

HOMEWORK DIOPHANTINE APPROXIMATION

IV

Due June 4

- 10 **18.** Using Bennett's result $|\sqrt[3]{2} - \xi| \geq \frac{1}{4}H(\xi)^{-2.45}$, compute explicit constants A, B such that the following holds:

for any solution $x, y \in \mathbb{Z}$ of $x^3 - 2y^3 = m$ we have $\max(|x|, |y|) \leq A|m|^B$.

- 5 **19.** Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be a square-free binary form of degree $n \geq 3$ and let $G(X, Y) = \sum_{i=0}^a \sum_{j=0}^b g_{ij} X^i Y^j \in \mathbb{Z}[X, Y]$ with $a + b \leq n - 3$. Prove that the Diophantine equation

$$F(x, y) = G(x, y) \quad \text{in } x, y \in \mathbb{Z} \text{ with } F(x, y) \neq 0$$

has only finitely many solutions.

- 20.** The *radical* $\text{rad}(N)$ of a non-zero integer N is the product of the primes dividing N . In the 1980's, Oesterlé and Masser formulated the

abc-conjecture. *For every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that for each triple of positive coprime integers a, b, c with $a + b = c$ we have $c \leq C(\varepsilon)(\text{rad}(abc))^{1+\varepsilon}$.*

Granville and Langevin proved independently that the abc-conjecture is equivalent to the following:

Granville-Langevin conjecture. *Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be a square-free binary form of degree $n \geq 3$. Then for every $\kappa > 2$ there is a constant $C(F, \kappa) > 0$ such that*

$$\text{rad}(F(x, y)) \geq C(F, \kappa) \max(|x|, |y|)^{n-\kappa}$$

for every $x, y \in \mathbb{Z}$ with $\gcd(x, y) = 1$, $F(x, y) \neq 0$.

- 5 a) Prove that the Granville-Langevin conjecture implies the abc-conjecture.

- 5 b) Prove that the Granville-Langevin conjecture implies Roth's Theorem.

Hint. Look at the idea of the proof of Fe'dlman's Theorem, i.e., Theorem 6.5 of the lecture notes.

- 5 c) An integer $n \neq 0$ is called *powerful* if every prime in the prime factorization of n occurs with exponent at least 2. In other words, n is powerful if it can be expressed as a^2b^3 for certain integers a, b with $(|a|, |b|) \neq (1, 1)$.

Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be a square-free binary form of degree at least 5 whose coefficients have gcd 1. Assuming the Granville-Langevin conjecture, prove that there are only finitely many pairs of integers x, y with $\gcd(x, y) = 1$ such that $F(x, y)$ is non-zero and powerful.

- 3 21.a) Let ξ_1, ξ_2 be distinct rational numbers. Prove that

$$|\xi_1 - \xi_2| \geq (H(\xi_1)H(\xi_2))^{-1}.$$

- 5 b) Let $\alpha \in \mathbb{R}$ and $\kappa > 2$, and consider the inequality

$$|\xi - \alpha| \leq H(\xi)^{-\kappa} \text{ in } \xi \in \mathbb{Q} \text{ with } \xi > \alpha. \quad (1)$$

Prove that if ξ_1, ξ_2 are distinct solutions of (1) with $H(\xi_2) \geq H(\xi_1)$, then

$$H(\xi_2) \geq H(\xi_1)^{\kappa-1}.$$

- 3 c) Let $A \geq 2, c > 1$. Prove that the number of solutions ξ of (1) with $A \leq H(\xi) < A^c$ is bounded above by $1 + (\log c / \log(\kappa - 1))$.

- 4 d) Let α be a real algebraic number of degree $d \geq 3$, and $\kappa > \frac{1}{2}d + 1$. By a sharpening of Thue's Theorem, there are effectively computable numbers $C > 0, \lambda_0 > 0$, such that if

$$|\xi - \alpha| \leq H(\xi)^{-\kappa} \text{ in } \xi \in \mathbb{Q} \quad (2)$$

has a solution ξ_1 with $H(\xi_1) \geq C$, then for every other solution ξ of (2) we have $H(\xi) \leq H(\xi_1)^{\lambda_0}$. Use this to compute an upper bound for the number of solutions of (2).

