3.6 Exercises

Exercise 3.6.1 *Let α be an algebraic number of degree $n \geq 2$. Prove that $\mu(\alpha) \leq n$. Hint: let f be the minimal polynomial of α and note that $|f(\alpha) - f(p/q)| \geq 1/q^n$ for any integers p, q with $q > 0$.*

Exercise 3.6.2 *Let*

$$\alpha = \sum_{n \geq 0} \frac{1}{2^{n!}}.$$

Prove that $\mu(\alpha) = \infty$.

Exercise 3.6.3 *Solve the equation $x^2 + 2 = 3^n$ in $x, n \in \mathbb{N}$. Hint: Use Skolem's method for a suitably chosen prime p and factorization in $\mathbb{Z}[\sqrt{-2}]$. Use the facts that $\mathbb{Z}[\sqrt{-2}]$ is a unique factorization domain and that $3 = (1 - \sqrt{-2})(1 + \sqrt{-2})$.*

Exercise 3.6.4 *Let $a_1, \ldots, a_{10}$ be integers with $0 < a_i < 100$ for all i. Show that there exist two disjoint and non-empty subsets $S, T \subset \{1, \ldots, 10\}$ such that*

$$\sum_{i \in S} a_i = \sum_{i \in T} a_i.$$

Exercise 3.6.5 *Let n be a positive integer and*

$$P_n(x) = x \sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{n-k} x^{2k}$$

$$Q_n(x) = \sum_{k=0}^{n} \binom{2n+1}{2k} 2^{n-k} x^{2k}.$$

Prove that

$$P_n(x) - \sqrt{2} Q_n(x) = (x - \sqrt{2})^{2n+1}.$$

Exercise 3.6.6 *Prove that there are no non-trivial polynomials $P(x), Q(x) \in \mathbb{Z}[x]$ of degree ≤ 2 such that*

$$P(x) - \sqrt[3]{2} Q(x) = O((x - \sqrt[3]{2})^2).$$

Construct non-trivial polynomials P, Q of degree ≤ 3 that satisfy the above condition (hint: find $\alpha \in \mathbb{Q}(\sqrt[3]{2})$ such that $(x - \alpha)(x - \sqrt[3]{2})^2$ has the desired form). Are they unique?
Prove that to any positive integer n there exist non-trivial polynomials $P, Q \in \mathbb{Q}[x]$ of degree $\leq n$ such that $P - \sqrt[3]{2} Q = O((x - \sqrt[3]{2})^N)$ where $N = \lfloor (2n+1)/3 \rfloor$.