SANDPILE GROWTH MODEL

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Consider two-dimensional lattice $\mathbb{Z}^2$ and fix a parameter $k \geq 4$. A sandpile configuration y is an element of $\mathbb{Z}_+^{\mathbb{Z}^2}$, i.e.,

$$y : \mathbb{Z}^2 \mapsto \{0, 1, 2, \ldots\} := \mathbb{Z}_+,$$

with $y(\boldsymbol{n}) = y_{\boldsymbol{n}}$ denoting the number of grains of sand present at site $\boldsymbol{n} \in \mathbb{Z}^2$.

A configuration is called *stable* if $y_{\boldsymbol{n}} < k$ for all $\boldsymbol{n} \in \mathbb{Z}^2$. If a configuration y is unstable at site $\boldsymbol{n}$, then the site $\boldsymbol{n}$ *topples*: k grains of sand are removed, and each of the 4 closest neighbors receives 1 grain of sand. We refer to this model as *critical* if $k = 4$ or *dissipative* if $k > 4$. In the dissipative case, some sand is lost with each toppling.

If y is a initial configuration with finite support, i.e., if the set $\{\boldsymbol{n} \in \mathbb{Z}^2 : \ y_{\boldsymbol{n}} > 0\}$ is finite, then y can always be stabilized. It is important to note that the resulting stable configuration is independent of the order in which unstable sites have been toppled (this is known as abelian property). Suppose the initial configuration is consists of N grain of sand placed at the origin, with the remaining cites being vacant. Let $y^{(N)}$ be the result of stabilization of a such configuration.

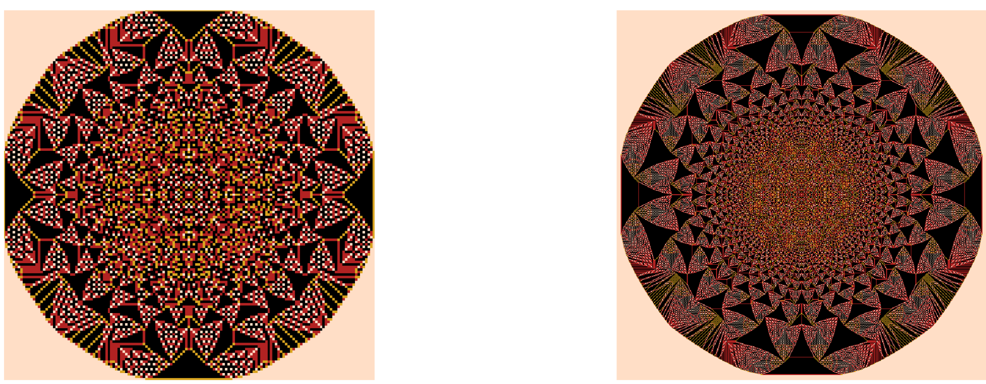

FIGURE 1. The resulting stabilized configuration for $N = 40000$ (left) and $N = 640000$ (right) in critical two-dimensional sandpile model. Right picture has been linearly scaled so that configurations occupy approximately the same volume in $\mathbb{R}^2$.

In this project asymptotic properties of $y^{(N)}$ as $N \to +\infty$ will be addressed. The most important open problem is the existence of the limiting shape. Namely, whether

$$\frac{\{\boldsymbol{n} \in \mathbb{Z}^2 : y_{\boldsymbol{n}}^{(N)} \neq 0\}}{N^{\frac{1}{2}}} \to \mathcal{S},$$

where $\mathcal{S} \subset \mathbb{R}^2$, $\mathcal{S} \neq \varnothing, \mathbb{R}^2$. Secondly, it is interesting to understand the self-similar structure of $y^{(N)}$ as $N \to \infty$. There are a number of other challenging questions and generalizations of this problem.

As a first step, the results of [1–3] must be understood and summarized. Computer experiments should a deeper insight into the properties of $y^{(N)}$. Finally, a novel link to harmonic analysis will be investigated.

REFERENCES

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