FOURTH HOMEWORK ASSIGNMENT
DEADLINE MAY 27.

Exercise 1. Let d_1, d_2 be distinct positive integers, none of them a square. Prove that there are only finitely many triples of integers x_1, x_2, x_3 such that

$$x_2^2 - d_1 x_1^2 = 1 \text{ and } x_3^2 - d_2 x_1^2 = 1.$$

Hint. Write $(x_2^2 - d_1 x_1^2)(x_3^2 - d_2 x_1^2)$ as a product of four linear forms and apply the Subspace Theorem.

Exercise 2. For a finite set of primes $S = \{p_1, \dots, p_t\}$, define $U_S := \{\pm p_1^{z_1} \cdots p_t^{z_t} : z_1, \dots, z_t \in \mathbb{Z}_{\geqslant 0}\}$.
Let $S_0, \dots, S_n$ be finite, non-empty, pairwise disjoint sets of primes and let $a_0, \dots, a_n$ be non-zero integers. Prove that the equation

$$a_0 x_0 + \cdots + a_n x_n = 0 \quad \text{in } x_0 \in U_{S_0}, \dots, x_n \in U_{S_n}$$

has only finitely many solutions.
Hint. Induction on n. In the induction step, write $a_1 x_1 + \cdots + a_n x_n = -a_0 x_0$. Let $S_0 \cup \cdots \cup S_n = \{p_1, \dots, p_t\}$. Apply the p-adic Subspace Theorem with $S = \{\infty, p_1, \dots, p_t\}$ and with appropriate linear forms in $x_1, \dots, x_n$ for $p \in S$ (so without x_0). The difference with Theorem 2.3 is that now, the solutions $x_0, \dots, x_n$ are pairwise coprime. Therefore, it is not necessary to restrict to non-degenerate solutions.

Exercise 3. Prove that there are only finitely many positive integers n such that

$$\frac{3^n - 1}{2^n - 1} \in \mathbb{Z}. \tag{1}$$

To this end, work out the following steps.

(a) Let n be a solution of (1) and define

$$\mathbf{x}_n = (x_{1n}, \dots, x_{5n}) := \left(4^n \frac{3^n - 1}{2^n - 1}, -6^n, -3^n, 2^n, 1\right) \in \mathbb{Z}^5.$$

Prove that $\sum_{i=1}^{5} x_{in} = \frac{3^n-1}{2^n-1}$ and that there exist $C > 0, \delta > 0$ independent of n such that $|\sum_{i=1}^{5} x_{in}| \leqslant C\|\mathbf{x}_n\|^{1-\delta}$.

(b) Prove that the set of vectors $\mathbf{x}_n$ such that n is a solution of (1) lies in a union of finitely many proper linear subspaces of $\mathbb{Q}^5$. Apply the p-adic Subspace Theorem to the vectors $\mathbf{x}_n$ with $S = \{\infty, 2, 3\}$ and with for each $p \in S$ the linear forms $X_1, \ldots, X_5$, $X_1 + \cdots + X_5$.

(c) Complete the exercise.