Ideals and Bands in Pre-Riesz Spaces

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Abstract. In a vector lattice, ideals and bands are well-investigated subjects. We study similar notions in a pre-Riesz space. The pre-Riesz spaces are exactly the order dense linear subspaces of vector lattices. Restriction and extension properties of ideals, solvex ideals and bands are investigated. Since every Archimedean directed partially ordered vector space is pre-Riesz, we establish properties of ideals and bands in such spaces.

Mathematics Subject Classification (2000). Primary 46A40; Secondary 06F20.

Keywords. band, disjointness, ideal, order dense subspace, partially ordered vector space, pre-Riesz space, solvex ideal.

1. Introduction

In the study of vector lattices one naturally encounters partially ordered vector spaces that are not lattices. For instance, spaces of operators between vector lattices often lack lattice structure. This problem is usually avoided by assuming Dedekind completeness of the codomain. Alternatively, one could try to extend the required notions from lattice theory to a more general class of partially ordered vector spaces. One approach is to reformulate relations involving absolute values or positive or negative parts as relations of suitable sets of upper bounds. Intrinsic definitions of several notions can thus be obtained, but it turns out to be difficult to set up their usual properties known from the vector lattice setting. Another approach is to embed the partially ordered vector space in a vector lattice and use the lattice structure of the ambient space. The latter approach is taken in [4] to study disjointness in partially ordered vector spaces. As a rule, the embedding method succeeds if the embedding is order dense. The partially ordered vector spaces that can be embedded order densely in a vector lattice have been characterized in [6] as the pre-Riesz spaces. Every directed Archimedean space is pre-Riesz, but there exist also non-Archimedean pre-Riesz spaces. Often a lattice notion in a vector lattice can be stated in several ways and their reformulation in partially ordered vector spaces may lead to different notions. It may be expected that the most useful generalizations will be those where the two approaches—direct reformulation and using embeddings—coincide. In the present paper we proceed the investigation of 'vector lattice notions' in pre-Riesz spaces from this point of view. We focus on ideals and bands, which are indispensable in the theory of vector lattices. We begin by intrinsic definitions and then study their properties relative to embeddings in vector lattices. In particular, we address extension and restriction. If a partially ordered vector space X is an order dense subspace of a vector lattice Y, can a generalized ideal or band in X be extended to an ideal or band in Y? And is the restriction of an ideal or band in Y a generalized ideal or band in X? Moreover, we determine what properties are similar to the lattice case and what properties fail. The main results are in Section 5.

2. Preliminaries

Let X be a real vector space and let K be a cone in X, that is, K is a wedge ($x, y \in K$, $\lambda, \mu \geq 0$ imply $\lambda x + \mu y \in K$) and $K \cap (-K) = \{0\}$. In X a partial order is introduced by defining $y \geq x$ if and only if $y - x \in K$. *In this paper,* (X, K) *is a partially ordered vector space.* Occasionally we write loosely X instead of (X, K), provided that K is fixed in advance. For two elements $y, z \in K$ with $y \leq z$ denote the according order interval by $[y, z] = \{x \in X\colon\ y \leq x \leq z\}$. A set $M \subset X$ is called *majorized* if there is $z \in X$ such that $x \leq z$ for all $x \in M$, and *order bounded* if there are $y, z \in X$ such that $M \subseteq [y, z]$. Denote for a subset $M \subseteq X$ the set of all upper bounds by

$$M^u = \{x \in X\colon\ x \geq m \text{ for all } m \in M\}\,.$$

The *positive-linear hull* in X of a subset $M \subseteq X$ is given by

$$\operatorname{pos} M = \{x \in X\colon\ \exists n \in \mathbb{N},\ \lambda_i \in [0, \infty),\ x_i \in M,\ i = 1, \ldots n \text{ with } x = \sum_{i=1}^{n} \lambda_i x_i\}\,.$$

The space (X, K) is called *Archimedean* if for every $x, y \in X$ with $nx \leq y$ for all $n \in \mathbb{N}$ one has $x \leq 0$. A set $M \subseteq X$ is called *directed* if for every $x, y \in M$ there is an element $z \in M$ such that $z \geq x$ and $z \geq y$. X is directed if and only if the cone K is *generating* in X, that is, $X = K - K$. X has the *Riesz decomposition property* if for every $y, x_1, x_2 \in K$ with $y \leq x_1 + x_2$ there exist $y_1, y_2 \in K$ such that $y = y_1 + y_2$ and $y_1 \leq x_1$, $y_2 \leq x_2$. A subset F of K is called a *base* of K if F is a (non-empty) convex set such that every element $x \in K$ with $x \neq 0$ has a unique representation $x = \lambda y$ with $y \in F$ and $\lambda \in (0, \infty)$. A net $\{x_\alpha\} \subseteq X$ is said to be decreasing (in symbols, $x_\alpha \downarrow$), whenever $\alpha \geq \beta$ implies $x_\alpha \leq x_\beta$. For $x \in X$ the notation $x_\alpha \downarrow x$ means that $x_\alpha \downarrow$ and $\inf_\alpha\{x_\alpha\} = x$ both hold. The meanings of $x_\alpha \uparrow$ and $x_\alpha \uparrow x$ are analogous. We say that a net $\{x_\alpha\}_\alpha \subset X$ *(o)-converges* to $x \in X$ (in symbols, $x_\alpha \xrightarrow{(o)} x$), if there is a net $\{y_\alpha\}_\alpha \subset X$ such that $y_\alpha \downarrow 0$ and for all α one has $\pm(x_\alpha - x) \leq y_\alpha$. The equivalence of $x_\alpha \xrightarrow{(o)} x$ and $x_\alpha - x \xrightarrow{(o)} 0$ is obvious. If a net (o)-converges, then the limit is unique. A set $M \subseteq X$ is called

(o)-*closed*, if for each net $\{x_\alpha\}_{\alpha\in A} \subseteq M$ which (o)-converges to $x \in X$ one has $x \in M$.

For standard notations in the case that X is a vector lattice see [1]. Recall that a vector lattice is *Dedekind complete* whenever every non-empty majorized subset has a supremum. By a subspace of a partially ordered vector space or a vector lattice we mean an arbitrary linear subspace with the inherited order. We do not require it to be a lattice or a sublattice. A subspace D of a partially ordered vector space X is called *majorizing* if for every $x \in X$ there is $y \in D$ such that $x \leq y$. A majorizing subspace of a directed partially ordered vector space is directed. We say that a subspace D of a vector lattice X *generates* X *as a vector lattice* if for every $x \in X$ there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in D$ such that $x = \bigvee_{i=1}^m a_i - \bigvee_{i=1}^n b_i$.

3. Ideals, solvex ideals, and bands

In [3] and [4] notions which are known in the theory of vector lattices are introduced in the general setting of a partially ordered vector space X. Instead of the supremum of two elements one considers the set of all common upper bounds of the elements.

