid q,\ \chi(p) = 1, \\ 1 & \text{if } p \nmid q,\ \chi(p) = -1,\ k \text{ even}, \\ 0 & \text{if } p \nmid q,\ \chi(p) = -1,\ k \text{ odd}. \end{cases}$$

Therefore, $f(p^k) \geqslant 0$ for all prime powers p^k. Since f is multiplicative, it follows that $f(n) \geqslant 0$ for all $n \in \mathbb{Z}_{>0}$.

The series $L_f(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, which is $F(s)$. So by Lemma 4.2.2, $L_f(s)$ has abscissa of convergence $\sigma_0(f) \leqslant 0$. On the other hand, from the above table and from the fact that f is multiplicative, it follows that if $n = m^2$ is a square, then $f(n) \geqslant 1$. Hence

$$L_f(\sigma) = \sum_{n=1}^{\infty} f(n) n^{-\sigma} \geqslant \sum_{m=1}^{\infty} m^{-2\sigma} = \infty \ \text{ if } \sigma \leqslant \tfrac{1}{2}.$$

So $\sigma_0(f) \geqslant \frac{1}{2}$. This gives a contradiction, and so our assumption that $L(1, \chi) = 0$ has to be false. □

4.3 Functional equations

Euler's Gamma function plays an important role in the functional equations of $\zeta(s)$, the L-functions, and various generalizations thereof. Euler's Gamma function is given by

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt \quad \text{for } z \in \mathbb{C} \text{ with } \operatorname{Re} z > 0,$$

where $t^z := e^{z \log t}$ for $z \in \mathbb{C}$, with $\log t$ the ordinary real natural logarithm of t. We collect here some properties:

Lemma 4.3.1. *(i)* $\Gamma(z)$ *defines an analytic function on* $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$*;*

(ii) $\Gamma(z+1) = z\Gamma(z)$ *for* $z \in \mathbb{C}$, $\operatorname{Re} z > 0$*;*

(iii) $\Gamma(n) = (n-1)!$ *for* $n = 1, 2, \ldots$ *and* $\Gamma(\frac{1}{2}) = \sqrt{\pi}$*;*

(iv) $\Gamma(z)$ *has an analytic continuation to* $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$ *with a simple pole with residue* $(-1)^n/n!$ *at* $z = -n$*, for* $n = 0, 1, 2, \ldots$*;*

(v) $\Gamma(z) \neq 0$ *for* $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$*;*

(vi) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$ *for* $z \in \mathbb{C} \setminus \mathbb{Z}$*;*

(vii) $\Gamma(2z) = \dfrac{2^{2z-1}}{\sqrt{\pi}} \cdot \Gamma(z)\Gamma(z + \frac{1}{2})$ *for* $z \in \mathbb{C},\ z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots$ *(duplication formula).*

Property (i) is an easy consequence of Theorem 0.6.25, and property (ii) an easy application of integration by parts. The interested reader may consult Section 4.4 for the proofs of (i)–(vii).

Now define

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s) \ = \ (s-1)\pi^{-s/2}\Gamma(\tfrac{1}{2}s+1)\zeta(s), \tag{4.3.1}$$

where we have used the identity $\frac{1}{2}s\Gamma(\frac{1}{2}s) = \Gamma(\frac{1}{2}s+1)$. In his memoir from 1859, Riemann proved the famous identity

$$\xi(s) = \tfrac{1}{2} + \tfrac{1}{2}s(s-1) \int_1^\infty \omega(t)\big(t^{(s/2)-1} + t^{-(s+1)/2}\big)dt \quad \text{if } \operatorname{Re} s > 1, \tag{4.3.2}$$

where $\omega(t) := \sum_{m=1}^\infty e^{-\pi m^2 t}$. Using Theorem 0.6.25 one shows that the integral is defined and analytic for all $s \in \mathbb{C}$. Moreover, the right-hand side is invariant under the substitution $s \mapsto 1-s$. This immediately leads to

Theorem 4.3.2. *The function ξ has an analytic continuation to $\mathbb{C}$. For this continuation we have $\xi(1-s) = \xi(s)$ for $s \in \mathbb{C}$ and moreover, $\xi(0) = \xi(1) = \frac{1}{2}$.*

For the interested reader we have included a proof of (4.3.2) and Theorem 4.3.2 in Sections 4.5, 4.6. See also H. Davenport, Multiplicative Number Theory, Chapter 8.

We deduce some consequences.

Corollary 4.3.3. *ζ has an analytic continuation to $\mathbb{C} \setminus \{1\}$, with a simple pole with residue 1 at $s = 1$.*

Proof. For $\zeta(s)$ with $\mathrm{Re}\, s > 1$ we have an expression

$$\zeta(s) = \frac{\xi(s)\pi^{s/2} \cdot \Gamma(\frac{1}{2}s + 1)^{-1}}{s-1}. \tag{4.3.3}$$

By Theorem 4.3.2, $\xi(s)$ has an analytic continuation to $\mathbb{C}$. Further, $\pi^{s/2}$ is analytic on $\mathbb{C}$, and moreover, $\Gamma(\frac{1}{2}s+1)^{-1}$ defines an analytic function on $\mathbb{C}$, since Γ has only poles and no zeros. Hence (4.3.3) gives an analytic continuation of $\zeta(s)$ to $\mathbb{C} \setminus \{1\}$. Using $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$, we get

$$\lim_{s \to 1}(s-1)\zeta(s) = \xi(1)\pi^{1/2}\Gamma(\tfrac{3}{2})^{-1} = 1,$$

which shows that the analytic continuation of $\zeta(s)$ defined by (4.3.3) has a simple pole with residue 1 at $s = 1$. □

Remark. On $\{s \in \mathbb{C} : \mathrm{Re}\, s > 0\} \setminus \{1\}$ the right-hand side of (4.3.3) coincides with the analytic continuation defined in (4.1.1) since analytic continuations to connected sets are uniquely determined.

Corollary 4.3.4. *ζ has simple zeros at $s = -2, -4, -6, \ldots$. ζ has no other zeros outside the critical strip $\{s \in \mathbb{C} : 0 < \mathrm{Re}\, s < 1\}$.*

Proof. By Corollary 4.1.4 and Theorem 4.2.1, we know that $\zeta(s) \neq 0$ for $s \in \mathbb{C}$ with $\mathrm{Re}\, s \geqslant 1$, $s \neq 1$. Further, $\lim_{s \to 1}(s-1)\zeta(s) = 1$, hence $(s-1)\zeta(s) \neq 0$ if $\mathrm{Re}\, s \geqslant 1$. We know also that $\pi^{s/2} \neq 0$ for $s \in \mathbb{C}$, that $\Gamma(\frac{1}{2}s+1)$ has a simple pole at $s = -2, -4, \ldots$ and $\Gamma(\frac{1}{2}s+1) \neq 0$ for $s \neq -2, -4, \ldots$. Using the second expression of (4.3.1) this implies $\xi(s) \neq 0$ for $s \in \mathbb{C}$ with $\mathrm{Re}\, s \geqslant 1$, and then also $\xi(s) \neq 0$ if

$\operatorname{Re} s \leqslant 0$ by Theorem 4.3.2. Together with what we observed above about $\pi^{s/2}$ and $\Gamma(\frac{1}{2}s+1)$, using (4.3.3), we obtain that for $s = s_0$ with $\operatorname{Re} s_0 \leqslant 0$,

$$\operatorname{ord}_{s=s_0}\zeta(s) = -\operatorname{ord}_{s=s_0}\Gamma(\tfrac{1}{2}s+1) = \begin{cases} 1 & \text{if } s_0 \in \{-2,-4,-6,\ldots\}, \\ 0 & \text{if } s_0 \notin \{-2,-4,-6,\ldots\}. \end{cases}$$

This proves Corollary 4.3.4. □

Corollary 4.3.5. *Suppose* ζ *has a zero* s_0 *with* $0 < \operatorname{Re} s_0 < 1$. *Then* $1-s_0$, $\overline{s_0}$ *and* $1-\overline{s_0}$ *are also zeros of* ζ.

