ANSWERS DIOPHANTINE APPROXIMATION 9-1-2020.

1 a) Let C be a central symmetric convex body in $\mathbb{R}^n$ and L a lattice in $\mathbb{R}^n$ such that $\mathrm{vol}(C) \geq 2^n d(L)$, where $\mathrm{vol}(C)$ is the volume of C and $d(L)$ the determinant of L. Then C contains a nonzero point of L

b) Consider the lattice
$L = \{(x_1, \ldots, x_d, y + \alpha_1 x_1 + \ldots + \alpha_d x_d) : y, x_1, \ldots, x_d \in \mathbb{Z}\}$
and the central symmetric convex body
$C = \{(u, v_1, \ldots, v_d) \in \mathbb{R}^{d+1} : |v_1| \leq H, \ldots, |v_d| \leq H, |u| \leq H^{-d}\}$

The lattice L has determinant $\left\| \begin{matrix} 1 & \alpha_1 & \ldots & \alpha_d \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{matrix} \right\| = 1$

and C has volume $2^{d+1} \cdot H^d \cdot H^{-d} = 2^{d+1}$. By a), C contains a non-zero point of L. This means precisely that there is $\underline{x} = (y, x_1, \ldots, x_d) \in \mathbb{Z}^{d+1}$ with $|x_1| \leq H, \ldots, |x_d| \leq H$, $|y + \alpha_1 x_1 + \ldots + \alpha_d x_d| \leq H^{-d}$.

c) Write $P = y + x_1 X + x_2 X^2 + \ldots + x_d X^d$. So $P(\theta) = y + x_1\theta + \ldots + x_d \theta^d$.
b) implies that there is nonzero $P \in \mathbb{Z}[X]$, with $|x_1| \leq H, \ldots, |x_d| \leq H$ and $|P(\theta)| \leq H^{-d}$. We still have to verify that also $|y| \leq H$. We have

$$|y| \leq |x_1\theta + \ldots + x_d\theta^d| + |P(\theta)| \underset{|\theta| \leq \frac{1}{2}}{\leq} H\left(\tfrac{1}{2} + (\tfrac{1}{2})^2 + \ldots + (\tfrac{1}{2})^d\right) + H^{-d}$$
$$= H(1 - 2^{-d}) + H^{-d} \underset{H \geq 2}{\leq} H - H^{1-d} + H^{-d} \leq H.$$

2 a) We first apply the Lindemann-Weierstrass theorem: let $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ be distinct and $\beta_1, \dots, \beta_n \in \overline{\mathbb{Q}}$ nonzero. Then $\beta_1 e^{\alpha_1} + \dots + \beta_n e^{\alpha_n} \neq 0$.
Suppose that $\log 2 =: \alpha$ is algebraic. Then $e^{\alpha} - 2e^{0} = 0$, contradicting the Lindemann-Weierstrass Theorem.
Suppose that $\sin 2 = \frac{e^{2i} - e^{-2i}}{2i} = \beta$ is algebraic. Then $e^{2i} - e^{-2i} - \beta \cdot 2i \cdot e^{0} = 0$, $\beta \cdot 2i \in \overline{\mathbb{Q}}$, and this again contradicts the Lindemann-Weierstrass Theorem.

Next we apply Baker's Theorem: let $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}^*$, for $i \leq n$ let $\log \alpha_i$ be any solution to $e^{z} = \alpha_i$ and suppose that $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$. Further, let $\beta_1, \dots, \beta_n \in \overline{\mathbb{Q}}^*$. Then $\beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n$ is transcendental.
Recall that $e^{\pi i} = -1$, so $\pi i = \log(-1)$. Clearly, $\log 2, \log(-1)$ are linearly independent over $\mathbb{Q}$, since $\log(-1)/\log 2 \notin \mathbb{R}$. So $\log 2 + \pi = \log 2 - i \log(-1)$ is transcendental.

b) Let $x_1, \dots, x_n \in \mathbb{C}$, linearly independent over $\mathbb{Q}$. Then $\operatorname{trdeg}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}) \geq n$.