Definition 3.1. [3, Definition 3.1] A subset M of a partially ordered vector space X is called *solid* if for every $x \in X$ and $y \in M$ the inclusion $\{x, -x\}^u \supseteq \{y, -y\}^u$ implies $x \in M$. A solid subspace of X is called an *ideal*.

If X is a vector lattice, these notions coincide with the usual ones. If some set can be considered as a subset of several spaces, then its solidness strongly depends on the space in which the upper bounds are taken. If there is risk of confusion, we write "solid in X". Since the solid hull of a convex set need not be convex, the solvex hull will be of interest.

Definition 3.2. [3, Definition 3.7] Let X be a partially ordered vector space. A set $M \subset X$ is called *solvex* if for every $x \in X$, $x_1, \ldots, x_n \in M$ and $\lambda_1, \ldots, \lambda_n \in (0, 1]$ with $\sum_{k=1}^n \lambda_k = 1$ such that

$$\{x, -x\}^u \supseteq \left\{ \sum_{k=1}^n \varepsilon_k \lambda_k x_k \colon\ \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u$$

one has that $x \in M$.

Lemma 3.3. [3, Lemma 3.8] *Let X be a partially ordered vector space. Every solvex set in X is solid and convex. If X is a vector lattice, then a set is solvex in X if and only if it is solid and convex.*

In order to obtain a representation of the solvex hull of a non-empty set, we need the following preliminary statement.

Lemma 3.4. [3, Lemma 3.28] *Let X be a partially ordered vector space. If $x, x_k \in X$ and $y_{k,i} \in X$, $i = 1, \ldots, m_k$, $k = 1, \ldots, n$, are such that*

$$\{x_k, -x_k\}^u \supseteq \left\{ \sum_{i=1}^{m_k} \varepsilon_i y_{k,i} \colon \varepsilon_1, \ldots, \varepsilon_{m_k} \in \{1, -1\} \right\}^u$$

for $k = 1, \ldots, n$, and

$$\{x, -x\}^u \supseteq \left\{ \sum_{k=1}^{n} \varepsilon_k x_k \colon \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u ,$$

then

$$\{x, -x\}^u \supseteq \left\{ \sum_{k=1}^{n} \sum_{i=1}^{m_k} \varepsilon_{k,i} y_{k,i} \colon \varepsilon_{k,i} \in \{1, -1\},\, k = 1, \ldots, n,\, i = 1, \ldots, m_k \right\}^u .$$

Let X be a partially ordered vector space and $M \subseteq X$. Then the set

$$S = \Bigg\{ x \in X \colon \exists\, x_1, \ldots, x_n \in M,\ \lambda_1, \ldots, \lambda_n \in (0,1] \text{ with } \sum_{k=1}^{n} \lambda_k = 1 \text{ such that } \{x, -x\}^u \supseteq \left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k x_k \colon \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u \Bigg\}$$

is the *solvex hull* of M, i. e. S is the smallest solvex set in X that contains M. Indeed, $M \subseteq S$, and if T is a solvex set containing M, then $S \subseteq T$. It remains to show that S is solvex. Let $x \in X$, $x_1, \ldots, x_n \in S$, $\lambda_1, \ldots, \lambda_n \in (0,1]$ with $\sum_{k=1}^{n} \lambda_k = 1$ be such that

$$\{x, -x\}^u \supseteq \left\{ \sum_{k=1}^{n} \varepsilon_k \lambda_k x_k \colon \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u .$$

For each $k \in \{1, \ldots, n\}$ there are $y_{k,1}, \ldots, y_{k,m_k} \in M$, $\lambda_{k,1}, \ldots, \lambda_{k,m_k} \in (0,1]$ with

$$\sum_{i=1}^{m_k} \lambda_{k,i} = 1$$

such that

$$\{x_k, -x_k\}^u \supseteq \left\{ \sum_{i=1}^{m_k} \varepsilon_i \lambda_{k,i} y_{k,i} \colon \varepsilon_1, \ldots, \varepsilon_{m_k} \in \{1, -1\} \right\}^u .$$

Then

$$\{\lambda_k x_k, -\lambda_k x_k\}^u \supseteq \left\{ \sum_{i=1}^{m_k} \varepsilon_i \lambda_k \lambda_{k,i} y_{k,i} \colon \varepsilon_1, \ldots, \varepsilon_{m_k} \in \{1, -1\} \right\}^u .$$

Due to Lemma 3.4, one has

$$\{x, -x\}^u \supseteq \left\{ \sum_{k=1}^{n} \sum_{i=1}^{m_k} \varepsilon_{k,i} \lambda_k \lambda_{k,i} y_{k,i} \colon \varepsilon_{k,i} \in \{1, -1\},\, k = 1, \ldots, n,\, i = 1, \ldots, m_k \right\}^u .$$

Since $y_{k,i} \in M$ and $\lambda_k \lambda_{k,i} \in (0,1]$ for $k = 1, \ldots, n,\ i = 1, \ldots, m_k$, and

$$\sum_{k=1}^{n} \sum_{i=1}^{m_k} \lambda_k \lambda_{k,i} = \sum_{k=1}^{n} \lambda_k \sum_{i=1}^{m_k} \lambda_{k,i} = \sum_{k=1}^{n} \lambda_k = 1\,,$$

one gets $x \in S$. So, S is solvex.

Lemma 3.5. *Let X be a partially ordered vector space and M a subspace of X. Then the solvex hull S of M is a subspace of X, and, hence, an ideal.*

Proof. Since M is a linear subspace, its solvex hull is given by

$$S = \Bigg\{ x \in X\colon\ \exists\, x_1, \ldots, x_n \in M \ \text{ such that}$$
$$\{x, -x\}^u \supseteq \left\{ \sum_{k=1}^{n} \varepsilon_k x_k\colon\ \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u \Bigg\}\,.$$

Let $x, y \in S$, we show $x + y \in S$. There are $x_1, \ldots, x_n, y_1, \ldots, y_m \in M$ such that

$$\{x, -x\}^u \supseteq \left\{ \sum_{k=1}^{n} \varepsilon_k x_k\colon\ \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u$$

and

$$\{y, -y\}^u \supseteq \left\{ \sum_{j=1}^{m} \delta_j y_j\colon\ \delta_1, \ldots, \delta_m \in \{1, -1\} \right\}^u\,.$$