Proof. We have $\xi(s_0) = 0$, and so $\xi(1-s_0) = 0$ by Theorem 4.3.2. But $\pi^{-s/2}\Gamma(\frac{1}{2}s)$ is non-zero and has no poles in the critical strip $0 < \operatorname{Re} s < 1$, hence $\zeta(1-s_0) = 0$. The function ζ assumes real values on $\mathbb{R}_{>1}$, so by Schwarz' reflection principle (Corollary 0.6.24 from the Prerequisites), $\overline{\zeta(s)} = \zeta(\overline{s})$ for $s \in \mathbb{C} \setminus \{1\}$, and in particular $\zeta(\overline{s_0}) = 0$, $\zeta(1-\overline{s_0}) = 0$. □

In Exercise 4.3 you will be asked to prove that $\zeta(s) < 0$ for $s \in \mathbb{R}$, $0 < s < 1$. Inserting $\xi(0) = \frac{1}{2}$ and $\Gamma(1) = 1$ in (4.3.3) one easily shows that $\zeta(0) = -\frac{1}{2}$.

There are also functional equations for L-functions $L(s,\chi)$, in the case that χ is a primitive character modulo an integer $q \geqslant 2$ (that is to say, χ is not induced by a character modulo d for any proper divisor d of q).

Notice that for any character χ modulo q we have $\chi(-1)^2 = \chi(1) = 1$, hence $\chi(-1) \in \{-1, 1\}$. A character χ is called *even* if $\chi(-1) = 1$, and *odd* if $\chi(-1) = -1$. There will be different functional equations for even and odd characters.

In Chapter 3 we defined the Gauss sum related to a character χ mod q by

$$\tau(1,\chi) = \sum_{a=0}^{q-1} \chi(a) e^{2\pi i a/q}.$$

By $\overline{\chi}$ we denote the complex conjugate of a character χ.

Theorem 4.3.6. *Let* q *be an integer with* $q \geqslant 2$, *and* χ *a primitive character mod* q. *Put*

$$\xi(s,\chi) := \left(\frac{q}{\pi}\right)^{s/2}\Gamma(\tfrac{1}{2}s)L(s,\chi), \qquad c(\chi) := \frac{\sqrt{q}}{\tau(1,\chi)} \quad \text{if } \chi \text{ is even,}$$

$$\xi(s,\chi) := \left(\frac{q}{\pi}\right)^{(s+1)/2}\Gamma\big(\tfrac{1}{2}(s+1)\big)L(s,\chi), \quad c(\chi) := \frac{i\sqrt{q}}{\tau(1,\chi)} \quad \text{if } \chi \text{ is odd.}$$

Then $\xi(s,\chi)$ has an analytic continuation to $\mathbb{C}$, and

$$\xi(1-s,\overline{\chi}) = c(\chi)\xi(s,\chi) \quad \textit{for } s \in \mathbb{C}.$$

Remark. According to Theorem 3.4.2, if χ is primitive then $|\tau(1,\chi)| = \sqrt{q}$, hence $|c(\chi)| = 1$. In general, it is a difficult problem to compute $c(\chi)$ for large values of q.

The proof of Theorem 4.3.6 is similar to that of that of the functional equation for $\zeta(s)$, but with some additional technicalities, see H. Davenport, Multiplicative Number Theory, Chapter 9.

In Exercise 4.6 you will be asked to deduce some consequences.

4.4 Properties of Euler's Gamma function

Euler's Gamma function is defined by the integral

$$\Gamma(z) := \int_0^\infty e^{-t}t^{z-1}dt \quad (z \in \mathbb{C},\, \operatorname{Re} z > 0).$$

Lemma 4.4.1. $\Gamma(z)$ *defines an analytic function on* $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Proof. This is standard using Theorem 0.6.25. Let $U := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. First, the function $F(t,z) := e^{-t}t^{z-1}$ is continuous, hence measurable on $\mathbb{R}_{>0} \times U$. Second, for each fixed $t > 0$, $z \mapsto e^{-t}t^{z-1}$ is analytic on U. Third, let K be a compact subset of U. Then there exist $\delta, R > 0$ such that $\delta \leqslant \operatorname{Re} z \leqslant R$ for $z \in K$. This implies that for $z \in K$, $t > 0$,

$$|e^{-t}t^{z-1}| \leqslant M(t) := \begin{cases} t^{\delta-1} & \text{for } 0 < t \leqslant 1, \\ e^{-t}t^{R-1} \leqslant Ce^{-t/2} & \text{for } t > 1, \end{cases}$$

where C is some constant. Now we have

$$\int_0^\infty M(t)dt = \int_0^1 t^{\delta-1}dt + C\int_1^\infty e^{-t/2}dt = \delta^{-1} + 2C \cdot e^{-1/2} < \infty.$$

Hence all conditions of Theorem 0.6.25 are satisfied, and thus, $\Gamma(z)$ is analytic on U. □

Using integration by parts, one easily shows that for $z \in \mathbb{C}$ with $\mathrm{Re}\, z > 0$,

$$\begin{aligned}\Gamma(z) &= z^{-1} \int_0^\infty e^{-t} dt^z \\ &= z^{-1} \left(\left[e^{-t} t^z \right]_{t=0}^{t=\infty} + \int_0^\infty e^{-t} t^z dt \right) = z^{-1}\Gamma(z+1),\end{aligned}$$

that is,

$$\Gamma(z+1) = z\Gamma(z) \quad \text{if } \mathrm{Re}\, z > 0. \tag{4.4.1}$$

One easily shows that $\Gamma(1) = 1$ and then by induction, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}_{>0}$.

We now show that Γ has a meromorphic continuation to $\mathbb{C}$.

Theorem 4.4.2. *The function Γ has a unique analytic continuation to $\mathbb{C}\backslash\{0,-1,-2,\ldots\}$. If we denote this continuation also by Γ, it has the following properties:*
(i) Γ has a simple pole with residue $(-1)^n/n!$ at $z = -n$ for $n = 0, 1, 2, \ldots$;
(ii) $\Gamma(z+1) = z\Gamma(z)$ for $z \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$.

Proof. Corollary 0.6.22 from the Prerequisites and the analyticity of the integral imply that the analytic continuation, if it exists, is unique. We proceed to construct this analytic continuation.

First let $z \in \mathbb{C}$ with $\mathrm{Re}\, z > 0$. By repeatedly applying (4.4.1) we get

$$\Gamma(z) = \frac{1}{z(z+1)\cdots(z+n-1)} \cdot \Gamma(z+n) \quad \text{for } \mathrm{Re}\, z > 0, \quad n = 1, 2, \ldots. \tag{4.4.2}$$

The right-hand side clearly defines an analytic continuation of Γ to $B_n := \{z \in \mathbb{C} : \mathrm{Re}\, z > -n,\ z \neq 0, 1, \ldots, 1-n\}$, for $n = 1, 2, \ldots$. According to Theorem 0.6.23 from the Prerequisites, Γ has an analytic continuation to $B := \mathbb{C} \setminus \{0,-1,-2,\ldots\}$, which coincides on B_n with the right-hand side of (4.4.2), for $n = 1, 2, \ldots$.

By (4.4.2) we have

$$\begin{aligned}\lim_{z\to -n} (z+n)\Gamma(z) &= \lim_{z\to -n} (z+n) \frac{1}{z(z+1)\cdots(z+n)} \Gamma(z+n+1) \\ &= \frac{1}{(-n)(-n+1)\cdots(-1)} \Gamma(1) = \frac{(-1)^n}{n!}.\end{aligned}$$

Hence Γ has a simple pole at $z = -n$ of residue $(-1)^n/n!$.

Both functions $\Gamma(z+1)$ and $z\Gamma(z)$ are analytic on B, and by (4.4.1), they are equal on the set $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ which has limit points in B. So by Corollary 0.6.22, $\Gamma(z+1) = z\Gamma(z)$ for $z \in B$. □

Before proving further properties of the Gamma function we introduce the related *Beta function* in two complex variables,

$$B(z,w) := \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \quad (z,w \in \mathbb{C},\ z,w,z+w \notin \{0,-1,-2,\ldots\}).$$

Theorem 4.4.3. *Assume* $\operatorname{Re} z, \operatorname{Re} w > 0$. *Then*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt.$$

Proof. We start with rewriting $\Gamma(z)\Gamma(w)$. We clearly have

$$\Gamma(z)\Gamma(w) = \int_0^\infty e^{-s}s^{z-1}ds \cdot \int_0^\infty e^{-t}t^{w-1}dt = \int_0^\infty \int_0^\infty e^{-s-t}s^{z-1}t^{w-1}dsdt.$$

We apply the substitution $s = uv$, $t = (1-u)v$. The map $(u,v) \mapsto (s,t)$ is a bijection from $(0,1)\times(0,\infty)$ to $(0,\infty)\times(0,\infty)$. The Jacobian determinant of this substitution is

$$\frac{\partial(s,t)}{\partial(u,v)} = \begin{vmatrix} \partial s/\partial u & \partial s/\partial v \\ \partial t/\partial u & \partial t/\partial v \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v.$$

So by the substitution rule,

$$\begin{aligned}
\Gamma(z)\Gamma(w) &= \int_0^\infty \int_0^1 e^{-v}(uv)^{z-1}((1-u)v)^{w-1}\left|\frac{\partial(s,t)}{\partial(u,v)}\right| dudv \\
&= \int_0^\infty \int_0^1 e^{-v}v^{z+w-1}u^{z-1}(1-u)^{w-1}dudv \\
&= \int_0^1 u^{z-1}(1-u)^{w-1}du \cdot \int_0^\infty e^{-v}v^{z+w-1}dv \\
&= \int_0^1 u^{z-1}(1-u)^{w-1}du \cdot \Gamma(z+w).
\end{aligned}$$

□

We use this to prove

Theorem 4.4.4. *We have* $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$ *for* $z \in \mathbb{C} \setminus \mathbb{Z}$.