22. Let $0 < \delta < 1$ and

$$\begin{aligned} L_1 &= X_1 + X_2\sqrt{2} + X_3\sqrt{3}, \quad L_2 = X_1 - X_2\sqrt{2} + X_3\sqrt{3}, \\ L_3 &= X_1 - X_2\sqrt{2} - X_3\sqrt{3}. \end{aligned}$$

Consider the inequality

$$|L_1(\mathbf{x})L_2(\mathbf{x})L_3(\mathbf{x})| \leq C\|\mathbf{x}\|^{-\delta} \text{ in } \mathbf{x} \in \mathbb{Z}^3. \quad (3)$$

5 a) In our course we proved that (3) has infinitely many solutions in the subspace given by $x_3 = 0$. Prove that (3) has infinitely many solutions in each of the subspaces $x_1 = 0, x_2 = 0$.

5 b) Prove that (3) has only finitely many solutions in $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$ with $x_1 x_2 x_3 \neq 0$.

Hint. You have to prove that each two-dimensional subspace of $\mathbb{Q}^3$ contains only finitely many solutions. Restricting to one such a subspace, you may eliminate one of the variables to get an inequality with three linear forms in two variables. Apply the “general position version” of the Subspace Theorem.

23. Let $K = \mathbb{Q}(\sqrt[5]{2})$. Then K has five embeddings in $\mathbb{C}$, given by $\sigma_i(\sqrt[5]{2}) = \rho^i \sqrt[5]{2}$ for $i = 0, \dots, 4$, where $\rho = e^{2\pi\sqrt{-1}/5}$. Consider the Diophantine equation

$$N_{K/\mathbb{Q}}(x_1 + x_2 \sqrt[5]{2} + x_3 (\sqrt[5]{2})^2) = 1 \text{ in } \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3. \quad (4)$$

4 a) Prove that the left-hand side of (4) is a product of linear forms in general position.

4 b) Let $\alpha, \beta \in K$ with $\beta \neq 0$ and $\alpha/\beta \notin \mathbb{Q}$. Prove that the linear forms $\sigma_i(\alpha)X + \sigma_i(\beta)Y$ ($i = 0, \dots, 4$) are in general position.

2 c) Prove that (4) has only finitely many solutions (you are allowed to use Theorem 7.4 but not Theorem 7.13).

24. Let $\alpha_1, \dots, \alpha_n$ be real algebraic numbers such that $1, \alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$ and let $C > 0, \delta > 0$. Consider the system of inequalities

$$|x_1 - \alpha_1 x_{n+1}| \leq C \|\mathbf{x}\|^{-(1/n)-\delta}, \dots, |x_n - \alpha_n x_{n+1}| \leq C \|\mathbf{x}\|^{-(1/n)-\delta} \quad (5)$$

in $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}) \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$.

3 a) Prove that the solutions of (5) lie in a union of finitely many proper linear subspaces of $\mathbb{Q}^{n+1}$.

7 b) Let T be an n -dimensional linear subspace of $\mathbb{Q}^{n+1}$. Prove that (5) has only finitely many solutions in $\mathbb{Z}^{n+1} \cap T$. Then prove that (5) has only finitely many solutions.

Hint. T is given by an equation $a_1 x_1 + \dots + a_{n+1} x_{n+1} = 0$ with $(a_1, \dots, a_{n+1}) \in$

$\mathbb{Q}^{n+1} \setminus \{\mathbf{0}\}$. Prove by an elementary argument, i.e., without using the Subspace Theorem, that the solutions $\mathbf{x} \in \mathbb{Z}^{n+1} \cap T$ of (5) are bounded. It is essential that $1, \alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$.

Total number of points: 75.

Grade: (Number of points)/7.5