Suppose u is an upper bound of the set

$$\left\{ \sum_{k=1}^{n} \varepsilon_k x_k + \sum_{j=1}^{m} \delta_j y_j\colon\ \varepsilon_1, \ldots, \varepsilon_n, \delta_1, \ldots, \delta_m \in \{1, -1\} \right\}\,,$$

i. e. for any $\varepsilon_1, \ldots, \varepsilon_n, \delta_1, \ldots, \delta_m \in \{1, -1\}$ we have $u \geq \sum_{k=1}^{n} \varepsilon_k x_k + \sum_{j=1}^{m} \delta_j y_j$. So,

$$u - \sum_{j=1}^{m} \delta_j y_j \geq \sum_{k=1}^{n} \varepsilon_k x_k\,,$$

hence $u - \sum_{j=1}^{m} \delta_j y_j$ is an upper bound of the set

$$\left\{ \sum_{k=1}^{n} \varepsilon_k x_k\colon\ \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}\,.$$

Therefore, $u - \sum_{j=1}^{m} \delta_j y_j \geq \pm x$. From this follows

$$u - x \geq \sum_{j=1}^{m} \delta_j y_j \quad \text{and} \quad u + x \geq \sum_{j=1}^{m} \delta_j y_j\,.$$

So, $u - x$ and $u + x$ are upper bounds of the set

$$\left\{ \sum_{j=1}^{m} \delta_j y_j \colon \ \delta_1, \ldots, \delta_m \in \{1, -1\} \right\}$$

and hence of $\{y, -y\}$. We conclude $u - x \geq y$ and $u + x \geq -y$, which implies $u \in \{x + y, -x - y\}^u$. Consequently,

$$\{x + y, -x - y\}^u \supseteq \left\{ \sum_{k=1}^{n} \varepsilon_k x_k + \sum_{j=1}^{m} \delta_j y_j \colon \ \varepsilon_1, \ldots, \varepsilon_n, \delta_1, \ldots, \delta_m \in \{1, -1\} \right\}^u ,$$

and we infer that $x + y \in S$. Further, if $x \in S$, then $-x \in S$. Also, $0 \in S$. Let $x \in S$ and $\lambda > 0$, we show $\lambda x \in S$. There are $x_1, \ldots, x_n \in M$ such that

$$\{x, -x\}^u \supseteq \left\{ \sum_{k=1}^{n} \varepsilon_k x_k \colon \ \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u .$$

If u is an upper bound of the set $\left\{\sum_{k=1}^{n} \varepsilon_k \lambda x_k \colon \ \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}\right\}$, then $\frac{1}{\lambda} u$ is an upper bound of $\left\{\sum_{k=1}^{n} \varepsilon_k x_k \colon \ \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}\right\}$, so $\frac{1}{\lambda} u \in \{x, -x\}^u$, which implies $u \in \{\lambda x, -\lambda x\}^u$. Hence,

$$\{\lambda x, -\lambda x\}^u \supseteq \left\{ \sum_{k=1}^{n} \varepsilon_k \lambda x_k \colon \ \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u$$

and, therefore, $\lambda x \in S$. $\square$

In a vector lattice, two elements x, y are disjoint, whenever $|x| \wedge |y| = 0$, which is equivalent to $|x + y| = |x - y|$ [1, Theorem 1.4(4)]. Recall the following notion of disjointness in a partially ordered vector space X.

Definition 3.6. [4] Let X be a partially ordered vector space. The elements $x, y \in X$ are called *disjoint*, in symbols $x \perp y$, if

$$\{x + y, -x - y\}^u = \{x - y, -x + y\}^u .$$

The *disjoint complement* of a subset $M \subseteq X$ is the set

$$M^d = \{y \in X \colon \ y \perp x \text{ for all } x \in M\} .$$

For $x, y \in X$ one has $y \perp x$ if and only if $x \perp y$, and, obviously, $x \perp 0$. Clearly, $x \perp y$ implies $-x \perp y$. If $x \perp x$, then $\{x + x, -x - x\}^u = \{x - x\}^u = K$, so $x, -x \leq 0$, which yields $x = 0$. Similarly, $x \perp (-x)$ implies $x = 0$. If $x, y \in K$ are such that $x \perp y$, and $z \geq x, y$, then $z \in \{x - y, -x + y\}^u = \{x + y\}^u$, so $z \geq x + y$. For $M \subseteq X$ the set M^d is non-empty since one has $0 \in M^d$. Further, if M and N are subsets of X such that $M \subseteq N$, then $M^d \supseteq N^d$. Observe that for $x, y \in X$ one has $x \perp y$ if and only if the four inclusions

$$(K \pm x) \cap (K \pm y) \subseteq K \tag{1}$$

are satisfied. Indeed, let $x, y \in X$ with $x \perp y$, i. e.

$$(K + x + y) \cap (K - x - y) = (K + x - y) \cap (K - x + y) . \tag{2}$$

Adding $x + y$, using that $K = 2K$, and dividing by 2, respectively, yields

$$\begin{aligned} (2) \ \Leftrightarrow (K + 2x + 2y) \cap K &= (K + 2x) \cap (K + 2y) \\ \Leftrightarrow (2K + 2x + 2y) \cap (2K) &= (2K + 2x) \cap (2K + 2y) \\ \Leftrightarrow (K + x + y) \cap K &= (K + x) \cap (K + y) . \end{aligned} \tag{3}$$

Moreover, one has $x \perp -y$, $-x \perp -y$ and $-x \perp y$. Using analogous calculations as in (3), one gets the four inclusions in (1). On the other hand, assume (1). From $(K + x) \cap (K - y) \subseteq K$ one gets

$$(K + x + y) \cap K \subseteq (K + y) ,$$

furthermore $(K - x) \cap (K + y) \subseteq K$ implies $K \cap (K + x + y) \subseteq (K + x)$, so

$$(K + x + y) \cap K \subseteq (K + x) \cap (K + y) .$$

One has $(K + x) \cap (K + y) \subseteq K$, moreover $(K - x) \cap (K - y) \subseteq K$ implies $(K + y) \cap (K + x) \subseteq (K + x + y)$, so

$$(K + x + y) \cap K \supseteq (K + x) \cap (K + y) .$$

Hence, (3) is satisfied, which means $x \perp y$. We conclude the section by giving the definition of a band in a partially ordered vector space.

Definition 3.7. [4, Definition 5.4] A linear subspace M of a partially ordered vector space X is called a *band* in X if $(M^d)^d = M$.

If X is an Archimedean vector lattice, then this notion of a band coincides with the usual one of an (o)-closed ideal.

Properties of ideals, solvex ideals and bands in certain partially ordered vector spaces will be derived in Section 5.

4. Pre-Riesz Spaces

In this section we provide the notion of a pre-Riesz space and recall that the pre-Riesz spaces are exactly the order dense subspaces of vector lattices. We give several examples of pre-Riesz spaces and their embeddings into vector lattices.

Definition 4.1. [6, Definition 1.1(viii), Theorem 4.15] A partially ordered vector space (X, K) is called *pre-Riesz* if for every $x, y, z \in X$ the inclusion $\{x + y, x + z\}^u \subseteq \{y, z\}^u$ implies $x \in K$.

