1 Exercises, March 1

Exercise 1.1.1 *Suppose $\alpha \in \mathbb{Q}$ is a zero of a monic polynomial (i.e. leading coefficient is 1) with coefficients in $\mathbb{Z}$. Prove that α is an algebraic integer. (Hint: use Gauss's lemma on the factorization of polynomials)*

Exercise 1.1.2 *Let K be an algebraic number field and $\mathcal{O}_K$ its ring of integers. Prove that $\eta \in \mathcal{O}_K$ is a unit if and only if $N(\eta) = \pm 1$.*

Exercise 1.1.3 *Let K be a number field and let I be an ideal in its ring of integers. In the following exercises use the multiplicative property of ideal norms.*

1. *Prove that $N(I) = 1 \iff I = \mathcal{O}_K$.*
2. *Prove that if $N(I)$ is a prime number then I is a prime ideal.*
3. *Suppose I is a prime ideal. Prove that there is a prime number $p \in \mathbb{Z}$ such that $I \cap \mathbb{Z} = (p)$.*
4. *Suppose that $[K : \mathbb{Q}] = n$ and I, p as in the previous question. Prove that there exists an integer f such that $N(I) = p^f$.*

Exercise 1.1.4 *Let $K = \mathbb{Q}(\sqrt{-23})$ and $\mathcal{O}_K = \mathbb{Z}[(1 + \sqrt{-23})/2]$ its ring of integers. Notice that $(2 + \sqrt{-23})(2 - \sqrt{-23}) = 2^2 + 23 = 3^3$.*

1. *Prove that for any algebraic integer $\alpha \in K$ with $\alpha \notin \mathbb{Z}$ we have $|\alpha| \geq \sqrt{6}$.*
2. *Prove that 3 and $2 + \sqrt{-23}$ are irreducible elements in $\mathcal{O}_K$.*
3. *Prove that $(3, 2 + \sqrt{-23})$ is not a principal ideal and that it is a prime ideal.*
4. *Prove that $(3, 2 + \sqrt{-23})^3$ is a principal ideal.*

Exercise 1.1.5 *Solve the Mordell equation $y^2 + 4 = x^3$. Consider the factorization $(y + 2i)(y - 2i) = x^3$ in $\mathbb{Z}[i]$ where $i = \sqrt{-1}$. Distinguish two cases y odd and y even.*

Exercise 1.1.6 *Solve the diophantine equation $x^2 + y^2 = z^3$ in $x, y, z \in \mathbb{Z}$ with $\gcd(x, y) = 1$. Hint: First prove that x, y cannot be both odd, then factor the left hand side in $\mathbb{Z}[i]$.*