DIOPHANTINE APPROXIMATION

Thursday January 9, 2020, 14:00-17:00

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- Write down your name (in CAPITALS), university and student number on each paper sheet. Take care that these are **VERY WELL READABLE**.
- To facilitate the grading, please give your answers in English.
- This exam consists of four pages, each with one exercise.
- The maximal number of points for each part of an exercise is indicated in the left margin. Grade = (total number of points)/10.
- Unless otherwise stated, you may use the theorems from the lecture notes without proof but you have to formulate the theorems you are using. You are not allowed to use the results from the exercises without proof.

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5 **1.** *a)* State Minkowski's first convex body theorem.

7.5 *b)* Let d, H be positive integers with $H \geq 2$ and $\alpha_1, \dots, \alpha_d$ real numbers. Using a), prove that there is a non-zero vector $\mathbf{x} = (y, x_1, \dots, x_d) \in \mathbb{Z}^{d+1}$ such that

$$|x_1| \leq H, \dots, |x_d| \leq H, \quad |y + \alpha_1 x_1 + \cdots + \alpha_d x_d| \leq H^{-d}.$$

7.5 *c)* Let d, H be as in b) and let θ be a real number with $|\theta| \leq 1/2$. Prove that there exists a non-zero polynomial $P \in \mathbb{Z}[x]$ of degree at most d and coefficients in the interval $[-H, H]$ such that

$$|P(\theta)| \leq H^{-d}$$

(the problem is to estimate the constant term $P(0)$).

2. In this exercise and below, the following notation is used:
$\overline{\mathbb{Q}} := \{\alpha \in \mathbb{C},\ \alpha \text{ algebraic}\}$,
$a^\alpha := e^{\alpha \log a}$ for any $\alpha \in \mathbb{C}$ and any integer $a \geq 2$, where $\log a$ is the ordinary natural logarithm of a.

10 *a)* Show that the following numbers are transcendental: $\log 2, \sin 2, \log 2 + \pi$. You may use all theorems that we discussed in the lecture.

5 *b)* State Schanuel's conjecture.

15 *c)* Prove that Schanuel's conjecture implies the following: let $\alpha \in \mathbb{C}$ be such that $3^\alpha - 2^\alpha \in \overline{\mathbb{Q}}$; then α is either rational or transcendental.
Hint: Assume that α is an irrational algebraic number and show that the numbers $\log 2, \log 3, \alpha \log 2, \alpha \log 3$ are linearly independent over $\mathbb{Q}$. Here you may use Schanuel's conjecture. You get 5 points for solving the hint.

3. Recall that the height of a rational number $a = x/y$, where x, y are integers with $\gcd(x, y) = 1$ is given by $H(a) := \max(|x|, |y|)$. In the exercises below you have to use the following variation on a result of Baker. Let $a_1, \dots, a_n$ be non-zero rational numbers and $b_1, \dots, b_n$ integers such that $a_1^{b_1} \cdots a_n^{b_n} \neq 1$. Put $H_i := H(a_i)$ for $i = 1, \dots, n$ and $B := \max(|b_1|, \dots, |b_n|)$. Then

$$|a_1^{b_1} \cdots a_n^{b_n} - 1| \geq \exp\Big(-C(n)\big(\prod_{i=1}^{n} \log(eH_i)\big) \log(eB)\Big),$$

where $C(n)$ is an effectively computable positive number, depending on n only.

12.5 *a)* Let $S = \{p_1, \dots, p_s\}$ be a set of primes and let A denote the set of integers of the shape $p_1^{k_1} \cdots p_s^{k_s}$ where $k_1, \dots, k_s$ are non-negative integers. Prove that there are effectively computable numbers $c_1, c_2 > 0$ depending on $p_1, \dots, p_s$ such that if x, y are any two elements of A with $x > y$ then

$$x - y \geq \frac{x}{c_1(\log x)^{c_2}}.$$

12.5 *b)* Let a, b be positive integers. Prove that there is an effectively computable positive number C depending on a, b with the following property: if z is an integer with $z > C$ then there are no integers $x, y \geq 2$ such that $0 < |ax^z - by^z| \leq \max(ax^z, by^z)^{1/2}$.

4. In the exercise below you have to use the following version of the p-adic Subspace Theorem. Let n be an integer ≥ 2, $C > 0$, $\delta > 0$, and let $p_1, \ldots, p_s$ be distinct prime numbers. Further, let

$$L_{i,\infty} = \alpha_{i,1}X_1 + \cdots + \alpha_{i,n}X_n \quad (i = 1, \ldots, n)$$

be n linearly independent linear forms with coefficients from $\overline{\mathbb{Q}}$ and for $j = 1, \ldots, s$, let

$$L_{i,p_j} = \alpha_{i,1}^{(p_j)}X_1 + \cdots + \alpha_{i,n}^{(p_j)}X_n \quad (i = 1, \ldots, n)$$

be n linearly independent linear forms with coefficients from $\mathbb{Q}$. Then the set of solutions $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ of

$$|L_{1,\infty}(\mathbf{x}) \cdots L_{n,\infty}(\mathbf{x})| \cdot \prod_{j=1}^{s} |L_{1,p_j}(\mathbf{x}) \cdots L_{n,p_j}(\mathbf{x})|_{p_j} \leq C\|\mathbf{x}\|^{-\delta}$$

is contained in a union $T_1 \cup \cdots \cup T_t$ of finitely many proper linear subspaces of $\mathbb{Q}^n$.

12.5 *a)* Let $U = \{\pm p_1^{k_1} \cdots p_s^{k_s} : k_1, \ldots, k_s \in \mathbb{Z}_{\geq 0}\}$, let $n \geq 2$, $C > 0$, $\delta > 0$ and let $\alpha_1, \ldots, \alpha_n$ be non-zero elements of $\overline{\mathbb{Q}}$. Prove that the set of solutions $\mathbf{x} = (x_1, \ldots, x_n)$ of

(*) $$|\alpha_1 x_1 + \cdots + \alpha_n x_n| \leq C\|\mathbf{x}\|^{1-\delta}, \quad x_1, \ldots, x_n \in U$$

is contained in a union of finitely many proper linear subspaces of $\mathbb{Q}^n$.

12.5 *b)* Let n, C, δ be as above but now assume that $n \geq 1$ and that $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$. Prove that (*) has only finitely many solutions.