c) We first solve the hint. Suppose $\log 2, \log 3, \alpha \log 2, \alpha \log 3$ are linearly independent over $\mathbb{Q}$. Then there are $a, b, c, d \in \mathbb{Q}$, not all 0, such that $a \log 2 + b \log 3 + (c \alpha \log 2 + d \alpha \log 3) = 0$ or $(a + b\alpha) \log 2 + (c + d\alpha) \log 3 = 0$. Schanuel's conjecture implies $\operatorname{trdeg}(\log 2, \log 3) = \operatorname{trdeg}(\log 2, \log 3, 2, 3) \geq 2$ provided $\log 2, \log 3$ are linearly independent over $\mathbb{Q}$. Suppose $\log 2, \log 3$ are linearly dependent over $\mathbb{Q}$. Then $a \log 2 + b \log 3 = 0$ for some $a, b \in \mathbb{Z}$, not both 0, hence $2^{a} = 3^{-b}$, which is impossible.
Now another application of Schanuel's conjecture gives $\operatorname{trdeg}(\log 2, \log 3, \alpha \log 2, \alpha \log 3, 2, 3, 2^{\alpha}, 3^{\alpha}) \geq 4$, so $\operatorname{trdeg}(\log 2, \log 3, 2^{\alpha}, 3^{\alpha}) \geq 4$. In particular, $2^{\alpha}, 3^{\alpha}$ are algebraically independent, so $3^{\alpha} - 2^{\alpha} \notin \overline{\mathbb{Q}}$.

3) a) Write $x = p_1^{k_1}\cdots p_s^{k_s}$, $y = p_1^{l_1}\cdots p_s^{l_s}$. By Baker's Theorem, we have

$$\left|1-\frac{y}{x}\right| = \left|1 - p_1^{l_1-k_1}\cdots p_s^{l_s-k_s}\right| \ge \exp\left(-C(S)\cdot\left(\prod_{i=1}^{s} \log(ep_i)\right)\log eB\right),$$

where $B=\max_{1\le i\le s}|k_i-l_i|$. $\quad = (eB)^{-C(S)\cdot\prod_{i=1}^{s} \log(ep_i)}$

Notice that $p_i^{k_i}\le x$, so $k_i \le \frac{\log x}{\log p_i}\le \frac{\log x}{\log 2}$, and $l_i \le \frac{\log y}{\log 2}\le\frac{\log x}{\log 2}$

So $B\le \frac{\log x}{\log 2}$. It follows that

$$x-y = x\cdot\left|1-\frac yx\right| \ge x\cdot\left(e\frac{\log x}{\log 2}\right)^{-C(S)\cdot\prod_{i=1}^{s}\log(ep_i)} \ge \frac{x}{c_1(\log x)^{c_2}},$$

with c_1,c_2 effectively computable in terms of $p_1,\dots,p_s$

b) Suppose there are integers $x,y\ge 2$ with $0<|ax^z - by^z|\le \max(ax^z,by^z)^{1/2}$? Assume without loss of generality that $x\ge y$. By Baker's Theorem,

$$\left|1-\frac{by^z}{ax^z}\right| = \left|1-\frac ba\cdot\left(\frac yx\right)^z\right| \ge \exp\left(-C\cdot(\log eH)(\log ex)\cdot\log(ez)\right),$$

where C is effectively computable, and $H=H\left(\frac ba\right)$. On the other hand,

$$\left|1-\frac{by^z}{ax^z}\right|\le\frac{\max(ax^z,by^z)^{1/2}}{ax^z}\le \max(a,1)\,x^{-z/2} = \exp\left(\log\max(a,1)-\frac z2\log x\right)$$

Comparing the upper and lower bound, we get

$$-C'(\log x+1)(\log z+1)\le C''-\frac z2\log x,$$ with C', C'' effectively computable in terms of a, b

Use that $\log x\ge\log 2$, and assume without loss of generality that $\frac z3\log 2\ge C''$. Then we get

$-C'''(\log x)\log z \le -\frac z3\log x$, or $z\le 3C'''\log z$. (C''' eff. computable)

Hence $z\le C''''$ with C'''' effectively computable.

