Waves through Obstacles in Discrete Spatial Domains

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Motivated by the study of physical structures such as crystals, grids of neurons and population patches, an increasing interest has arisen in mathematical modelling techniques that reflect the underlying spatial discreteness. The main goal of this project is to deepen our understanding of the differences and similarities between such spatially discrete systems and their traditional continuous counterparts.

Lattice Differential Equations

In this project, we will consider the two dimensional Nagumo lattice differential equation (LDE), which is given by

$$\dot{u}_{ij} = \alpha_{ij}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}] + g(u_{ij}; a), \qquad (i,j) \in \mathbb{Z}^2, \tag{1}$$

with prototypical cubic nonlinearity $g(u; a) = u(1-u)(u-a)$ for some $a \in (0,1)$. The Nagumo equation is a phenomenological model in which two stable equilibria compete for dominance in a spatial domain. In modelling contexts one often thinks of these equilibria as representing material phases or chemical or biological species. Upon choosing $\alpha_{ij} = h^{-2} > 0$, the LDE (1) can be seen as the discretization of the heavily studied planar PDE

$$\partial_t u(x,y,t) = (\partial_x^2 + \partial_y^2)u(x,y,t) + g\big(u(x,y,t); a\big), \qquad (x,y) \in \mathbb{R}^2, \tag{2}$$

on a square lattice with internode distance h.

Travelling waves

A natural place to start the analysis of (1) is to assume that all the coefficients are equal ($\alpha_{ij} = \alpha > 0$) and to look for planar travelling wave solutions that connect the two stable equilibria $u \equiv 0$ and $u \equiv 1$. Such waves can be written as

$$u_{ij}(t) = \Phi(i\cos\theta + j\sin\theta + ct); \quad \Phi(-\infty) = 0, \qquad \Phi(\infty) = 1, \tag{3}$$

in which the angle θ indicates the direction of propagation of the wave relative to the horizontal axis of $\mathbb{Z}^2$. Substituting this Ansatz into (1), we find that the wave profile $\Phi : \mathbb{R} \to \mathbb{R}$ necessarily satisfies the system

$$c\Phi'(\xi) \quad = \quad \alpha\big[\Phi(\xi + \cos\theta) + \Phi(\xi - \cos\theta) + \Phi(\xi + \sin\theta) + \Phi(\xi - \sin\theta) - 4\Phi(\xi)\big] + g(\Phi(\xi), a). \tag{4}$$

It is known [3] that (4) admits solutions (c, Φ) for all $0 < a < 1$ and $\theta \in [0, 2\pi]$. In addition, the wave speed c is unique once a and θ are fixed, but the profile Φ is only unique (up to translations) if $c \neq 0$.

Propogation Failure This condition on c reflects a crucial difference between the LDE (1) and its continuous counterpart (2). Indeed, the wave speed c appears in travelling wave MFDE (4) in front of the highest derivative, which should be contrasted to the travelling wave ODE

$$c\Phi'(\xi) = \Phi''(\xi) + g(\Phi(\xi); a) \tag{5}$$

that is associated to the PDE (2).

Fig. 2 illustrates the far-reaching consequences that this singular dependence can have: upon fixing the angle θ, the wave speed c may vanish for all parameter values a in some interval $[a_*(\theta), \frac{1}{2}]$, with $a_*(\theta) < \frac{1}{2}$. This phenomenon is called propagation failure and is present throughout a wide range of discrete systems. It can be interpreted as the consequence of an energy barrier caused by the gaps, which must be overcome in order to allow propagation.