Proof. Let $A := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. Notice that $\mathbb{C} \setminus \mathbb{Z}$ is connected, A has a limit point in $\mathbb{C} \setminus \mathbb{Z}$, and both $\Gamma(z)\Gamma(1-z)$ and $\pi/\sin \pi z$ are analytic on $\mathbb{C} \setminus \mathbb{Z}$. So by Corollary 0.6.21 it suffices to prove that both functions are equal on A. By Theorem 4.4.3 we have $\Gamma(z)\Gamma(1-z) = \int_0^1 t^{z-1}(1-t)^{-z}dt$ for $z \in A$. Hence it suffices to prove that

$$\int_0^1 t^{z-1}(1-t)^{-z}dt = \frac{\pi}{\sin \pi z} \quad \text{for } 0 < \operatorname{Re} z < 1. \tag{4.4.3}$$

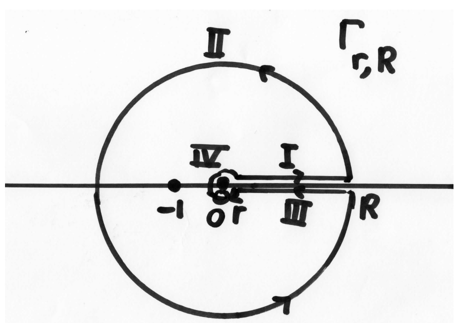
We can rewrite the integral as $\int_0^1 \left(\frac{t}{1-t}\right)^z t^{-1}dt$. We substitute $x = \frac{t}{1-t}$. With this choice, $t = \frac{x}{x+1}$, $dt = (x+1)^{-2}dx$, and x runs from 0 to ∞. Thus, the integral is equal to

$$J(z) := \int_0^\infty \frac{x^{z-1}}{x+1} dx.$$

We compute $J(z)$ by integrating the analytic function $f(w) := \frac{w^{z-1}}{w+1}$ along the closed, simple, positively oriented contour $\Gamma_{r,R}$ described below and letting $r \downarrow 0$, $R \to \infty$. Here we define

$$w^{z-1} := e^{(z-1)\log w}, \quad \log w := \log|w| + i\arg w,$$

where we take $0 \leqslant \arg w < 2\pi$.



Let $R > r > 0$. The contour $\Gamma_{r,R}$ consists of four parts. Part I is a line segment just above the real line from r to R traversed from left to right. Part II is the circle with center 0 and radius R, traversed counterclockwise. Part III is a line segment just below the real line traversed from R to r (so from right to left). Part IV is the circle with center 0 and radius r, traversed clockwise.

The Residue Theorem gives

$$\begin{aligned}\int_{\Gamma_{r,R}} f(w)dw &= 2\pi i\cdot \mathrm{Res}_{w=-1} f(w) = 2\pi i\cdot w^{z-1}|_{w=-1} \\ &= 2\pi i\cdot e^{(z-1)\log(-1)} = 2\pi i\cdot e^{(z-1)\pi i} = -2\pi i\cdot e^{\pi i z}\end{aligned} \tag{4.4.4}$$

for all $R > r > 0$.

We compute the integral in another way. Part I of $\Gamma_{r,R}$ was defined to be a line segment just above the real line, traversed from r to R. If we move I towards the segment on the real line from r to R the value of the integral doesn't change, so we may as well assume that I is precisely the segment on the real line line from r to R. Then on I we have $\arg w = 0$ and $w = x$ with $r \leqslant x \leqslant R$ so

$$\int_I f(w)dw = \int_r^R \tfrac{x^{z-1}}{x+1}dx.$$

Part III of $\Gamma_{r,R}$ was defined to be a line segment just below the real line, traversed from R to r. Like above, the value of the integral doesn't change if we move III towards the segment on the real line, traversed from R to r, so we may as well assume that III is equal to this line segment. But note that we have arrived at III after having traversed the circle II, so $\arg w$ has become 2π. Thus, on III we have $\log w = \log|w| + 2\pi i$. Writing again x for w, where x moves from R to r, we get

$$\int_{III} f(w)dw = \int_R^r \frac{e^{(z-1)(\log x+2\pi i)}}{x+1}dx = \int_R^r \frac{x^{z-1}e^{2\pi i z}}{x+1}dx = -e^{2\pi i z}\int_r^R \frac{x^{z-1}}{x+1}dx.$$

Here we have used $e^{2\pi i} = 1$. For the integrals over II and IV we only give estimates. We have

$$\left|\int_{II} f(w)dw\right| \leqslant \text{length}\, II\cdot \sup_{w\in II}|f(w)|.$$

On II we have

$$\begin{aligned}|w^{z-1}| &= |e^{(z-1)(\log|w|+i\arg w)}| = e^{\mathrm{Re}\,((z-1)(\log|w|+i\arg w))} \\ &= e^{(\mathrm{Re}\, z-1)\log|w|-(\mathrm{Im}\, z-1)\arg w} \leqslant R^{\mathrm{Re}\, z-1}\cdot e^{2\pi(|\mathrm{Im}\, z|+1)} = C\cdot R^{\mathrm{Re}\, z-1}\end{aligned}$$

with C independent of R, hence

$$|f(w)| \leqslant C\cdot \frac{R^{\mathrm{Re}\, z-1}}{R-1}.$$

Since $\operatorname{Re} z < 1$ we conclude that

$$\left|\int_{II} f(w)dw\right| \leqslant 2\pi R \cdot C \cdot \frac{R^{\operatorname{Re} z-1}}{R-1} \to 0 \text{ as } R \to \infty.$$

Finally, on IV we have by a similar estimate,

$$|w^{z-1}| \leqslant C \cdot r^{\operatorname{Re} z-1}, \quad |f(w)| \leqslant C \cdot \frac{r^{\operatorname{Re} z-1}}{1-r},$$

and thus, since $\operatorname{Re} z > 0$,

$$\left|\int_{II} f(w)dw\right| \leqslant 2\pi r \cdot C \cdot \frac{r^{\operatorname{Re} z-1}}{1-r} \to 0 \text{ as } r \downarrow 0.$$

Combining our results for I,II,III,IV, we get

$$\int_{\Gamma_{r,R}} f(w)dw = \int_I + \int_{II} + \int_{III} + \int_{IV} \to (1 - e^{2\pi iz})J(z) \quad \text{as } r \downarrow 0,\ R \to \infty.$$

Using the other expression for the integral obtained in (4.4.4), this leads to

$$J(z) = \frac{-2\pi i \cdot e^{\pi iz}}{1 - e^{2\pi iz}} = \frac{2\pi i}{e^{\pi iz} - e^{-\pi iz}} = \frac{\pi}{\sin \pi z}.$$

□

Corollary 4.4.5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Proof. Substitute $z = \frac{1}{2}$ in Theorem 4.4.4, and use $\Gamma(\frac{1}{2}) > 0$. □

Corollary 4.4.6. *(i)* $\Gamma(z) \neq 0$ *for* $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.
(ii) $1/\Gamma$ *is analytic on* $\mathbb{C}$*, and* $1/\Gamma$ *has simple zeros at* $z = 0, -1, -2, \ldots$ *and is non-zero elesewhere.*

Proof. (i) Recall that $\Gamma(n) = (n-1)! \neq 0$ for $n = 1, 2, \ldots$. Further, by Theorem 4.4.4 we have $\Gamma(z)\Gamma(1-z)\sin \pi z = \pi \neq 0$ for $z \in \mathbb{C} \setminus \mathbb{Z}$.

(ii) By (i), the function $1/\Gamma$ is analytic and non-zero on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$. Further, at $z = 0, -1, -2, \ldots$, Γ has a simple pole, hence $1/\Gamma$ is analytic and has a simple zero. □

We give another expression for the Gamma function.