4 a) It suffices to prove that for $i=1,\dots,n$, the solutions of $(*)$ with $\|\underline{x}\|=|x_i|$ lie in only finitely many proper linear subspaces of $\mathbb{Q}^n$. We explain the argument for $i=n$. Multiply $(*)$ with $|x_1\cdots x_{n-1}|\cdot|x_1\cdots x_n|_{p_1}\cdots|x_1\cdots x_n|_{p_s}$. Then if $\underline{x}$ satisfies $(*)$, then

$$|(\alpha_1 x_1+\dots+\alpha_n x_n)x_1\cdots x_{n-1}|\cdot|x_1\cdots x_n|_{p_1}\cdots|x_1\cdots x_n|_{p_s}$$
$$\le C\|\underline{x}\|^{1-\delta}\cdot\overbrace{|x_1\cdots x_{n-1}|\cdot|x_1\cdots x_{n-1}|_{p_1}\cdots|x_1\cdots x_{n-1}|_{p_s}}^{=1}\ \overbrace{|x_n|_{p_1}\cdots|x_n|_{p_s}}^{\le|x_n|^{-1}}$$
$$\le C\|\underline{x}\|^{-\delta}$$

By the p-adic Subspace Theorem, the solutions of the latter inequality, hence those of $(*)$ with $\|\underline{x}\|=|x_n|$, lie in finitely many proper linear subspaces of $\mathbb{Q}^n$.

b) We proceed by induction on n. First let $n=1$. Then $(*)$ becomes
$$|\alpha x_1|\le C\cdot|x_1|^{1-\delta}\qquad\text{or}\qquad|x_1|^{\delta}\le C/|\alpha|,$$

implying that there are only finitely many possibilities for x_1.

Now let $n\geq 2$ and suppose inequalities of type $(*)$ with fewer than n unknowns have only finitely many solutions. By a) the solutions $\underline{x}=(x_1,\dots,x_n)$ of $(*)$ lie in finitely many proper linear subspaces of $\mathbb{Q}^n$ which we may assume to be of dimension $n-1$. So we have to assume that each of these subspaces contains only finitely many solutions. Let T be one of these subspaces, say given by an equation $a_1x_1+\dots+a_nx_n=0$ with $a_1,\dots,a_n\in\mathbb{Q}$, not all 0. Assume for instance that $a_n\neq 0$. Then for $\underline{x}\in T$ we have $x_n=b_1x_1+\dots+b_{n-1}x_{n-1}$, with $b_1,\dots,b_{n-1}\in\mathbb{Q}$. Substituting this into $(*)$ we get

$$|(\alpha_1 + b_1\alpha_n)x_1 + \dots + (\alpha_{n-1} + b_{n-1}\alpha_n)x_{n-1}| \le C\|\underline{x}\|^{-n-\delta}$$

$$\le C'\|\underline{x}'\|^{-n-\delta} \quad \text{with } \underline{x}' = (x_1, \dots, x_{n-1}) \qquad (**)$$

since $|x_n| \le |b_1||x_1| + \dots + |b_{n-1}||x_{n-1}| \le (|b_1| + \dots + |b_{n-1}|)\|\underline{x}'\|$.

~~By the~~ Notice that $\alpha_1 + b_1\alpha_n, \dots, \alpha_{n-1} + b_{n-1}\alpha_n$ are linearly independent over $\mathbb{Q}$. For let $c_1, \dots, c_{n-1} \in \mathbb{Q}$ with $c_1(\alpha_1 + b_1\alpha_n) + \dots + c_{n-1}(\alpha_{n-1} + b_{n-1}\alpha_n) = 0$. Then $c_1\alpha_1 + \dots + c_{n-1}\alpha_{n-1} + (c_1b_1 + \dots + c_{n-1}b_{n-1})\alpha_n = 0$, implying $c_1 = \dots = c_{n-1} = 0$.

So by the induction hypothesis, (**) has only finitely many solutions $\underline{x}' = (x_1, \dots, x_{n-1})$ with $x_1, \dots, x_{n-1} \in \mathcal{U}$. Since for $\underline{x} = (x_1, \dots, x_n) \in T$, $x_n = b_1x_1 + \dots + b_{n-1}x_{n-1}$, we have that x_n is determined by $(x_1, \dots, x_{n-1})$. Hence (*) has only finitely many solutions $\underline{x} = (x_1, \dots, x_n)$ with $x_1, \dots, x_n \in \mathcal{U}$.