# EXAM ANALYTIC NUMBER THEORY

**Thursday January 21, 2021, 14:00-17:00**

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- You are allowed to use the results from the lecture notes and the results from the exercises, unless otherwise stated. But you have to formulate the results you are using.
- To facilitate the grading, please give your answers in English.
- The maximal number of points for each part of an exercise is indicated in the left margin. Grade is (number of points)/10.

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**1.** Define the arithmetic function $f$ by $f(n) = \sum_{d^3|n} \mu(d)d$, where the sum is taken over all positive divisors $d$ of $n$ with $d^3$ dividing $n$.

5 *a)* Prove that $f$ is multiplicative.

5 *b)* Prove that $\sum_{n\leq x} f(n) = \sum_{d\leq x^{1/3}} \mu(d)d[x/d^3]$.

10 *c)* Prove that $\sum_{n\leq x} f(n) = 6x/\pi^2 + O(x^{2/3})$ as $x \to \infty$.

**2.** Let $q = p_1^{k_1} \cdots p_t^{k_t}$ where $p_1, \ldots, p_t$ are distinct primes $> 2$ and $k_1, \ldots, k_t$ positive integers. Denote by $R(q)$ the group of real characters modulo $q$. Recall that

$$(\mathbb{Z}/q\mathbb{Z})^* \cong \langle g_1 \rangle \times \cdots \times \langle g_t \rangle,$$

where $g_i \mod p_i^{k_i}$ is the generator of the cyclic group $(\mathbb{Z}/p_i^{k_i}\mathbb{Z})^*$, and

$$R(q) = \{\chi_1^{u_1} \cdots \chi_t^{u_t} : u_1, \ldots, u_t \in \{0, 1\}\}$$

where $\chi_i$ is the character modulo $p_i^{k_i}$ with $\chi_i(g_i) = -1$.
We call $n \in \mathbb{Z}$ a quadratic residue modulo $q$ if $\gcd(n, q) = 1$ and $x^2 \equiv n (\mathrm{mod}\, q)$ is solvable.

7 *a)* Let $n \in \mathbb{Z}$ be a residue class modulo $q$ such that $\gcd(n, q) = 1$ and $n$ is not a quadratic residue modulo $q$. Prove that there is $\chi \in R(q)$ with $\chi(n) = -1$.

8 *b)* Prove that

$$\sum_{\chi \in R(q)} \chi(n) = \begin{cases} 2^t \text{ if } n \text{ is a quadratic residue modulo } q, \\ 0 \text{ otherwise.} \end{cases}$$

10 *c)* Let $M, N$ be integers with $1 \leq M+1 < M+N < q$, denote by $N'$ the number of integers in $\{M+1, \ldots, M+N\}$ that are coprime with $q$, and denote by $Q$ the number of quadratic residues in $\{M+1, \ldots, M+N\}$. Prove that

$$\left| Q - \frac{N'}{2^t} \right| \leq 3\sqrt{q} \log q.$$

Use the Polyá-Vinogradov inequality for character sums:
$|\sum_{n=M+1}^{N} \chi(n)| \leq 3\sqrt{q} \log q$ for $\chi \in G(q)$, $\chi \neq \chi^{(0)}(q)$.

**3.** Define $\Omega(1) = 0$, and for $n = p_1^{k_1} \cdots p_t^{k_t}$, where $p_1, \dots, p_t$ are distinct primes and $k_1, \dots, k_t$ positive integers, put $\Omega(n) = k_1 + \cdots + k_t$.

10 *a)* Let $q$ be an integer $\geq 2$ and $\chi$ a character modulo $q$. Prove that

$$\sum_{n=1}^{\infty} (-1)^{\Omega(n)} \chi(n) n^{-s} = \frac{L(2s, \chi^2)}{L(s, \chi)} \quad \text{for } s \in \mathbb{C} \text{ with } \operatorname{Re} s > 1.$$

3 *b)* Let $a$ be an integer with $\gcd(a, q) = 1$. Prove that

$$\sum_{n=1,\, n \equiv a (\operatorname{mod} q)}^{\infty} (-1)^{\Omega(n)} n^{-s} = \varphi(q)^{-1} \sum_{\chi \in G(q)} \overline{\chi(a)} \frac{L(2s, \chi^2)}{L(s, \chi)} \quad \text{for } s \in \mathbb{C} \text{ with } \operatorname{Re} s > 1.$$

12 *c)* Let $a, q$ be positive integers with $q \geq 2$, $\gcd(a, q) = 1$. Prove that

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x,\, n \equiv a (\operatorname{mod} q)} (-1)^{\Omega(n)} = 0.$$

5 *d)* Prove that the limit in c) is 0 also if $a, q$ are positive integers with $\gcd(a, q) > 1$.

**4.** Part a) is independent of the rest of the exercise. Recall that for $z \in \mathbb{C}$ we use the notation $e(z) := e^{2\pi i z}$.

6 *a)* Let $n$ be a positive integer. Let $R(n)$ be the number of representations of $n$ as a sum of two squares and three cubes of positive integers. Define

$$f_k(\alpha) := \sum_{1 \le x \le n^{1/k}} e(\alpha x^k),$$

where $k \in \{2, 3\}$. Show that

$$R(n) = \int_0^1 f_2(\alpha)^2 f_3(\alpha)^3 e(-\alpha n) \mathrm{d}\alpha.$$

10 *b)* Let $\alpha \in \mathbb{R}$. Define

$$f(\alpha) := \sum_{1 \le x, y \le N} e(\alpha(x^3 + 2xy^2)),$$

where here and below, all summations are over integers. Use Cauchy's inequality to show that

$$|f(\alpha)|^2 \le N \sum_{1 \le y_1 \le N} \sum_{1 \le y_2 \le N} \sum_{1 \le x \le N} e(2\alpha x(y_1^2 - y_2^2)).$$

9 *c)* Prove that for every $\epsilon > 0$,

$$|f(\alpha)|^2 \ll_\epsilon N^3 + N^{1+\epsilon} \sum_{1 \le |z| \le 2(N^2-1)} \Big| \sum_{1 \le x \le N} e(\alpha x z) \Big|.$$