Theorem 4.4.7. *For* $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ *we have*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! \cdot n^z}{z(z+1)\cdots(z+n)}.$$

Proof. Define

$$F_n(z) := \frac{n! \cdot n^z}{z(z+1)\cdots(z+n)}.$$

We prove by induction that for every non-negative integer m we have $\Gamma(z) = \lim_{n\to\infty} F_n(z)$ for $z \in \mathbb{C}$ with $\operatorname{Re} z > -m$ and (if $m > 0$) $z \neq 0, -1, \dots, 1-m$. We first consider the case $m = 0$, i.e., $z \in \mathbb{C}$ and $\operatorname{Re} z > 0$. We skip a few details, which are left to the reader. Using repeated integration by parts, one shows that

$$\int_0^1 (1-u)^n u^{z-1} du = \frac{n!}{z(z+1)\cdots(z+n)},$$

so

$$F_n(z) = n^z \int_0^1 (1-u)^n u^{z-1} du = \int_0^n \left(1 - \tfrac{t}{n}\right)^n t^{z-1} dt = \int_0^\infty g_n(t) dt,$$

where $g_n(t) = \left(1 - \frac{t}{n}\right)^n t^{z-1}$ if $0 < t \leqslant n$ and $g_n(t) = 0$ if $t > n$. Now we have $(1 - x^{-1})^x \leqslant e^{-1}$ for all $x \geqslant 1$, and $\lim_{x\to\infty}(1 - x^{-1})^x = e^{-1}$. This implies

$$\lim_{n\to\infty} g_n(t) = e^{-t} t^{z-1}, \quad |g_n(t)| \leqslant e^{-t} t^{\operatorname{Re} z - 1} \quad \text{for } t > 0,\ n > 0,$$

while moreover, $\int_0^\infty e^{-t} t^{\operatorname{Re} z - 1} dt = \Gamma(\operatorname{Re} z) < \infty$. So by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} F_n(z) = \lim_{n\to\infty} \int_0^\infty g_n(t) dt = \int_0^\infty e^{-t} t^{z-1} dt = \Gamma(z).$$

We now carry out the induction step. Let $m \geqslant 0$ be an integer and suppose that $\Gamma(z) = \lim_{n\to\infty} F_n(z)$ for $z \in \mathbb{C}$ with $\operatorname{Re} z > -m$ and (if $m > 0$) $z \neq 0, -1, \dots, 1-m$. Let $z \in \mathbb{C}$ with $\operatorname{Re} z > -m-1$ and $z \neq 0, \dots, m$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{F_n(z+1)}{zF_n(z)} &= \lim_{n\to\infty} \frac{n! \cdot n^{z+1}}{(z+1)\cdots(z+n+1)} \cdot \frac{(z+1)\cdots(z+n+1)}{(n+1)!(n+1)^z} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{z+1} = 1.
\end{aligned}$$

By the induction hypothesis we know that $\Gamma(z+1) = \lim_{n\to\infty} F_n(z+1)$, and so

$$\lim_{n\to\infty} F_n(z) = \lim_{n\to\infty} \frac{F_n(z+1)}{z} = \frac{\Gamma(z+1)}{z} = \Gamma(z).$$

This completes the induction step and thus the proof of Theorem 4.4.7. □

We deduce some consequences. Recall that the *Euler-Mascheroni constant* γ is given by

$$\gamma := \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} \right) - \log N.$$

Corollary 4.4.8. *We have*

$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+z/n} \quad \textit{for } z \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}.$$

Proof. Let $z \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$. Then for $N \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned}
\Gamma(z) &= \lim_{N\to\infty} \frac{N^z \cdot N!}{z(z+1)\cdots(z+N)} = z^{-1} \lim_{N\to\infty} \frac{e^{z \log N}}{(1+z)(1+z/2)\cdots(1+z/N)} \\
&= z^{-1} \lim_{N\to\infty} e^{(\log N - 1 - \frac{1}{2} - \cdots - \frac{1}{N})z} \prod_{n=1}^{N} \frac{e^{z/n}}{1+z/n} \\
&= e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+z/n}.
\end{aligned}$$

□

Corollary 4.4.9. *For the logarithmic derivative of* Γ *we have*

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)} \quad \textit{for } z \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}.$$

Proof. Apply Corollary 0.6.30 from the Prerequisites. □

As another consequence, we derive an infinite product expansion for $\sin \pi z$.

Corollary 4.4.10. *We have*

$$\sin \pi z = \pi z \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad \textit{for } z \in \mathbb{C}.$$

Proof. For $z \in \mathbb{C}$ we have by Theorem 4.4.4, Corollary 4.4.6 and Corollary 4.4.8,

$$\begin{aligned}
\sin \pi z &= \frac{\pi}{\Gamma(z)\Gamma(1-z)} = \frac{\pi}{\Gamma(z)(-z)\Gamma(-z)} \\
&= \pi(-z)^{-1} e^{\gamma z} z \prod_{n=1}^{\infty} \left(e^{-z/n}\left(1+\tfrac{z}{n}\right)\right) \cdot e^{-\gamma z}(-z) \prod_{n=1}^{\infty} \left(e^{z/n}\left(1-\tfrac{z}{n}\right)\right) \\
&= \pi z \prod_{n=1}^{\infty} \left(1-\tfrac{z}{n}\right)\left(1+\tfrac{z}{n}\right) = \pi z \cdot \prod_{n=1}^{\infty} \left(1-\tfrac{z^2}{n^2}\right).
\end{aligned}$$

□

We finish with another important consequence of Theorem 4.4.7, the so-called *duplication formula.*

Corollary 4.4.11. *We have*

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \cdot \Gamma(z)\Gamma(z+\tfrac{1}{2}) \quad \textit{for } z \in \mathbb{C},\ z \neq 0, -\tfrac{1}{2}, -1, -\tfrac{3}{2}, -2, \ldots.$$

Proof. Let $A = \mathbb{C} \setminus \{0, -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots\}$. We show that the function $F(z) := 2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})/\Gamma(2z)$ is constant on A. Substituting $z = \frac{1}{2}$ gives that the constant is $2\sqrt{\pi}$, and then Corollary 4.4.11 follows.

Let $z \in A$. To get nice cancellations in the numerator and denominator, we use the expressions

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \frac{n! \cdot n^z}{z(z+1)\cdots(z+n)} = \lim_{n\to\infty} \frac{2^{n+1} \cdot n! \cdot n^z}{2z(2z+2)\cdots(2z+2n)}, \\
\Gamma(z+\tfrac{1}{2}) &= \lim_{n\to\infty} \frac{n! \cdot n^{z+1/2}}{(z+1/2)(z+3/2)\cdots(z+n+1/2)} \\
&\qquad\qquad = \lim_{n\to\infty} \frac{2^{n+1} \cdot n! \cdot n^{z+1/2}}{(2z+1)(2z+3)\cdots(2z+2n+1)}, \\
\Gamma(2z) &= \lim_{n\to\infty} \frac{(2n+1)! \cdot (2n+1)^{2z}}{2z(2z+1)\cdots(2z+2n+1)}
\end{aligned}$$

(i.e., in Theorem 4.4.7 we substitute $2z$ for z and take the limit over the odd integers).

Thus,

$$\begin{aligned} F(z) &= \frac{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)} \\ &= 2^{2z}\lim_{n\to\infty}\left\{\frac{2^{2n+2}(n!)^2 n^{2z+1/2}}{2z(2z+1)\cdots(2z+2n+1)}\cdot\frac{2z(2z+1)\cdots(2z+2n+1)}{(2n+1)!\cdot(2n+1)^{2z}}\right\} \\ &= \lim_{n\to\infty}\left\{\frac{2^{2n+2}(n!)^2\sqrt{n}}{(2n+1)!}\right\} \end{aligned}$$

since

$$\lim_{n\to\infty}\frac{2^{2z}\cdot n^{2z}}{(2n+1)^{2z}} = \lim_{n\to\infty} e^{2z\log(2n/(2n+1))} = 1.$$

This shows that indeed $F(z)$ is constant. □

Remark. By an argument similar to the proof of Corollary 4.4.11 (exercise), one can derive the *multiplication formula of Legendre-Gauss*,

$$\Gamma(nz) = (2\pi)^{-(n-1)/2} n^{nz-1/2}\Gamma(z)\Gamma\big(z+\tfrac{1}{n}\big)\cdots\Gamma\big(z+\tfrac{n-1}{n}\big) \quad \text{for } n\in\mathbb{Z}_{\geqslant 2}.$$

4.5 The Jacobi theta function

There are various methods to prove Theorem 4.3.2, see E.C. Titchmarsh, The theory of the Riemann zeta function. In the next section we will give Riemann's proof based on a functional equation for the Jacobi theta function

$$\theta(z) = \sum_{m=-\infty}^{\infty} e^{-\pi m^2 z}.$$

In the present section we derive this functional equation. For this, we need the following version of the Poisson summation formula for infinite sums.