Clearly, each vector lattice is pre-Riesz, since the inclusion in Definition 4.1 reduces to the inequality $(x + y) \vee (x + z) \geq y \vee z$, so $x + (y \vee z) \geq y \vee z$, which implies $x \geq 0$.

Proposition 4.2. [6, Theorem 1.7(ii)] *Every pre-Riesz space is directed and every directed Archimedean partially ordered vector space is pre-Riesz.*

As an example of a pre-Riesz space which is not Archimedean consider the space $\mathbb{R}^2$ equipped with the cone

$$K = \{(x_1, x_2)^T : x_2 > 0\} \cup \{(x_1, 0)^T : x_1 \geq 0\}.$$

We call a linear subspace D of a partially ordered vector space X *order dense* in X if for every $x \in X$ we have

$$x = \inf\{y \in D \colon\ y \geq x\},$$

that is, each x is the greatest lower bound of the set $\{y \in D \colon\ y \geq x\}$ in X (see [2, p. 360]). It is straightforward that D is order dense in X if and only if for every $x \in X$ one has $x = \sup\{y \in D \colon\ y \leq x\}$. Clearly, an order dense subspace is majorizing. Recall that a linear map $i \colon X \to Y$, where X and Y are partially ordered vector spaces, is called *bipositive* if for every $x \in X$ one has $i(x) \geq 0$ if and only if $x \geq 0$. An *embedding map* is required to be linear and bipositive, which implies injectivity. X and Y are called *order isomorphic* if there is a bipositive surjective linear map $i \colon X \to Y$. The pre-Riesz spaces are exactly the order dense linear subspaces of vector lattices.

Theorem 4.3. [6, Corollaries 4.9–11 and Theorems 3.5, 3.7, 4.13] *Let X be a partially ordered vector space. The following statements are equivalent:*

(i) X is pre-Riesz.
(ii) There exist a vector lattice Y and a bipositive linear map $i\colon\ X \to Y$ such that $i(X)$ is order dense in Y.
(iii) There exist a vector lattice Y and a bipositive linear map $i\colon\ X \to Y$ such that $i(X)$ is order dense in Y and generates Y as a vector lattice.

All spaces Y as in (iii) are order isomorphic. A pair (Y, i) as in (iii) is called a *Riesz completion* of X. As it is unique up to isomorphism we will speak of *the* Riesz completion of X and denote it by X^ρ. If a pre-Riesz space (X, K) is Archimedean, then X^ρ is Archimedean as well. We consider examples of pre-Riesz spaces. We compute, for instance, the Riesz completions of $\mathbb{R}^3$ with the ice-cream cone and of the space of symmetric 2×2 matrices ordered by the cone of positive semidefinite matrices. The Examples 4.4, 4.5, 4.6, and 4.7 describe different representations of the same partially ordered vector space. The Examples 4.7 and 4.8 will be continued in Section 5 to construct counterexamples.

Example 4.4. In the present example, we consider subspaces of the space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$, ordered by the natural cone $\{x \in C(\mathbb{R}) \colon\ x(t) \geq 0 \text{ for all } t \in \mathbb{R}\}$. Let $X = P_2(\mathbb{R})$ be the ordered vector space of all real polynomial functions on $\mathbb{R}$ of at most degree 2. Observe that X is directed. We show that X is order dense in the vector lattice

$$V = \{v \in C(\mathbb{R}) \colon\ \text{ there is } p \in X \text{ such that } |v| \leq p\}.$$

Let $v \in V$ and $p \in X$ be such that $|v(t)| < p(t)$ for all $t \in \mathbb{R}$. Fix an arbitrary point $t_0 \in \mathbb{R}$ and $\varepsilon > 0$ such that $v(t_0) + \varepsilon < p(t_0)$. Since v is continuous, there is $\delta > 0$ such that for all $t \in (t_0 - \delta, t_0 + \delta)$ one has $v(t) \leq v(t_0) + \varepsilon$. One obtains a polynomial $p_\varepsilon \in X$ from the equations $p_\varepsilon(t_0) = v(t_0)+\varepsilon$, $p_\varepsilon(t_0-\delta) = p_\varepsilon(t_0+\delta) = \max\{p(t_0-\delta), p(t_0+\delta)\}$. The polynomial p_ε attains its minimum in t_0 and satisfies $p_\varepsilon(t) \geq p(t)$ for all $t \in \mathbb{R} \setminus (t_0-\delta, t_0+\delta)$. So, $p_\varepsilon \geq v$. It follows that

$$\inf\{p(t_0)\colon\ p \in X,\ p \geq v\} \leq p_\varepsilon(t_0) = v(t_0) + \varepsilon$$

and, as ε tends to zero, $v(t_0) = \inf\{p(t_0)\colon\ p \in X,\ p \geq v\}$. Hence, $v = \inf\{p \in X\colon\ p \geq v\}$. We conclude that X is order dense in V.

A function y on $\mathbb{R}$ is called a piecewise polynomial function of at most degree 2 if there are $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in \mathbb{R}$ such that $t_1 < t_2 < \ldots < t_n$ and y is a polynomial function of at most degree 2 on $(-\infty, t_1]$, $[t_n, \infty)$ and $[t_i, t_{i+1}]$ for each $i = 1, \ldots, n-1$. For such a function y we define

$$L(y) = \lim_{t\to-\infty} y(t)/t^2 \text{ and } R(y) = \lim_{t\to\infty} y(t)/t^2.$$

Denote

$$\begin{aligned} Y = \{y \in C(\mathbb{R})\colon\ y \ &\text{is a piecewise polynomial function of at most degree 2} \\ &\text{and } L(y) = R(y)\}\,. \end{aligned}$$

Clearly, Y is a subspace of V and $X \subset Y$, so X is order dense also in Y. In addition, Y is a sublattice of V. To see that X generates Y, consider a function $y \in Y$ with the corresponding $t_1 < t_2 < \ldots < t_n$ and $p_i \in X$, $i = 0, \ldots, n$, such that $y = p_0$ on $(-\infty, t_1]$, $y = p_i$ on $[t_i, t_{i+1}]$, $i = 1, \ldots, n-1$, and $y = p_n$ on $[t_n, \infty)$. For $i \in \{1, \ldots, n-1\}$, one can construct a function $y_i = p_i \vee z$ such that $y_i \geq y$ and $y_i = y$ on $[t_i, t_{i+1}]$, by choosing a parabola $z \in X$ through $(t_i, y(t_i))$ and $(t_{i+1}, y(t_{i+1})$ with the coefficient of t^2 sufficiently large. One can construct y_0 as pointwise maximum of at most three elements of X with $y_0 = p_0$ on $(-\infty, t_1]$ and $y_0 \geq y$. Indeed, if $a = L(y) = R(y) \leq 0$, one can take for y_0 the pointwise maximum of p_0 and a sufficiently steep line. If $a > 0$ one can take $y_0 = p_0 \vee v \vee w$, where $v(t) = at^2 + pt + q$ is such that $v(t_n) = y(t_n)$ and $v'(t_n)$ is so large that $v \leq p_0$ on $(-\infty, t_1]$ and $v \geq p_n$ on $[t_n, \infty)$. Further, w is a parabola through $(t_1, y(t_1))$ and $(t_n, y(t_n))$ such that $w \geq y$ on $[t_1, t_n]$ and $w \leq p_0$ on $(-\infty, t_1]$. One can construct y_n similarly. Now $y = \bigwedge_{i=0}^n y_i = -\bigvee_{i=0}^n (-y_i)$. Thus, X generates Y, and we infer that Y is the Riesz completion of X.

