Waves in Discrete Spatial Domains

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Motivated by the study of physical structures such as crystals, grids of neurons and population patches, an increasing interest has arisen in mathematical modelling techniques that reflect the underlying spatial discreteness. In this project, we will consider the propagation of electrical signals through nerve fibres. Such fibres are insulated by a myeline coating that admits gaps at the so-called nodes of Ranvier [3], which are regularly spaced along the fibre; see Fig. 1. Excitations of the nerve effectively jump from one node to the next, through a process called saltatory conduction [1].

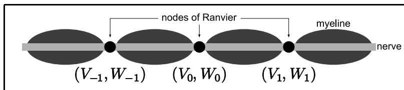

Fig. 1: Myelinated nerve fibres.

Electro-chemical analysis [2] leads to the discrete FitzHugh–Nagumo equation

$$\begin{aligned} \dot{V}_j(t) &= \alpha[V_{j+1}(t) + V_{j-1}(t) - 2V_j(t)] + g\big(V_j(t); a\big) - W_j(t), \\ \dot{W}_j(t) &= \epsilon\big(V_j(t) - \gamma W_j(t)\big), \end{aligned} \tag{1}$$

posed on the 1-dimensional lattice $j \in \mathbb{Z}$. The variable V_j encodes the potential at the j-the node of Ranvier, W_j denotes a recovery component and the cubic $g(v; a) = v(1-v)(v-a)$ describes the ionic interactions.

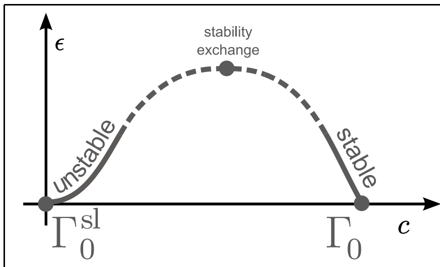

Fig. 2: Slow and fast pulses for (2).

Relatively little is known about the lattice differential equation (LDE) (1). Indeed, the spatial discreteness in the model is often ignored. Upon writing $\alpha = h^{-2}$ and treating h as the distance between nodes of Ranvier, one typically takes the limit $h \to 0$ to arrive at the FitzHugh–Nagumo PDE

$$\begin{aligned} V_t &= V_{xx} + g\big(V; a\big) - W, \\ W_t &= \epsilon(V - \gamma W). \end{aligned} \tag{2}$$

In this project we will investigate the differences between the LDE (1) and its PDE counterpart (2). We will be primarily interested in so-called travelling pulses, which are solutions to (1) that can be written as

$$(V_j, W_j)(t) = (v, w)(j + ct), \qquad \lim_{\xi \to \pm\infty} (v, w)(\xi) = (0, 0). \tag{3}$$

Here $c \in \mathbb{R}$ indicates the speed of the pulse and the functions $v : \mathbb{R} \to \mathbb{R}$ and $w : \mathbb{R} \to \mathbb{R}$ determine the shape of the pulse, which remains fixed. For the PDE (2) it is known[1] that one can construct a branch of fast ($c \sim 1$) stable pulses and a branch of slow ($c \approx 0$) unstable pulses that can be connected to each other by changing the parameter $\epsilon > 0$; see Figure 2.

We expect the situation to be less clear in the case of the LDE (1). To get a first glimpse why, substitute the Ansatz (3) into (1) to obtain the travelling wave differential equation

$$\begin{aligned} cv'(\xi) &= \alpha[v(\xi+1) + v(\xi-1) - 2v(\xi)] + g\big(v(\xi)\,;\, a\big) - w(\xi), \\ cw'(\xi) &= \epsilon\big(v(\xi) - \gamma w(\xi)\big). \end{aligned} \tag{4}$$

Notice the shifted arguments ($\xi + 1$ and $\xi - 1$) appearing in the equation for v. Notice also that when $c = 0$, all derivatives vanish and the equation transforms into a recursion relation. Both these features do not occur when studying the PDE (2).

Using both numerical and analytical techniques, depending on taste, the aim of this project is to investigate the modelling consequences of these mathematical curiosities.

References

[1] A. F. Huxley and R. Stampfli (1949), Evidence for Saltatory Conduction in Peripheral Meylinated Nerve Fibres. *J. Physiology* **108**, 315–339.

[2] J. Keener and J. Sneyd (1998), *Mathematical Physiology.* Springer–Verlag, New York.

[3] L. A. Ranvier (1878), *Lećons sur l'Histologie du Système Nerveux, par M. L. Ranvier, recueillies par M. Ed. Weber.* F. Savy, Paris.

[1] The curve in Figure 2 has been confirmed numerically for all $0 < a < \frac{1}{2}$ but theoretically only for $a \approx \frac{1}{2}$.