3 Third assignment

1. Let a be a positive integer > 20. We will solve $1+a^x = (a+1)^y$ in integers $x, y \geq 0$ using the polynomial method given in the proof of the Siegel-Mahler Theorem and show that $x = y = 1$ is the only solution. Assume that we are given a solution x, y and suppose that $x > 1$.

   (a) Show that $x \geq y \geq 0.9x$.

   (b) Show that x is odd.

   (c) Let $x = 2n + 1$. Using Corollary 4.3.5 construct integers A, B, C such that $aA + a^{2n+1}B = a(a+1)^{2n}C$ and $A \neq B$ and $|A|, |B|, |C| \leq (6\sqrt{3})^n(a+1)^n$.

   (d) Replace in the above equality $a^{2n+1} = a^x$ by $(a+1)^y - 1$ and show that we get a contradiction. (Hint: see proof of Corollary 4.3.6).

2. Let $f, g \in \mathbb{Z}[x]$ be two polynomials and suppose that g has at least three distinct zeros and f, g have no common zeros. We introduce the concept of resultant in the following two questions.

   (a) Suppose that f, g have degrees m, n respectively. Show that there exist non-trivial polynomials $A, B \in \mathbb{Z}[x]$ of degrees at most $n-1, m-1$ and an integer R such that $Af + Bg = R$. (Hint: use (linear) algebra). Note that in a similar way one can prove the existence of non-trivial polynomials $C, D \in \mathbb{Z}[x]$ of degrees at most $n-1, m-1$ and $S \in \mathbb{Z}$ such that $Cf + Dg = Sx^{m+n-1}$.

   (b) Let R, S be the integers of the previous question. Show that $R, S \neq 0$ (the smallest possible positive R is called the *resultant* of f and g).

   (c) Prove that the number of rational numbers a/b such that $f(a/b)/g(a/b) \in \mathbb{Z}$ is finite. (Hint: Homogenize $f(a/b), g(a/b)$ by multiplication with appropriate powers of b so that they become integers. Then use parts (a) and (b))

3. In this exercise you are asked to apply the estimate of Laurent, Mignotte and Nesterenko (see Exercises of 'linear form in logarithm' notes) to more advanced equations.

   (a) Prove that the equation
   $$x^n - 2y^n = 1$$
   in unknowns x, y with $x \geq 2, y \geq 2$ has no solutions if $n > 10000$. (Hint: Applying Laurent-Mignotte-Nesterenko to an appropriate linear form in two logarithms you will get a lower estimate depending on x, y and n. But you can derive also an upper estimate which depends on x, y, n. Comparison of the two estimates leads to an upper bound for n independent of x, y.

   (b) Let a, b, c be positive integers. Prove that there is a number $C > 0$, effectively computable in terms of a, b, c, such that the equation
   $$ax^n - by^n = c$$

has no solutions in $x, y \geq 2$ if $n > C$. In the case $a = b$ give an elementary proof, without using the result of Laurent-Mignotte-Nesterenko.

(c) Prove that the equation

$$y^z = \binom{x}{3}$$

integers x, y, z with $x > 3, y \geq 2, z \geq 2$ has only finitely many solutions. (Hint: factor the right hand side).