Next we consider the 3-dimensional icecream cone and show that its Riesz completion is the space Y of the previous example.

Example 4.5. Let the space $X = \mathbb{R}^3$ be equipped with the so-called 3-dimensional ice-cream cone $K_3 = \{(x_1, x_2, x_3)^T\colon\ x_1^2 + x_2^2 \leq x_3^2,\ x_3 \geq 0\}$. We claim that X is order isomorphic to the space $P_2(\mathbb{R})$ (equipped with the pointwise ordering) in Example 4.4. Let $a, b, c \in \mathbb{R}$ and consider the element $p(t) = at^2 + bt + c$ of $P_2(\mathbb{R})$. In the case $a \neq 0$ the zeros of p are given by

$$t_{1/2} = \tfrac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right).$$

So, p is positive if and only if either (i) $0 < a$ and $b^2 - 4ac \leq 0$, or (ii) $a = b = 0$ and $0 \leq c$. Consider the linear bijection $j\colon \mathbb{R}^3 \to P_2(\mathbb{R})$ defined by

$$(x_1, x_2, x_3)^T \mapsto p(t) = at^2 + bt + c\,,$$

where $a = x_1 + x_3,\ b = 2x_2,\ c = x_3 - x_1$. We show that j is bipositive. Let $x = (x_1, x_2, x_3)^T \in K_3$, i. e. $x_1^2 + x_2^2 \leq x_3^2$ and $0 \leq x_3$. Due to $x_1 = \frac{a-c}{2},\ x_2 = \frac{b}{2}$ and $x_3 = \frac{a+c}{2}$, the inequality $x_1^2 + x_2^2 \leq x_3^2$ is satisfied if and only if $b^2 \leq 4ac$. In particular, one has $0 \leq ac$. In combination with $0 \leq x_3 = \frac{a+c}{2}$ one gets $0 \leq a$ and $0 \leq c$. If $0 < a$, we have the condition in (i). On the other hand, if $a = 0$ we obtain $b = 0$, and, hence, the condition in (ii). So, the polynomial $j(x)$ is positive with respect to the pointwise ordering.

Vice versa, let p be a positive element of $P_2(\mathbb{R})$. In the case (i) one obtains $x_1^2 + x_2^2 \leq x_3^2$, in particular $|x_1| \leq |x_3|$, and from $0 < a = x_1 + x_3$ follows $0 < x_3$, so $x \in K_3$. In the case (ii), one has $x_2 = 0$ and $x_3 = -x_1$, so $x_1^2 = x_3^2$. Now $0 \leq c = 2x_3$ yields $x \in K_3$. We conclude that the Riesz completion of $(\mathbb{R}^3, K_3)$ is the space Y in Example 4.4.

Example 4.6. Let X be the space of all symmetric 2×2-matrices, ordered by the cone K of all matrices in X which are positive semidefinite. We show that X is order isomorphic to $P_2(\mathbb{R})$ in Example 4.4. Consider the bijection

$$\begin{array}{rcl} j\colon P_2(\mathbb{R}) & \to & X \\ p(t) = at^2 + bt + c & \mapsto & \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}. \end{array}$$

The matrix $A = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$ is in K if and only if the eigenvalues of A, which are given by

$$\lambda_{1/2} = \tfrac{1}{2}(a + c \pm \sqrt{(a+c)^2 + b^2 - 4ac}\,)\,,$$

are both non-negative. Furthermore, one has $0 \leq \lambda_1, \lambda_2$ if and only if (iii) $0 \leq a+c$ and $b^2 - 4ac \leq 0$. Indeed, if $0 \leq \lambda_1, \lambda_2$, then $0 \leq \lambda_1 + \lambda_2 = a + c$. Moreover, the inequality $a + c \geq \sqrt{(a+c)^2 + b^2 - 4ac}$ implies $0 \geq b^2 - 4ac$. On the other hand, assuming (iii), it is straightforward that $0 \leq \lambda_1, \lambda_2$.

The condition (i) in Example 4.5 yields $0 \leq ac$ and hence $0 \leq c$, so (iii) is satisfied. Clearly, (ii) implies (iii) as well. On the other hand, (iii) implies (i) or (ii). So, $p \in P_2(\mathbb{R})$ is positive if and only if $A = j(p) \in K$, i. e. j is a bipositive bijection, and we conclude that the two spaces are order isomorphic. Hence, the Riesz completion of X is again the space Y in Example 4.4.

Example 4.7. Let $S = \{(\xi, \eta)\colon\ \xi^2 + \eta^2 = 1\}$ be the unit circle in $\mathbb{R}^2$ and let X be the space of restrictions to S of all affine functions from $\mathbb{R}^2$ into $\mathbb{R}$, that is,

$$X = \{x : S \to \mathbb{R}\colon\ \exists \mu_1, \mu_2, \mu_3 \in \mathbb{R} \text{ such that } x(\xi, \eta) = \mu_1\xi + \mu_2\eta + \mu_3\ \forall (\xi, \eta) \in S\}.$$

Let K be the cone of functions that are pointwise non-negative on S. This partially ordered vector space (X, K) is described in [3, Example 3.6]. X is directed and Archimedean and, hence, pre-Riesz. X can be considered as a partially ordered subspace of $C(S)$. Due to [3, 3.6 (b)], X is order dense in $C(S)$. So, the lattice subspace of $C(S)$ generated by X is the Riesz completion X^ρ of X. The space (X,K) is isomorphic to the space $(\mathbb{R}^3, K_3)$ of Example 4.5 (and, therefore, to the space $P_2(\mathbb{R})$ of Example 4.4). Indeed, consider the linear bijection $j\colon \mathbb{R}^3 \to X$ defined by

$$(\mu_1, \mu_2, \mu_3)^T \mapsto x(\xi, \eta) = \mu_1 \xi + \mu_2 \eta + \mu_3 .$$

A straightforward calculation yields

$$\min\{x(\xi,\eta)\colon\ (\xi,\eta)\in S\} = \mu_3 - \sqrt{\mu_1^2+\mu_2^2}.$$

Hence x is pointwise non-negative on S if and only if

$$\mu_3 \geq 0 \text{ and } \mu_3^2 \geq \mu_1^2 + \mu_2^2.$$

So, j is bipositive. The spaces Y of Example 4.4 and X^ρ are two different representations of the Riesz completion of X.