Theorem 4.5.1. *Let f be a complex function such that:*
(i) there is $\delta>0$ such that $f(z)$ is analytic on $U(\delta):=\{z\in\mathbb{C}: |\mathrm{Im}\, z|<\delta\}$;
(ii) there are $\varepsilon>0, r>0$ such that

$$|f(z)| \leqslant |z|^{-1-\varepsilon} \quad \textit{for } z\in U(\delta),\ |z|\geqslant r.$$

Then

$$\sum_{m=-\infty}^{\infty} f(m) = \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt.$$

The idea is to apply Theorem 3.6.5 to the function $F(z) := \sum_{m=-\infty}^{\infty} f(z+m)$. We first prove some properties of this function. We keep assuming that f has the properties (i), (ii) from Theorem 4.5.1.

Lemma 4.5.2. *(i) The function $F(z)$ is analytic on an open subset of $\mathbb{C}$ containing $[0,1]$.*
(ii) For every $n \in \mathbb{Z}$ we have $\int_0^1 F(t)e^{-2\pi int}dt = \int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt$.
(iii) $F(0) = F(1) = \sum_{m=-\infty}^{\infty} f(m)$.

Proof. (i) Let $U := \{z \in \mathbb{C} : -\delta < \operatorname{Re} z < 1+\delta, |\operatorname{Im} z| < \frac{1}{2}\delta\}$. By taking δ sufficiently small we may assume that $|z| \leqslant 2$ for $z \in U$. We prove that there are positive reals A_m such that $\sum_{m=-\infty}^{\infty} A_m$ converges and $|f(z+m)| \leqslant A_m$ for $z \in U$, $m \in \mathbb{Z}$. Since all summands $f(z+m)$ are analytic, by Corollary 0.6.29 it then follows that $F(z)$ is analytic on U. Note that for $z \in U$, $m \in \mathbb{Z}$ with $|m| \geqslant r+3$ we have $|z+m| \geqslant |m| - |z| \geqslant |m| - 2 > r$ and $|f(z+m)| \leqslant (|m|-2)^{-1-\varepsilon} =: A_m$. To treat the remaining m, let $R := \sup\{|f(z)| : |\operatorname{Im} z| \leqslant \frac{1}{2}\delta, |z| \leqslant r+5\}$; this quantity is finite since $|f(z)|$ is continuous on the region indicated. Then for $z \in U$, $m \in \mathbb{Z}$ with $|m| < r+3$ we have $|f(z+m)| \leqslant R =: A_m$. Clearly, $\sum_{m=-\infty}^{\infty} A_m$ converges.

(ii) We apply the Fubini-Tonelli Theorem. We have

$$\int_0^1 \Big(\sum_{m=-\infty}^{\infty} |f(t+m)e^{-2\pi int}| \Big) dt \leqslant \int_0^1 \Big(\sum_{m=-\infty}^{\infty} A_m \Big)) dt < \infty,$$

so

$$\begin{aligned}
\int_0^1 F(t)e^{-2\pi int}dt &= \int_0^1 \Big(\sum_{m=-\infty}^{\infty} f(t+m) \Big) e^{-2\pi int}dt = \sum_{m=-\infty}^{\infty} \int_0^1 f(t+m)e^{-2\pi int}dt \\
&= \sum_{m=-\infty}^{\infty} \int_0^1 f(t+m)e^{-2\pi in(t+m)}dt = \sum_{m=-\infty}^{\infty} \int_m^{m+1} f(t)e^{-2\pi int}dt \\
&= \int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt.
\end{aligned}$$

In the last step we have used that $\int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt$ converges, due to our assumption $|f(z)| \leqslant |z|^{-1-\varepsilon}$ for $z \in U(\delta)$, $|z| \geqslant r$.

(iii) Obvious, since $\sum_{m=-\infty}^{\infty} f(m)$ converges. □

Proof of Theorem 4.5.1. By combining Theorem 3.6.15 with Lemma 4.5.2 we obtain

$$\begin{aligned}\sum_{m=-\infty}^{\infty} f(m) &= \tfrac{1}{2}\big(F(0)+F(1)\big) = \lim_{N\to\infty} \sum_{n=-N}^{N} \int_0^1 F(t)e^{-2\pi int}dt \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt.\end{aligned}$$

□

Recall that the Jacobi theta function is given by

$$\theta(z) := \sum_{m=-\infty}^{\infty} e^{-\pi m^2 z} \quad (z \in \mathbb{C},\ \mathrm{Re}\, z > 0).$$

Verify yourself that $\theta(z)$ converges and is analytic on $\{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$.

Theorem 4.5.3. *$\theta(z^{-1}) = \sqrt{z} \cdot \theta(z)$ for $z \in \mathbb{C}$, $\mathrm{Re}\, z > 0$, where $\sqrt{z}$ is chosen such that $|\arg \sqrt{z}| < \pi/4$.*

Remark. Let $A := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$. We may choose the argument of $z \in A$ such that $|\arg z| < \pi/2$. Then indeed, we may choose $\sqrt{z}$ such that $|\arg \sqrt{z}| < \pi/4$.

Proof. Both $\theta(z^{-1})$ and $\sqrt{z}\theta(z)$ are analytic on A. In view of Corollary 0.6.21, it suffices to prove the identity in Theorem 4.5.3 on any infinite subset of A of our choice having a limit point in A. For this subset we take $\mathbb{R}_{>0}$. Thus, it suffices to prove that

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2/x} = \sqrt{x} \cdot \sum_{m=-\infty}^{\infty} e^{-\pi m^2 x} \quad \text{for } x > 0.$$

We apply Theorem 4.5.1 to $f(z) := e^{-\pi z^2/x}$ with $x > 0$ fixed. Verify that f satisfies all conditions of that Theorem. Thus, for any $x > 0$,

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2/x} = \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-\infty}^{\infty} e^{-(\pi t^2/x)-2\pi int}dt.$$

We compute the integrals by substituting $u = t\sqrt{x}$. Thus,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-(\pi t^2/x)-2\pi int} dt &= \sqrt{x}\cdot\int_{-\infty}^{\infty} e^{-\pi u^2-2\pi in\sqrt{x}\cdot u} du \\ &= \sqrt{x}\cdot\int_{-\infty}^{\infty} e^{-\pi(u+in\sqrt{x})^2-\pi n^2 x} du \\ &= \sqrt{x}e^{-\pi n^2 x}\int_{-\infty}^{\infty} e^{-\pi(u+in\sqrt{x})^2} du.\end{aligned}$$

In the lemma below we prove that the last integral converges and is equal to 1. Then it follows that

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2/x} = \lim_{N\to\infty}\sum_{n=-N}^{N}\sqrt{x}e^{-\pi n^2 x} = \sqrt{x}\sum_{n=-\infty}^{\infty} e^{-\pi n^2 x},$$

since the last series converges. This proves our Theorem. □

Lemma 4.5.4. *Let $z \in \mathbb{C}$. Then $\int_{-\infty}^{\infty} e^{-\pi(u+z)^2} du = 1$.*

Proof. The following proof was suggested to me by Michiel Kosters. Let

$$F(z) := \int_{-\infty}^{\infty} e^{-\pi(u+z)^2} du.$$

We show that this defines an analytic function on $\mathbb{C}$. We apply Theorem 0.6.25. First, $(u,z)\mapsto e^{-\pi(u+z)^2}$ is continuous, hence measurable, on $\mathbb{R}\times D(0,R)$. Second, for every fixed $u\in\mathbb{R}$, $z\mapsto e^{-\pi(u+z)^2}$ is analytic on $\mathbb{C}$. Third, let K be a compact subset of $\mathbb{C}$, and choose $R>0$ such that $|z|\leqslant R$ for $z\in K$. Then for $z\in K$ we have

$$\begin{aligned}|e^{-\pi(u+z)^2}| &= e^{-\mathrm{Re}\,\pi(u+z)^2} = e^{-(\pi u^2+2\pi u\mathrm{Re}\,z+\pi\mathrm{Re}\,z^2)} \\ &\leqslant e^{-\pi u^2+2\pi Ru+\pi R^2} = e^{-\pi(u-R)^2+2\pi R^2},\end{aligned}$$

and $\int_{-\infty}^{\infty} e^{-\pi(u-R)^2+2\pi R^2} du$ converges. So by Theorem 0.6.25, F is analytic on $\mathbb{C}$.