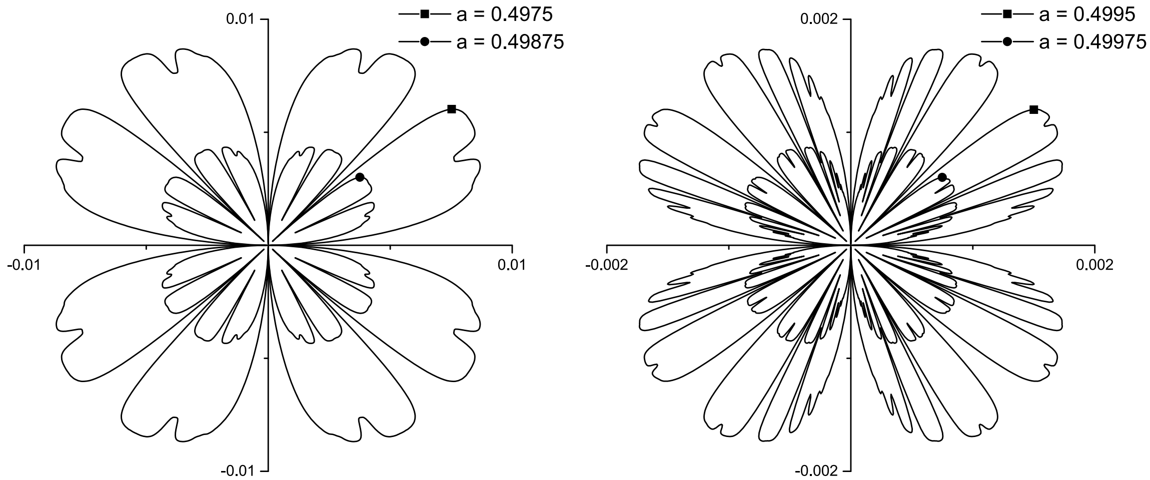

Fig. 1: Plotted are the pairs $\big(c(\theta, a)\cos\theta, c(\theta, a)\sin\theta\big)$ for travelling wave solutions to the LDE (1) with $\alpha_{ij} = 1$ and $g(u; a) = 10u(1-u)(u-a)$ at different values of the detuning parameter a. In particular, these figures can be seen as polar plots where the radial distance gives the wave speed for the angle under consideration. The angular dependence becomes more and more pronounced as $a \to \frac{1}{2}$. The spikes towards the center indicate the presence of propagation failure in certain directions.

Anistropy Embedding the square lattice $\mathbb{Z}^2$ into the continuous space $\mathbb{R}^2$ necessarily breaks the spatial isotropy of $\mathbb{R}^2$ since two preferred lattice directions are chosen. Indeed, in (4) the angle θ appears explicitly, in contrast to (5).

As shown in Figure 1, the dependence of c on a and θ can be rather delicate, which is a direct consequence of the rotational symmetry that one breaks when embedding the square lattice $\mathbb{Z}^2$ into the continuous space $\mathbb{R}^2$. Indeed, the corresponding polar plots for the PDE (2) are all circles as c no longer depends on the direction θ.

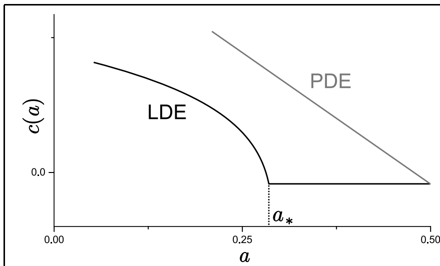

Fig. 2: Typical plot of the wave speed c as a function of a for a fixed value of θ.

Obstacles

In this project we intend to investigate the impact of lattice impurities on the propagation of planar waves. In particular, we will study (1) with diffusion coefficients that are homogeneous everywhere except at $(0,0)$, i.e., $\alpha_{ij} = \alpha + \beta\delta_{i0}\delta_{j0}$ with $\alpha > 0$ and $\beta > 0$. The main question is in what sense the travelling wave for the 'pure' system scatters when it 'hits' the obstacle. In particular, does the wave recover its shape after it has passed the obstacle and if so, how fast?

Results of this type for the Nagumo PDE (2) have only become available very recently [1]. For the LDE (1) partial results have also been developed for general directions [2], but they are rather messy and require that waves are able to travel in all directions (which in the context of Figure 1 means that the spikes are not allowed.) In this project, we will focus on the special cases of waves travelling in horizontal, vertical and diagonal directions. In these cases the analysis becomes much nicer and more concrete results can be expected. In particular, it is very interesting to see if waves travel in diagonal directions survive the impact of the obstacle if waves in the horizontal and vertical directions are blocked.

References

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[3] J. Mallet-Paret (1999), The Global Structure of Traveling Waves in Spatially Discrete Dynamical Systems. *J. Dyn. Diff. Eq.* **11**, 49–128.