In the following example we discuss an embedding of the space $\mathbb{R}^n$, equipped with a finitely generated cone, into a vector lattice.

Example 4.8. Consider the Euclidean space $X = \mathbb{R}^n$ with the scalar product $\langle \cdot, \cdot \rangle$. We identify the space X' of all linear functionals on X with $\mathbb{R}^n$, as usual. A cone K in X is called *finitely generated* if there are finitely many elements $x_1, \ldots, x_r \in X$ such that $K = \operatorname{pos}\{x_1, \ldots, x_r\}$, that is, K equals the positive-linear span of $x_1, \ldots, x_r$. Let K be a generating and finitely generated cone in X, i. e. $r \geq n$. K is closed and hence Archimedean, so (X,K) is pre-Riesz. Moreover, $\operatorname{int}(K) \neq \varnothing$, which implies that the dual wedge $K' = \{f \in X'\colon f(K) \subseteq [0,\infty)\}$ is a cone. For a fixed element $u \in \operatorname{int}(K)$ the set $F = \{x \in K'\colon \langle u, x\rangle = 1\}$ is a base of K'. F has finitely many extreme points $f_1, \ldots, f_k$, where $k \geq n$, and one has $K' = \operatorname{pos}\{f_1, \ldots, f_k\}$. For K we get the representation

$$K = \{x \in \mathbb{R}^n :\ f_i(x) \geq 0 \ \text{ for all } \ i = 1, \ldots, k\}. \tag{4}$$

The map $i\colon \mathbb{R}^n \to \mathbb{R}^k$ given by

$$i\colon\ x \mapsto (f_1(x), \ldots, f_k(x))^T \tag{5}$$

is bipositive and hence injective, i. e. i embeds the partially ordered vector space $(\mathbb{R}^n, K)$ into the vector lattice $(\mathbb{R}^k, \mathbb{R}^k_+)$. (This embedding is sometimes called the *Königstein embedding.*) We consider the space $\mathbb{R}^3$ equipped with a cone K which is generated by four elements. The cone

$$K = \operatorname{pos}\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix} \right\}$$

has a representation (4) with respect to the functionals

$$f_1 = \begin{pmatrix}-1\\-1\\1\end{pmatrix},\ f_2 = \begin{pmatrix}1\\-1\\1\end{pmatrix},\ f_3 = \begin{pmatrix}1\\1\\1\end{pmatrix},\ f_4 = \begin{pmatrix}-1\\1\\1\end{pmatrix}.$$

The embedding (5) yields $(x_1, x_2, x_3)^T \mapsto x_1 b^{(1)} + x_2 b^{(2)} + x_3 b^{(3)}$, where

$$b^{(1)} = \begin{pmatrix}-1\\1\\1\\-1\end{pmatrix}, b^{(2)} = \begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}, b^{(3)} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}.$$

For sake of convenience we use the vectors

$$\tfrac{1}{2}(b^{(3)} - b^{(2)}),\ \tfrac{1}{2}(b^{(1)} + b^{(3)}) \ \text{ and } \ \tfrac{1}{2}(b^{(2)} + b^{(3)})$$

to span $i(\mathbb{R}^3)$, i. e.

$$i(\mathbb{R}^3) = \operatorname{span}\left\{ \begin{pmatrix} 1\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix} \right\}.$$

The linear subspace $i(\mathbb{R}^3)$ of $\mathbb{R}^4$ is order dense in $\mathbb{R}^4$ (with the standard order). Indeed, for

$$x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$$

consider

$$\begin{aligned} u &= x_1 \begin{pmatrix} 1\\1\\0\\0 \end{pmatrix} + (x_2 - x_1) \begin{pmatrix} 0\\1\\1\\0 \end{pmatrix} + (|x_2 - x_1| + |x_3| + |x_4|) \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \\ (x_2-x_1)+|x_2-x_1|+|x_3|+|x_4| \\ |x_2-x_1|+|x_3|+|x_4| \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} v &= (|x_1| + |x_2| + |x_3 - x_4|) \begin{pmatrix} 1\\1\\0\\0 \end{pmatrix} + (x_3 - x_4) \begin{pmatrix} 0\\1\\1\\0 \end{pmatrix} + x_4 \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix} \\ &= \begin{pmatrix} |x_1|+|x_2|+|x_3-x_4| \\ |x_1|+|x_2|+|x_3-x_4|+(x_3-x_4) \\ x_3 \\ x_4 \end{pmatrix}. \end{aligned}$$

Then $u, v \in i(\mathbb{R}^3)$, $u \geq x$, $v \geq x$ and $x = u \wedge v$. So, $i(\mathbb{R}^3)$ is order dense in $\mathbb{R}^4$. Moreover, $(\mathbb{R}^4, i)$ is the Riesz completion of the partially ordered vector space $(\mathbb{R}^3, K)$.

5. Restriction and extension properties

As generalizations of ideals and bands in vector lattices we defined ideals, solvex ideals, (o)-closed ideals and bands in partially ordered vector spaces. These notions are investigated in a pre-Riesz space and are related to the corresponding notions in the Riesz completion. As a pre-Riesz space is an order dense subspace of the Riesz completion, we consider a vector lattice Y and an order dense subspace X of Y. We address a restriction property (R) and an extension property (E) as below, where (P) for a subset of X or Y stands for "being an ideal", "being a band" etc.

(R) If $J \subseteq Y$ has the property (P) in Y, then $J \cap X$ has the property (P) in X.
(E) If $I \subseteq X$ has the property (P) in X, then there is $J \subseteq Y$ such that J has the property (P) in Y and $I = J \cap X$.

The results in the present section will be as follows:

(P)	(R)	(E)
being an ideal	yes, Proposition 5.3	no, Example 5.7
being an (o)-closed ideal	yes, Propositions 5.1 (iii), 5.3	no, Example 5.7
being a solvex ideal	yes, Proposition 5.5 (i)	yes, Proposition 5.6
being a band	no, Example 5.13	yes, Proposition 5.12

We start with a statement concerning (o)-convergence and infer the restriction property for (o)-closed sets.

Proposition 5.1. *Let X be an order dense subspace of a partially ordered vector space Y.*

(i) If D is a subset of X such that the infimum $\inf_X D$ in X exists, then the infimum $\inf_Y D$ in Y exists and equals $\inf_X D$.

(ii) If $\{x_\alpha\}$ is a net in X and $x \in X$ such that $x_\alpha \xrightarrow{(o)} x$ in X, then $x_\alpha \xrightarrow{(o)} x$ in Y.