Knowing that F is analytic on $\mathbb{C}$, in order to prove that $F(z)=1$ for $z\in\mathbb{C}$ it is sufficient to prove, for any infinite set $S\subset\mathbb{C}$ with a limit point in $\mathbb{C}$, that $F(z)=1$ for $z\in S$. For the set S we take $\mathbb{R}$. For $z\in\mathbb{R}$ we obtain, by substituting $v=u+z$,

$$F(z) = \int_{-\infty}^{\infty} e^{-\pi(u+z)^2} du = \int_{-\infty}^{\infty} e^{-\pi v^2} dv = 2\int_{0}^{\infty} e^{-\pi v^2} dv.$$

Now a second substitution $t = \pi v^2$ yields

$$F(z) = \pi^{-1/2} \int_0^\infty e^{-t} t^{-1/2} dt = \pi^{-1/2} \Gamma(\tfrac{1}{2}) = 1.$$

$\square$

4.6 Proof of the functional equation for the Riemann zeta function

Define

$$\xi(s) := \tfrac{1}{2} s(s-1) \pi^{-s/2} \Gamma(\tfrac{1}{2}s) \zeta(s).$$

Theorem 4.3.2. *The function ξ has an analytic continuation to $\mathbb{C}$. For this continuation, we have $\xi(1-s) = \xi(s)$ for $s \in \mathbb{C}$ and moreover, $\xi(0) = \xi(1) = \frac{1}{2}$.*

Proof (Riemann). Let for the moment $s \in \mathbb{C}$, $\operatorname{Re} s > 1$. Recall that

$$\Gamma(\tfrac{1}{2}s) = \int_0^\infty e^{-t} t^{(s/2)-1} dt.$$

Substituting $t = \pi n^2 u$ gives

$$\Gamma(\tfrac{1}{2}s) = \int_0^\infty e^{-\pi n^2 u} (\pi n^2 u)^{(s/2)-1} d(\pi n^2 u) = \pi^{s/2} n^s \int_0^\infty e^{-\pi n^2 u} u^{(s/2)-1} du.$$

Hence

$$\pi^{-s/2} \Gamma(\tfrac{1}{2}s) n^{-s} = \int_0^\infty e^{-\pi n^2 u} u^{(s/2)-1} du,$$

and so, by summing over n,

$$\pi^{-s/2} \Gamma(\tfrac{1}{2}s) \zeta(s) = \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 u} \cdot u^{(s/2)-1} du.$$

We justify that the infinite integral and infinite sum can be interchanged. We use the following special case of the Fubini-Tonelli theorem: if $\{f_n : (0,\infty) \to \mathbb{C}\}_{n=1}^\infty$ is a sequence of measurable functions such that $\sum_{n=1}^\infty \int_0^\infty |f_n(u)| du$ converges, then all integrals $\int_0^\infty f_n(u)du$ $(n \geqslant 1)$ converge, the series $\sum_{n=1}^\infty f_n(u)$ converges almost everywhere on $(0,\infty)$ and moreover,

$$\sum_{n=1}^\infty \int_0^\infty f_n(u)du, \quad \int_0^\infty \left(\sum_{n=1}^\infty f_n(u)\right) du$$

converge and are equal. In our situation we have that indeed (putting $\sigma := \mathrm{Re}\, s$)

$$\begin{aligned}\sum_{n=1}^\infty \int_0^\infty |e^{-\pi n^2 u} \cdot u^{(s/2)-1}|du &= \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 u} u^{(\sigma/2)-1} du \\ &= \sum_{n=1}^\infty \pi^{-\sigma/2}\Gamma(\tfrac{1}{2}\sigma)n^{-\sigma} \quad \text{(reversing the above argument)} \\ &= \pi^{-\sigma/2}\Gamma(\tfrac{1}{2}\sigma)\zeta(\sigma)\end{aligned}$$

converges. Thus, we conclude that for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$,

$$\text{(4.6.1)} \qquad \pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s) = \int_0^\infty \omega(u) \cdot u^{(s/2)-1} du, \quad \text{where } \omega(u) = \sum_{n=1}^\infty e^{-\pi n^2 u}.$$

Recall that $\theta(u) = \sum_{n=-\infty}^\infty e^{-\pi n^2 u} = 1 + 2\omega(u)$.

We want to replace the right-hand side of (4.6.1) by something that converges for every $s \in \mathbb{C}$. Obviously, for $s \in \mathbb{C}$ with $\mathrm{Re}\, s < 0$ there are problems if $u \downarrow 0$. To overcome these, we split the integral $\int_0^\infty$ into $\int_1^\infty + \int_0^1$ and then transform $\int_0^1$ into an integral $\int_1^\infty$ by means of a substitution $v = u^{-1}$. After this substitution, the integral contains a term $\omega(v^{-1})$. By Theorem 4.5.3, we have

$$\begin{aligned}\omega(v^{-1}) &= \tfrac{1}{2}(\theta(v^{-1}) - 1) = \tfrac{1}{2}v^{1/2}\theta(v) - \tfrac{1}{2} \\ &= \tfrac{1}{2}v^{1/2}\big(2\omega(v) + 1\big) - \tfrac{1}{2} = v^{1/2}\omega(v) + \tfrac{1}{2}v^{1/2} - \tfrac{1}{2}.\end{aligned}$$

We work out in detail the approach sketched above. We keep for the moment our assumption $\mathrm{Re}\, s > 1$. Thus,

$$\begin{aligned}\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s) &= \int_1^\infty \omega(u)u^{(s/2)-1}du - \int_1^\infty \omega(v^{-1})v^{1-s/2}dv^{-1} \\ &= \int_1^\infty \omega(u)u^{(s/2)-1}du + \int_1^\infty \big(v^{1/2}\omega(v) + \tfrac{1}{2}v^{1/2} - \tfrac{1}{2}\big)v^{1-s/2}v^{-2}dv \\ &= \int_1^\infty \tfrac{1}{2}\big(v^{-(s+1)/2} - v^{-(s/2)-1}\big)dv + \int_1^\infty \omega(v)\big(v^{(s/2)-1} + v^{-(s+1)/2}\big)dv\end{aligned}$$

where we have combined the terms without ω into one integral, and the terms involving ω into another integral. Since we are still assuming $\operatorname{Re} s > 1$, the first integral is equal to

$$\tfrac{1}{2}\left[-\frac{2}{s-1}v^{-(s-1)/2} + \frac{2}{s}v^{-s/2}\right]_1^\infty = \frac{1}{s-1} - \frac{1}{s} = \frac{1}{s(s-1)}.$$

Hence

$$\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \omega(v)\big(v^{(s/2)-1} + v^{-(s+1)/2}\big)dv.$$

For our function $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{1}{2}s)\zeta(s)$ this gives

$$\text{(4.3.2)} \qquad \xi(s) = \tfrac{1}{2} + \tfrac{1}{2}s(s-1)\int_1^\infty \omega(v)\big(v^{(s/2)-1} + v^{-(s+1)/2}\big)dv \quad \text{if } \operatorname{Re} s > 1.$$

Assume for the moment that $F(s) := \int_1^\infty \omega(v)\big(v^{(s/2)-1} + v^{-(s+1)/2}\big)dv$ defines an analytic function on $\mathbb{C}$. Then we can use the right-hand side of (4.3.2) to define the analytic continuation of ξ to $\mathbb{C}$. By substituting $1-s$ for s in the right-hand side, we see that $\xi(1-s) = \xi(s)$. Further, we observe that $\xi(0) = \xi(1) = \frac{1}{2}$.

