1 First Assignment

1. (a) In the projective plane $\mathbb{P}^2$ we are given the points $(1:0:0), (0:1:0), (0:0:1), (1:1:1)$. Determine all projective curves of degree 2 which pass through these points.

   (b) Given any five points in $\mathbb{P}^2$. Show that there exists a curve of degree 2 passing through these points.

   (c) Suppose that among five points in $\mathbb{P}^2$ there are three points on a straight line. Show that any quadratic curve through the five points is a union of to straight lines.

2. In this problem we determine the solutions of

$$(xy + xz + yz)^2 - 4xyz(x + y + z) = 0 \tag{C}$$

   in $x, y, z \in \mathbb{Z}$. Consider the projective curve C given by this equation.

   (a) Show that the points $(1:0:0), (0:1:0), (0:0:1)$ are singular points of C.

   (b) Prove that these are the only singular points.

   (c) Consider the family of quadratic curves Q_t given by the projective equation

$$xy + t(x - y)z = 0$$

   with parameter t. Show that every curve Q_t passes through the singular points of C and determine the points of intersection between C and Q_t for every t.

   (d) Determine a rational parametrisation of C. Then prove that the full solution set of (C) is given by the integer multiples of the solutions of the form $(a^2b^2, (a-b)^2a^2, (a-b)^2b^2)$ where $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$.

   (e) (Bonus problem) Consider the diophantine equation

$$(xy + xz + yz)^2 - 3xyz(x + y + z) = 0.$$

   So the 4 in equation (C) changed into 3. Prove or disprove the following claim: all solutions $x, y, z \in \mathbb{Z}$ are given by integer scalar multiples of $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$.

3. We consider the algebraic number field $K = \mathbb{Q}(\zeta_5)$ where ζ_5 is a non-trivial fifth root of unity (e.g. $\zeta_5 = e^{2\pi i/5}$).

   (a) Determine the minimal polynomial for ζ_5. Then determine the number of real and the number of complex embeddings of K.

   (b) Show that the following elements of K are algebraic integers,

$$\zeta_5, \ u_k = \frac{1 - \zeta_5^k}{1 - \zeta_5}, \ 1/u_k = \frac{1 - \zeta_5}{1 - \zeta_5^k}$$

   where k is any integer not divisible by 5. Thus we can conclude that u_k is a unit in the integers of K for all $k = 1, 2, 3, 4$ (cyclotomic units).

(c) Prove that $5 = \prod_{k=1}^{4}(1-\zeta_5^k)$. Then show that there is unit u in the integers of K such that $5 = u(1-\zeta_5)^4$.

(d) Show that $\zeta_5^2 + \zeta_5^{-2} = \eta$, where η is the golden ratio $\eta = (1+\sqrt{5})/2$. Hence $\mathbb{Q}(\sqrt{5})$ is a subfield of $\mathbb{Q}(\zeta_5)$.

(e) Let α be an integer in $\mathbb{Q}(\sqrt{5})$ and let α' be its conjugate. Show that if $\alpha \neq 0, \pm 1$ then $\max(|\alpha|, |\alpha'|) \geq \eta$.

(f) Use Dirichlet's theorem for units in $\mathbb{Q}(\sqrt{5})$ and the lower bound just given to show that the unit group of the integers in $\mathbb{Q}(\sqrt{5})$ are given by $\pm\eta^k$ with $k \in \mathbb{Z}$.

(g) Prove that the roots of unity in $\mathbb{Q}(\sqrt{5})$ are given by $\pm\zeta_5^k$ with $k = 0, 1, 2, 3, 4$.

(h) Prove that if u is a unit in $\mathbb{Q}(\zeta_5)$ then $u\overline{u}$ is a unit in $\mathbb{Q}(\sqrt{5})$ ($\overline{u}$ is the complex conjugate of u). Determine $u_2\overline{u}_2$ where $u_2 = 1 + \zeta_5$.

(i) Use Dirichlet's theorem and the previous problems to prove that the group of units in $\mathbb{Q}(\zeta_5)$ is given by $\pm\zeta_5^k(1+\zeta_5)^l,\ k = 0, 1, 2, 3, 4; l \in \mathbb{Z}$.

4. We solve the diophantine equation $x^2 - 2y^2 = z^3$ in $x, y, z \in \mathbb{Q}$ and in integers x, y, z with $\gcd(x, y) = 1$.

(a) Choose $r, s \in \mathbb{Q}$ arbitrarily. Show that $x = r(r^2-2s^2), y = (r^2-2s^2)s, z = r^2-2s^2$ is a rational solution of our equation. Prove that any rational solution of $x^2-2y^2 = z^3$ is of this form.

(b) Let $x, y, z \in \mathbb{Z}$ be a solution of $x^2 - 2y^2 = z^3$ with $\gcd(x, y) = 1$. Prove that x is odd.

(c) We are given that $\mathbb{Z}[\sqrt{2}]$ is a unique factorization domain and its unit group is given by $\pm(-1+\sqrt{2})^k,\ k \in \mathbb{Z}$. By factorization of $x^2 - 2y^2$ over $\mathbb{Z}[\sqrt{2}]$ solve the equation $x^2 - 2y^2 = z^3$ in integers x, y, z with $\gcd(x, y) = 1$.