(iii) If $J \subseteq Y$ is (o)-closed in Y, then $J \cap X$ is (o)-closed in X.

Proof. (i) Put $w = \inf_X D$, so $w \in X \subseteq Y$ and $w \leq d$ for all $d \in D$. Let $v \in Y$ be such that $v \leq d$ for all $d \in D$. Since X is order dense in Y, one has $v = \sup\{x \in X\colon\ x \leq v\}$. Let $x \in X$ be such that $x \leq v$, then $x \leq d$ for all $d \in D$, hence $x \leq \inf_X D = w$. So, w is an upper bound for all $x \in X$ with $x \leq v$, which implies $v \leq w$. Consequently, $w = \inf_Y D$.

(ii) Assume that $x_\alpha \xrightarrow{(o)} x$ in X, i. e. there is a net $\{y_\alpha\} \subset X$ such that $y_\alpha \downarrow 0$ and for all α one has $\pm(x_\alpha - x) \leq y_\alpha$. Then $y_\alpha \downarrow$ in Y and $y_\alpha \geq 0$ for all α. Due to (i), one gets $\inf\{y_\alpha\} = 0$ in Y, so $y_\alpha \downarrow 0$ in Y. Hence, $x_\alpha \xrightarrow{(o)} x$ in Y.

(iii) is an immediate consequence of (ii). □

Observe that Proposition 5.1 (ii) in particular implies that the embedding $i\colon X \to X^\rho$ is (o)-continuous.

In general, the converse statement of (ii) is not true. We give an example where the (o)-convergence in Y does not imply the (o)-convergence in X, even if the limit of the sequence is in X.

Example 5.2. Consider the vector lattice

$$Y = \{y = (y_i)_{i\in\mathbb{Z}} \in l^\infty(\mathbb{Z})\colon\ \lim_{i\to\infty} y_i \ \text{ exists }\}$$

and its subspace

$$X = \{x = (x_i)_{i\in\mathbb{Z}} \in Y\colon\ \sum_{k=1}^{\infty} \frac{x_{-k}}{2^k} = \lim_{i\to\infty} x_i\}\,.$$

For $i \in \mathbb{Z}$ denote $e^{(i)} = (e^{(i)}_j)_{j\in\mathbb{Z}}$ with $e^{(i)}_j = 1$ if $i = j$ and $e^{(i)}_j = 0$ otherwise.

(a) X is order dense in Y.

Let $y=(y_i)_{i\in\mathbb{Z}}\in Y$ and
$$\alpha=\lim_{i\to\infty} y_i-\sum_{k=1}^{\infty}\frac{y_{-k}}{2^k}\,.$$
If $\alpha\geq 0$, put
$$x^{(1)}=y+2\alpha e^{(-1)}\quad\text{and}\quad x^{(2)}=y+4\alpha e^{(-2)}\,.$$
Then $x^{(1)},x^{(2)}\in X$, and $y=x^{(1)}\wedge x^{(2)}$. If $\alpha<0$, then put
$$x^{(m)}=y-\alpha\sum_{i=m}^{\infty}e^{(i)}$$
for all $m\in\mathbb{N}$. One has $x^{(m)}\in X$ for all $m\in\mathbb{N}$, and, moreover, $y=\inf\{x^{(m)}\colon\ m\in\mathbb{N}\}$.

(b) The sequence $(e^{(m)})_{m\in\mathbb{N}}\subset X$ (o)-converges to 0 in Y, but it does not (o)-converge in X.

For $m\in\mathbb{N}$ put $y^{(m)}=\sum_{i=m}^{\infty}e^{(i)}$. Clearly, $y^{(m)}\in Y$ for all $m\in\mathbb{N}$, and $y^{(m)}\downarrow_m 0$ in Y. Since $\pm e^{(m)}\leq y^{(m)}$ for all $m\in\mathbb{N}$, one has $e^{(m)}\xrightarrow{(o)}0$ in Y. We show by way of contradiction that the sequence $(e^{(m)})_{m\in\mathbb{N}}$ does not (o)-converge in X. Assume that there is $v\in X$ such that $e^{(m)}\xrightarrow{(o)}v$ in X. Then $e^{(m)}\xrightarrow{(o)}v$ in Y, hence $v=0$. So, there is a sequence $(v^{(m)})_{m\in\mathbb{N}}\subset X$ such that $v^{(m)}\downarrow_m 0$ and $e^{(m)}\leq v^{(m)}$ for all $m\in\mathbb{N}$. If $i\geq m$, then $v^{(m)}\geq v^{(i)}\geq e^{(i)}$. Hence, $v_i^{(m)}\geq 1$ for all $i\geq m$. This implies $\lim_{i\to\infty}v_i^{(m)}\geq 1$, and, consequently, $\sum_{k=1}^{\infty}\frac{v_{-k}^{(m)}}{2^k}\geq 1$ for all $m\in\mathbb{N}$. We do not have $v_{-k}^{(m)}\downarrow_m 0$ for every $k\in\mathbb{N}\setminus\{0\}$, since otherwise we would get $\sum_{k=1}^{\infty}\frac{v_{-k}^{(m)}}{2^k}\downarrow_m 0$ by monotone convergence. So, there is $k\in\mathbb{N}\setminus\{0\}$ and $\delta>0$ such that $v_{-k}^{(m)}\geq\delta$ for all $m\in\mathbb{N}$. Put $w=\delta e^{(-k)}-2\delta e^{(-k-1)}$, then $w\in X$ and $w\leq\delta e^{(-k)}\leq v^{(m)}$ for all $m\in\mathbb{N}$. Moreover, $w\not\leq 0$, which contradicts $\inf\{v^{(m)}\colon\ m\in\mathbb{N}\}=0$.

Next we state the restriction property for solid sets (cf. also [4, Lemma 5.2]).

Proposition 5.3. *Let Y be a directed partially ordered vector space, X an order dense subspace of Y and J a solid subset of Y. Then the set $J\cap X$ is solid in X.*

Proof. Let $x\in X$ and $y\in J\cap X$ be such that $\{x,-x\}^u\cap X\supseteq\{y,-y\}^u\cap X$. Since Y is directed, the set $\{y,-y\}^u$ is non-empty. If $v\in Y$ is such that $v\geq y$, $v\geq -y$, then
$$\{u\in X\colon\ u\geq v\}\subseteq\{y,-y\}^u\cap X\subseteq\{x,-x\}^u\cap X\,,$$
which implies $v=\inf\{u\in X\colon\ u\geq v\}\geq x$ and, analogously, $v\geq -x$. So, $\{x,-x\}^u\supseteq\{y,-y\}^u$. As J is solid in Y, it follows that $x\in J$, so $x\in J\cap X$. Hence $J\cap X$ is solid in X. □

Corollary 5.4. *Let X be a pre-Riesz space and X^ρ its Riesz completion with the corresponding embedding map i. If J is an ideal in X^ρ, then $\{x\in X\colon\ i(x)\in J\}$ is an ideal in X.*

For solvex sets both the restriction and the extension property are valid.