It remains to prove that F defines an analytic function on $\mathbb{C}$. We apply as usual Theorem 0.6.25. We check that $f(v,s) := \omega(v)\big(v^{(s/2)-1} + v^{-(s+1)/2}\big)$ satisfies the conditions of that theorem.

a) $f(v,s)$ is measurable on $(1,\infty)\times\mathbb{C}$. For $\omega(v) = \sum_{n=1}^\infty e^{-\pi n^2 v}$ is measurable, being a pointwise convergent series of continuous, hence measurable functions, and also $v^{(s/2)-1} + v^{-(s+1)/2}$ is measurable, since it is continuous.

b) $s \mapsto \omega(v)\big(v^{(s/2)-1} + v^{-(s+1)/2}\big)$ is analytic on $\mathbb{C}$ for every fixed v. This is obvious.

c) Let K be a compact subset of $\mathbb{C}$. Then there is a measurable function $M_K(v)$ on $(1,\infty)$ such that $|f(v,s)| \leqslant M_K(v)$ for $s \in K$ and $\int_1^\infty M_K(v)dv < \infty$. Indeed, choose $A > 0$ such that $|\operatorname{Re} s| \leqslant A$ for $s \in K$. we first have for $v \in (1,\infty)$

$$\begin{aligned} 0 \leqslant \omega(v) &\leqslant e^{-\pi v}\big(1 + e^{-3\pi v} + e^{-8\pi v} + \cdots\big) \\ &\leqslant e^{-\pi v}\cdot\sum_{k=0}^\infty e^{-3k\pi v} = \frac{e^{-\pi v}}{1-e^{-3\pi v}} \leqslant 2e^{-\pi v} \end{aligned}$$

and second, for $v \in (1,\infty)$, $s \in K$,

$$|v^{(s/2)-1} + v^{-(s+1)/2}| \leqslant v^{(A/2)-1} + v^{(A-1)/2} \leqslant 2v^{(A-1)/2}.$$

Hence

$$|f(v,s)| \leqslant 4e^{-\pi v} v^{(A-1)/2} =: M_K(v).$$

Further,

$$\int_1^\infty M_K(v)dv \leqslant 4 \int_0^\infty e^{-v} v^{(A-1)/2} dv \leqslant 4 \cdot \Gamma((A+1)/2) < \infty.$$

So $f(v,s)$ satisfies all conditions of Theorem 0.6.25, and it follows that the function $F(s) = \int_1^\infty f(v,s)dv$ is indeed analytic on $\mathbb{C}$. $\square$

4.7 Exercises

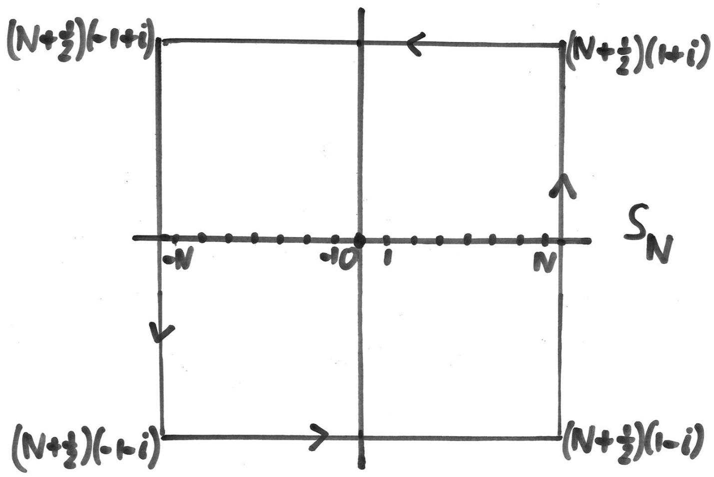
Before listing our exercises, we state and prove a general result to compute infinite series of values of rational functions, which is needed in some of the exercises.

Let $f = P/Q$ be a rational function, where $P, Q \in \mathbb{C}[z]$ are polynomials with $\deg Q \geqslant \deg P + 2$, such that P and Q have no common zeros. Denote by S the set of poles of f, i.e., zeros of Q. Then the series $\sum_{m \in \mathbb{Z} \setminus S} f(m) = \lim_{M,N \to \infty} \sum_{m=-M,\, m \notin S}^{N} f(m)$ converges absolutely. The following result allows us to compute such a series.

Theorem 4.7.1. *We have*

$$\sum_{m \in \mathbb{Z} \setminus S} f(m) = -2\pi i \sum_{\alpha \in S} \operatorname{res}\Big(\alpha,\, \frac{f(z)}{e^{2\pi i z} - 1}\Big).$$

Remark. Notice that this theorem gives in general a non-trivial result, except when f is *odd*, i.e., $f(-z) = -f(z)$ for $z \in \mathbb{C} \setminus S$. In this case, S is symmetric about 0, i.e., if $\alpha \in S$ then also $-\alpha \in S$, and the series is easily seen to be equal to 0.



Let N be an integer $\geqslant 1$ and let S_N be the square through the four points

$$(N+\tfrac{1}{2})(1+i),\ (N+\tfrac{1}{2})(-1+i),\ (N+\tfrac{1}{2})(-1-i),\ (N+\tfrac{1}{2})(1-i),$$

traversed counterclockwise. Assume that N is large enough such that the set S of poles of f lies in the interior of C_N. The idea behind the proof of Theorem 4.7.1 is as follows:

1) Compute $\oint_{S_N} \frac{f(z)}{e^{2\pi iz}-1} \cdot dz$, using the Residue Theorem from complex analysis. The crucial observation here will be that $\mathrm{res}\Big(m,\, \frac{f(z)}{e^{2\pi iz}-1}\Big) = \frac{f(m)}{2\pi i}$ for $m \in \mathbb{Z}\setminus S$ (see below).

2) Prove that $\lim_{N\to\infty} \oint_{S_N} \frac{f(z)}{e^{2\pi iz}-1} \cdot dz = 0$. It will be important to notice here that $|e^{2\pi iz}-1|$ cannot be too small on S_N. This is the reason why we have chosen to integrate over this particular contour S_N.

In the next lemmas we work out the details of 1) and 2). We assume that N is large enough such that the interior of S_N contains all poles of f.

Lemma 4.7.2. *We have*

$$\oint_{S_N} \frac{f(z)}{e^{2\pi iz}-1} dz = \sum_{m=-N,\, m\notin S}^{N} f(m) + 2\pi i \sum_{\alpha\in S} \mathrm{res}\Big(\alpha,\, \frac{f(z)}{e^{2\pi iz}-1}\Big).$$

Proof. We apply the Residue Theorem. Note that $e^{2\pi iz}-1$ is en entire function, which has simple zeros at the integers, and is non-zero elsewhere. So by Lemma

0.6.15 (iv),

$$\operatorname{res}\Big(m,\ \frac{f(z)}{e^{2\pi iz}-1}\Big) = \frac{f(m)}{2\pi i} \quad \text{for } m \in \mathbb{Z} \setminus S.$$

Now by the Residue Theorem,

$$\oint_{S_N} \frac{f(z)}{e^{2\pi iz}-1} dz = 2\pi i \sum_{\alpha \in \{-N,\ldots,N\} \cup S} \operatorname{res}\Big(\alpha,\ \frac{f(z)}{e^{2\pi iz}-1}\Big),$$

implying the lemma. □

The next lemma is needed in 2).

Lemma 4.7.3. *There is a constant $c > 0$, independent of N, such that $|e^{2\pi iz}-1| \geqslant c$ holds for all integers $N \geqslant 1$ and all $z \in S_N$.*

Proof. We consider the four edges of the square separately. First consider the edge from $(N+\frac{1}{2})(-1-i)$ to $(N+\frac{1}{2})(1-i)$. This can be parametrized by $(N+\frac{1}{2})(t-i)$ with $-1 \leqslant t \leqslant 1$. So for the points z on this edge we have

$$\begin{aligned} |e^{2\pi iz}-1| &= |e^{2\pi i(N+\frac{1}{2})(t-i)}-1| = |e^{2\pi i(N+\frac{1}{2})t} e^{2\pi(N+\frac{1}{2})}-1| \\ &\geqslant e^{2\pi(N+\frac{1}{2})}-1 \geqslant e^{3\pi}-1. \end{aligned}$$

Next, consider the edge from $(N+\frac{1}{2})(1-i)$ to $(N+\frac{1}{2})(1+i)$. This can be parametrized by $(N+\frac{1}{2})(1+it)$ with $-1 \leqslant t \leqslant 1$. So for the points z on this edge we have

$$|e^{2\pi iz}-1| = |e^{2\pi i(N+\frac{1}{2})(1+it)}-1| = |-e^{-2\pi(N+\frac{1}{2})t}-1| \geqslant 1.$$

Here we have used that $e^{2\pi i(N+\frac{1}{2})} = -1$. The other two edges can be treated in the same manner. □

Lemma 4.7.4. *We have*

$$\lim_{N\to\infty} \oint_{S_N} \frac{2\pi i f(z)}{e^{2\pi iz}-1} \cdot dz = 0.$$

