EXAM ANALYTIC NUMBER THEORY

Thursday January 26, 2023, 13:45-16:45

- Write down your name, student number and university on each sheet of paper you use. Take care that these are very well readable.
- You are allowed to use the results from the lecture notes and the results from the exercises, unless otherwise stated. But you have to formulate the results you are using.
- To facilitate the grading, please give your answers in English.
- There are four exercises, one per page. On page 5 there are some theorems that you have to use.
- If in a particular part of an exercise you have to use some earlier parts, you are allowed to use these also if you were not able to solve them.
- The maximal number of points for each part of an exercise is indicated in the left margin. Grade is (number of points)/10.

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10 **1.** *a)* Let $C > 0, \sigma > 0$ and let $f : \mathbb{Z}_{>0} \to \mathbb{C}$ be an arithmetic function such that $|\sum_{n\le x} f(n)| \le Cx^\sigma$ for all $x > 0$. Prove that $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ has abscissa of convergence $\le \sigma$.

10 *b)* For $n = p_1^{k_1} \cdots p_t^{k_t}$ (unique prime factorization) define
$\Omega(n) := k_1 + \cdots + k_t$.
Assume that for every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that
$|\sum_{n\le x}(-1)^{\Omega(n)}| \le C_\varepsilon x^{1/2+\varepsilon}$ for all $\varepsilon > 0$. Prove that $\zeta(s) \ne 0$ for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1/2$ (you don't have to prove this, but this is equivalent to the Riemann Hypothesis).
Hint. Express $\sum_{n=1}^{\infty}(-1)^{\Omega(n)}n^{-s}$ *in terms of the zeta-function.*

2. A positive integer n is called squarefull if for every prime p we have $p|n \Rightarrow p^2|n$. For convenience, the integer 1 is also called squarefull.

7 *a)* Prove that every squarefull integer n can be expressed uniquely as $n = n_1^3 n_2^2$, where n_1, n_2 are positive integers with n_1 squarefree.

6 *b)* Let $f(n) = 1$ if n is squarefull and $f(n) = 0$ otherwise. Prove that there is $C > 0$ such that $\sum_{n \le x} f(n) \le C x^{1/2}$ for all $x > 0$. Use a).

7 *c)* Prove that $\sum_{n=1}^{\infty} f(n) n^{-s} = \frac{\zeta(3s)\zeta(2s)}{\zeta(6s)}$ for all $s \in \mathbb{C}$ with $\operatorname{Re} s > 1/2$.

10 *d)* Prove that $\lim_{x \to \infty} x^{-1/2} \sum_{n \le x} f(n)$ exists, and compute this limit.

3. Let a and k be positive coprime integers. Let

$$\mathcal{A} := \{n \leq x \ : \ n \equiv a \pmod{k}\},$$

$$\mathcal{P} := \{p \ : \ (p,k) = 1\},$$

and

$$P := \prod_{\substack{p \in \mathcal{P} \\ p < z}} p.$$

5 *a)* Let d be a squarefree number composed of primes in $\mathcal{P}$. Show:

$$\#\mathcal{A}_d = \frac{x}{k} \cdot \frac{1}{d} + O(1).$$

5 *b)* Show that, for all $0 < z < x$, we have

$$\pi(x;k,a) \leq z + S(\mathcal{A}, \mathcal{P}, z).$$

10 *c)* Use Selberg's sieve (stated on p. 5) to show that, for all $k < z$,

$$S(\mathcal{A}, \mathcal{P}, z) \leq \frac{x}{\varphi(k) \log z} + O(z^2).$$

Hint: Begin by proving that,

$$\frac{\varphi(\overline{P})}{\overline{P}} = \frac{\varphi(k)}{k},$$

where

$$\overline{P}(z) := \prod_{\substack{p \notin \mathcal{P} \\ p < z}} p.$$

In order to bound $V(z)$, use Lemma 2 on p. 5.

