DIOPHANTINE APPROXIMATION

Tuesday January 23, 14:00-17:00

• Write down your name and student number on each sheet. Take care that these are **VERY WELL READABLE**.
• Indicate whether you are doing Bachelor wiskunde, Master mathematics, Algant, or any other program and if not from Leiden, from which other university you are coming.
• There are four exercises on three pages. In each exercise you are allowed to use all theorems from the lecture notes, unless otherwise stated. Formulate the theorems you are using. You are not allowed to use without proof the results from the homework exercises, so if you need one you have to prove it.
• The maximal number of points for each part of an exercise is indicated in the left margin. Grade is (number of points)/4.

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1. Schanuel's conjecture asserts that if $x_1, \ldots, x_n$ are complex numbers that are linearly independent over $\mathbb{Q}$, then $\mathrm{trdeg}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n$. You are asked to prove the assertions in the exercises below under the assumption of Schanuel's conjecture.

6 *a)* Let α, β be complex algebraic numbers such that $\alpha \neq 0, 1$ and $\deg \beta = d \geq 2$. Prove that $\alpha^{\beta}, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}$ are algebraically independent. Here we fix a solution $\log \alpha$ of $e^z = \alpha$ and then put $\alpha^w := e^{w \log \alpha}$ for $w \in \mathbb{C}$.
Are $\alpha^{\beta}, \alpha^{\beta^2}, \ldots, \alpha^{\beta^d}$ algebraically independent?

4 *b)* Let α be algebraic with $\alpha \notin \mathbb{Q}(i)$. Prove that e, π, e^{π} and $e^{\pi\alpha}$ are algebraically independent.

4 **2.** *a)* Let C be a central symmetric convex body in $\mathbb{R}^n$ and $\mathbf{a}_1, \ldots, \mathbf{a}_r \in C$. Prove that

$$D := \left\{ \sum_{i=1}^{r} x_i \mathbf{a}_i \, : \, x_1, \ldots, x_r \in \mathbb{R}, \, \sum_{i=1}^{r} |x_i| \leq 1 \right\} \subseteq C.$$

2 *b)* Define the successive minima $\lambda_1, \ldots, \lambda_n$ of a central symmetric convex body C in $\mathbb{R}^n$ with respect to a lattice L in $\mathbb{R}^n$ and formulate Minkowski's second convex body theorem.

4 *c)* Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be linearly independent points in L such that $\mathbf{v}_i \in \lambda_i C$ for $i = 1, \ldots, n$ and let M be the lattice with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Prove that $(L : M) \leq n!$.
You may use that $O_n := \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |x_i| \leq 1\}$ has volume $2^n/n!$.

3. You have to use Baker's theorem, which asserts that if $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers in $\mathbb{C}$ and $b_1, \ldots, b_n$ are integers such that $\alpha_1^{b_1} \cdots \alpha_n^{b_n} \neq 1$, then

$$|\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1| \geq (eB)^{-C},$$

where $B = \max(|b_1|, \ldots, |b_n|)$ and C is effectively computable in terms of $\alpha_1, \ldots, \alpha_n$.

Let A, B be integers with $A^2+4B < 0$ and let $\{u_n\}_{n=0}^{\infty}$ be a sequence of integers such that u_0, u_1 are not both 0 and $u_n = Au_{n-1} + Bu_{n-2}$ for all $n \geq 2$.

5 *a)* Let $\alpha, \overline{\alpha}$ (where $\overline{z}$ denotes the complex conjugate of a complex number z) be the complex zeros of $X^2 - AX - B$. Prove that there is a non-zero algebraic number $\gamma \in \mathbb{C}$ such that $u_n = \gamma\alpha^n + \overline{\gamma} \cdot \overline{\alpha}^n$ for $n \geq 0$.

5 *b)* Assume that $\alpha/\overline{\alpha}$ is not a root of unity. Prove that there are effectively computable positive numbers n_0, c_1, c_2, depending on A, B, u_0 and u_1, such that $|u_n| \geq |\alpha|^n / c_1 n^{c_2}$ for all $n \geq n_0$.

2 **4.** *a)* Formulate the Subspace Theorem.

3 *b)* Let $\alpha_1 X + \alpha_2 Y$, $\beta_1 X + \beta_2 Y$ be two linearly independent linear forms with algebraic coefficients in $\mathbb{C}$. Prove that for every $C > 0$, $\delta > 0$, the inequality

$$0 < |(\alpha_1 x_1 + \alpha_2 x_2)(\beta_1 x_1 + \beta_2 x_2)| \leq C \|\mathbf{x}\|^{-\delta}$$

has only finitely many solutions. What if we allow the left-hand side to be 0?

5 *c)* Let $n \geq 2$, and let $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ be algebraic numbers in $\mathbb{C}$ such that the determinants $\Delta_{ij} := \alpha_i \beta_j - \alpha_j \beta_i$ $(1 \leq i < j \leq n)$ are linearly independent over $\mathbb{Q}$. Prove that for every $C > 0$, $\delta > 0$ there are only finitely many points $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ with

$$0 < |(\alpha_1 x_1 + \cdots + \alpha_n x_n)(\beta_1 x_1 + \cdots + \beta_n x_n)| \leq C \cdot \|\mathbf{x}\|^{2-n-\delta}.$$