Proof. We use the general inequality

$$\left| \int_\gamma g(z) dz \right| \leqslant L(\gamma) \cdot \sup_{z \in \gamma} |g(z)|,$$

where γ is a path in $\mathbb{C}$, $g:\gamma\to\mathbb{C}$ is a continuous function, and $L(\gamma)$ denotes the length of γ. Our assumption $f=P/Q$ with $\deg Q\geqslant \deg P+2$ implies that there are $C,r>0$ such that $|f(z)|\leqslant C\cdot|z|^{-2}$ for $z\in\mathbb{C}$ with $|z|>r$. Taking $\gamma=S_N$ with $N>r$ and $g(z)=f(z)/(e^{2\pi iz}-1)$ we get

$$\begin{aligned}\left|\oint_{S_N}\frac{2\pi i f(z)}{e^{2\pi iz}-1}\cdot dz\right| &\leqslant L(S_N)\cdot\sup_{z\in S_N}\frac{|f(z)|}{|e^{2\pi iz}-1|}\\ &\leqslant 4(2N+1)\cdot\frac{C\cdot N^{-2}}{c}\to 0\ \text{ as } N\to\infty.\end{aligned}$$

□

Proof of Theorem 4.7.1. Take the formula in Lemma 4.7.2 and let $N\to\infty$. □

Exercise 4.1. *a) The Bernoulli numbers B_n are given by* $\dfrac{z}{e^z-1}=\displaystyle\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n$ *($z\in\mathbb{C}$, $|z|<2\pi$). Using Theorem 4.7.1 prove that*

$$\zeta(2k)=(-1)^{k-1}2^{2k-1}\frac{B_{2k}}{(2k)!}\cdot\pi^{2k}\quad \textit{for } k=1,2,\ldots.$$

b) Show that $\frac{1}{2}z+\frac{z}{e^z-1}$ is an even function, and conclude that $B_1=-\frac{1}{2}$, $B_3=B_5=\cdots=0$.

c) Prove that $B_n=-\frac{1}{n+1}\sum_{l=0}^{n-1}\binom{n+1}{l}B_l$ for $n\geqslant 1$.

Remark. Theorem 4.7.1 cannot be used to compute the "odd zeta values" $\zeta(2k+1)$ ($k=1,2,\ldots$) since z^{-2k-1} is an odd function. About these odd zeta values not so much is known. Apéry proved in 1978 that $\zeta(3)$ is irrational. Zudilin proved in 2001 that at least one among the numbers $\zeta(5),\zeta(7),\zeta(9),\zeta(11)$ is irrational, but he could not indicate which one. Rivoal proved in 2000 that infinitely many among $\zeta(5),\zeta(7),\zeta(9),\ldots$ are irrational. At the moment there is not a single odd integer $k\geqslant 5$ for which irrationality of $\zeta(k)$ could be proved. It is of course believed that $\zeta(k)$ is irrational for every odd integer $k\geqslant 3$.

Exercise 4.2. *Let q be an integer $\geqslant 2$, and a an integer with $1\leqslant a\leqslant q/2$.*

a) Show that $\sum_{m=-\infty}^{\infty}(qm+a)^{-2}$ *converges and prove that*

$$\sum_{m=-\infty}^{\infty}(qm+a)^{-2}=\frac{(\pi/q)^2}{(\sin\pi a/q)^2}$$

by means of Theorem 4.7.1.

b) Let χ *be a non-principal character modulo* q *with* $\chi(-1)=1$*. Express* $L(2,\chi)$ *as a finite sum.*

Exercise 4.3. *In this exercise you have to use the identity*

$$\zeta(s)=\frac{1}{s-1}+1-s\int_1^{\infty}(x-[x])x^{-1-s}dx \quad (\operatorname{Re}s>0).$$

a) Prove that $\zeta(s)<0$ *for* $s\in\mathbb{R}$ *with* $0<s<1$*.*

b) Around $s=1$*,* $\zeta(s)$ *has a Laurent series expansion*

$$\frac{1}{s-1}+\sum_{n=0}^{\infty}a_n(s-1)^n.$$

Prove that $a_0=\gamma$*, where* γ *is the Euler-Mascheroni constant,*

$$\gamma=\lim_{N\to\infty}\Big(\sum_{n=1}^{N}\frac{1}{n}-\log N\Big).$$

Hint. *Explain that* $a_0=1-\int_1^{\infty}(x-[x])dx/x^2$ *and compute the integral.*

Exercise 4.4. *For* $\alpha\in\mathbb{C}$ *with* $\operatorname{Re}\alpha>0$ *one defines the* Hurwitz zeta function

$$\zeta(s;\alpha):=\sum_{n=0}^{\infty}(n+\alpha)^{-s}.$$

So $\zeta(s;1)=\zeta(s)$*.*

a) Prove that $\zeta(s;\alpha)$ *converges for all* $s\in\mathbb{C}$ *with* $\operatorname{Re}s>1$*.*

b) Prove that $\zeta(s;\alpha)$ *has an analytic continuation to* $\{s\in\mathbb{C}:\ \operatorname{Re}s>0,\ s\neq1\}$*, with a simple pole with residue* 1 *at* $s=1$*.*

c) Let q be an integer with $q \geqslant 2$. Prove that for $a \in \mathbb{Z}$ with $1 \leqslant a < q$ and $\gcd(a,q) = 1$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$, $s \neq 1$ we have

$$\zeta(s; a/q) = \frac{q^s}{\varphi(q)} \sum_{\chi \in G(q)} \overline{\chi(a)} L(s, \chi).$$

Exercise 4.5. *In this exercise you are asked to work out an alternative proof of the following theorem: $\zeta(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$, with a pole with residue 1 at $s = 1$. Equivalently, if $G(s) := (s-1)(\zeta(s) - 1)$, then $G(s)$ has an analytic continuation to $\mathbb{C}$ and $G(1) = 1$.*

a) For $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ and $k \in \mathbb{Z}_{\geqslant 0}$, define $F_k(s) := \int_1^\infty (x - [x])^k x^{-s-k} dx$. Prove that $F_0(s) = \frac{1}{s-1}$ and

$$F_k(s) = \frac{1}{k+1}\Big(\zeta(s+k) - 1 + (s+k)F_{k+1}(s)\Big).$$

Hint. *First express $F_{n,k}(s) := \int_n^{n+1} (x-n)^k x^{-s-k} dx$ in terms of $F_{n,k+1}(s)$ and then sum over n.*

b) Let $r \in \mathbb{Z}_{>0}$. Prove that for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ we have

$$G(s) = 1 - \sum_{k=1}^{r} \frac{(s-1)s(s+1)\cdots(s+k-2)G(s+k)}{(k+1)!} - \frac{(s-1)s(s+1)\cdots(s+r)}{(r+1)!} F_{r+1}(s).$$

c) Prove by induction on r that for every $r \in \mathbb{Z}_{\geqslant 0}$, the function $G(s)$ has an analytic continuation to $U_r := \{s \in \mathbb{C} : \operatorname{Re} s > -r\}$, and then deduce that $G(s)$ has an analytic continuation to $\mathbb{C}$. Show also that $G(1) = 1$ and $G(0) = \frac{3}{2}$, hence $\zeta(0) = -\frac{1}{2}$.

Exercise 4.6. *a) Using Theorem 4.3.2 and Lemma 4.3.1, prove that*

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos(\tfrac{1}{2}\pi s)\Gamma(s) \cdot \zeta(s) \quad \text{for } s \in \mathbb{C} \setminus \{0,1\}.$$

b) Prove that $\zeta(1-2k) = -\frac{B_{2k}}{2k}$ for $k = 1, 2, \ldots$
(in particular, '$1 + 2 + 3 + \cdots = -\frac{1}{12}$').

c) Determine the sign of $\zeta(s)$ *for* $s \in \mathbb{R}$.

d) Compute $\zeta'(0)$. *Use the above functional equation in combination with Exercise 4.3 b) and Corollary 4.4.9.*

Exercise 4.7. *Let* q *be an integer* $\geqslant 2$ *and* χ *a primitive character mod* q. *Using Theorem 4.3.6 and Lemma 4.3.1, prove the following:*

a) $L(s, \chi)$ *has an analytic continuation to* $\mathbb{C}$.

b) if χ *is even, then* $L(s, \chi)$ *has simple zeros at* $s = 0, -2, -4, \ldots$ *and* $L(s, \chi) \neq 0$ *if* $\operatorname{Re} s < 0$, $s \notin \{-2, -4, \ldots\}$;
if χ *is odd, then* $L(s, \chi)$ *has simple zeros at* $s = -1, -3, -5, \ldots$ *and* $L(s, \chi) \neq 0$ *if* $\operatorname{Re} s < 0$, $s \notin \{-1, -3, -5, \ldots\}$.

c) Prove a) and b) in the case that χ *is non-principal, but not necessarily primitive.*