5 *d)* Prove the Brun Titchmarsh inequality:

$$\pi(x;k,a) \ll \frac{x}{\varphi(k) \log(x/k)}$$

for all $k < x$. *Hint: Take* $z = \left(\frac{x}{k \log(x/k)}\right)^{\frac{1}{2}}$.

4. Let $\tau(n) = \sum_{d|n} 1$ be the divisor counting function.

8 *a)* Show that $\tau(n) = 2 \sum_{\substack{d|n \\ d \le \sqrt{n}}} 1 - \delta(n)$, where

$$\delta(n) = \begin{cases} 1, & \text{if } n \text{ is a square} \\ 0, & \text{otherwise.} \end{cases}$$

7 *b)* Prove that for any $a \in \mathbb{Z}$, we have

$$\sum_{p \le x} \delta(p+a) \ll_a \sqrt{x}.$$

Deduce from this that

$$\sum_{p \le x} \tau(p+a) = 2 \sum_{d \le \sqrt{x}} \pi(x; d, -a) + O_a(\sqrt{x}).$$

(We replace $d \le \sqrt{x+a}$ by $d \le \sqrt{x}$ with an error term of $O_a(\sqrt{x})$.)

10 *c)* Show that there is some positive constant c such that

$$\sum_{p \le x} \tau(p+a) = cx + O_a\Big(\frac{x \log\log x}{\log x}\Big).$$

Hint: Split the sum in the following way:

$$\sum_{d \le \sqrt{x}} \pi(x; d, -a) = \sum_{d \le \frac{x^{1/2}}{(\log x)^8}} \pi(x; d, -a) + \sum_{\frac{x^{1/2}}{(\log x)^8} \le d \le \sqrt{x}} \pi(x; d, -a).$$

For the first sum, use Theorem 3. For the second sum, use part (d) of Exercise 3. You may also use that there is a constant $c_0 > 0$ with

$$\sum_{n \le x} \frac{1}{\varphi(n)} = c_0 \log x + O(1)$$

for all $x \ge 1$, and that

$$\mathrm{Li}(x) = \frac{x}{\log x} + O\Big(\frac{x}{(\log x)^2}\Big).$$

Useful Theorem Statements

Theorem 1. (Selberg's sieve)**.** *Let $X > 0$ and let f be a multiplicative function satisfying $f(p) > 1$ for any prime $p \in \mathcal{P}$ such that for any squarefree integer d composed of primes in $\mathcal{P}$ we have*

$$\#\mathcal{A}_d = \frac{X}{f(d)} + R_d$$

for some real number R_d. Let f_1 be the function satisfying

$$f(n) = \sum_{d|n} f_1(d)$$

that is uniquely determined by the Möbius inversion formula. Let

$$V(z) := \sum_{\substack{d \le z \\ d|P}} \frac{\mu^2(d)}{f_1(d)}.$$

Then

$$S(\mathcal{A}, \mathcal{P}, z) \le \frac{X}{V(z)} + O\left(\sum_{\substack{d_1, d_2 \le z \\ d_1, d_2 | P}} |R_{[d_1,d_1]}| \right).$$

Lemma 2. *Let $\tilde{f}$ be the completely multiplicative function defined by $\tilde{f}(p) := f(p)$ for all primes p and let $\overline{P}(z) := \prod_{\substack{p \notin \mathcal{P} \\ p < z}} p$. Then:*

$$f(\overline{P}(z))V(z) \ge f_1(\overline{P}(z)) \sum_{e \le z} \frac{1}{\tilde{f}(e)}.$$

Theorem 3. (Corollary to Bombieri-Vinogradov)**.** *We have*

$$\sum_{d \le \frac{x^{1/2}}{(\log x)^8}} \max_{(a,d)=1} \left| \pi(x; d, a) - \frac{\mathrm{Li}(x)}{\varphi(d)} \right| \ll \frac{x}{(\log x)^2}.$$