Proposition 5.5. *Let Y be a directed partially ordered vector space and X an order dense subspace of Y.*

(i) If J is a solvex subset of Y, then $J \cap X$ is solvex in X.
(ii) If I is a solvex subset of X and J is its solvex hull in Y, then $J \cap X = I$.

Proof. (i) Let $x \in X$, $x_1, \ldots, x_n \in J \cap X$ and $\lambda_1, \ldots, \lambda_n \in (0, 1]$ with $\sum_{k=1}^n \lambda_k = 1$ be such that

$$\{x, -x\}^u \cap X \supseteq \left\{ \sum_{k=1}^n \varepsilon_k \lambda_k x_k \colon \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u \cap X \,.$$

If $v \in Y$ is an upper bound of the set $\{\sum_{k=1}^n \varepsilon_k \lambda_k x_k \colon \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}\}$, then

$$\{u \in X \colon u \geq v\} \subseteq \left\{ \sum_{k=1}^n \varepsilon_k \lambda_k x_k \colon \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u \cap X \subseteq \{x, -x\}^u \cap X \,,$$

which implies $v = \inf\{u \in X \colon u \geq v\} \geq x$ and, analogously, $v \geq -x$. So,

$$\{x, -x\}^u \supseteq \left\{ \sum_{k=1}^n \varepsilon_k \lambda_k x_k \colon \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u \,.$$

As J is solvex in Y, it follows that $x \in J$, so $x \in J \cap X$. Hence $J \cap X$ is solvex in X.

(ii) We assume that I is solvex in X. The set J is given by

$$J = \left\{ y \in Y \colon \exists x_1, \ldots, x_n \in I,\ \lambda_1, \ldots, \lambda_n \in (0, 1] \text{ with } \sum_{k=1}^n \lambda_k = 1 \right.$$
$$\left. \text{such that } \{y, -y\}^u \supseteq \left\{ \sum_{k=1}^n \varepsilon_k \lambda_k x_k \colon \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u \right\} \,.$$

Clearly, $I \subseteq J$, so $I \subseteq J \cap X$. It remains to establish $J \cap X \subseteq I$. Let $x \in J \cap X$. Then there are $x_1, \ldots, x_n \in I$ and $\lambda_1, \ldots, \lambda_n \in (0, 1]$ with $\sum_{k=1}^n \lambda_k = 1$ such that

$$\{x, -x\}^u \supseteq \left\{ \sum_{k=1}^n \varepsilon_k \lambda_k x_k \colon \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u \,.$$

Therefore,

$$\{x, -x\}^u \cap X \supseteq \left\{ \sum_{k=1}^n \varepsilon_k \lambda_k x_k \colon \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \right\}^u \cap X \,,$$

and, as I is solvex in X, it follows $x \in I$. ☐

The ideals in X which satisfy the extension property are exactly the solvex ideals.

Proposition 5.6. *Let Y be a vector lattice and let X be an order dense subspace of Y. For an ideal I in X the following two statements are equivalent:*

(i) There is an ideal J in Y such that $I = J \cap X$.
(ii) I is solvex in X.

Proof. (i) $\Rightarrow$ (ii): J is solid and convex, as any ideal in a vector lattice. Hence, J is solvex by Lemma 3.3, so $J \cap X$ is solvex due to Proposition 5.5 (i).

(ii) $\Rightarrow$ (i): Let J be the solvex hull of I in Y. Due to Lemma 3.5, J is an ideal in Y. Further, according to Proposition 5.5 (ii), $J \cap X = I$. □

Since in the vector lattice Y every ideal is solvex, Proposition 5.6 establishes the extension property for solvex ideals.

Next we present an example of a pre-Riesz space X and an ideal I in X such that I is not solvex in X. According to Proposition 5.6 there is no ideal J in the Riesz completion X^ρ of X such that $I = J \cap X$. The ideal I in the example is even (o)-closed.

Example 5.7. We continue Example 4.7, i. e. S is the unit circle in $\mathbb{R}^2$ and X is the space of restrictions to S of all affine functions from $\mathbb{R}^2$ into $\mathbb{R}$, equipped with the pointwise order. Recall that X is pre-Riesz and order dense in $C(S)$. Consider in X the subset

$$I = \{x : S \to \mathbb{R} \colon \text{ there are } \mu_1, \mu_2 \in \mathbb{R} \text{ such that } x(\xi, \eta) = \mu_1 \xi + \mu_2 \eta \ \forall (\xi, \eta) \in S\}.$$

(a) I is an ideal in X.

Clearly, I is a linear subspace of X, so it remains to show that I is solid. Let

$$y = \mu_1 \xi + \mu_2 \eta \in I, \quad y \neq 0.$$

The zeros of y are

$$s_1 = \left(\frac{\mu_2}{\sqrt{\mu_1^2+\mu_2^2}}, \frac{-\mu_1}{\sqrt{\mu_1^2+\mu_2^2}} \right) \quad \text{and} \quad s_2 = \left(\frac{-\mu_2}{\sqrt{\mu_1^2+\mu_2^2}}, \frac{\mu_1}{\sqrt{\mu_1^2+\mu_2^2}} \right).$$

Let $x \in X$ be such that $\{x, -x\}^u \supseteq \{y, -y\}^u$. In the following calculation, $|z|_{C(S)}$ denotes the pointwise absolute value of a function $z \in C(S)$. Due to [3, Example 3.6(b)], we have

$$\begin{aligned} |x(s_1)| &= \inf\{u(s_1) \colon u \in X, u \geq |x|_{C(S)}\} \\ &= \inf\{u(s_1) \colon u \in X, u \geq \pm x\} \\ &\leq \inf\{u(s_1) \colon u \in X, u \geq \pm y\} \\ &= \inf\{u(s_1) \colon u \in X, u \geq |y|_{C(S)}\} \\ &= |y(s_1)| = 0, \end{aligned}$$

so $x(s_1) = 0$, and, similarly, $x(s_2) = 0$. Hence there is $\lambda \in \mathbb{R}$ such that $x = \lambda y$, i. e. $x \in I$.

(b) I is not solvex.

Let J be the smallest ideal in X^ρ which contains I. In particular, for the functions $x_1(\xi,\eta) = \xi$ and $x_2(\xi,\eta) = \eta$, $(\xi,\eta) \in S$, we have $|x_1| \vee |x_2| \in J$. There is $\delta > 0$ such that for all $s \in S$ one has $(|x_1| \vee |x_2|)(s) \geq \delta$, so all constant functions belong to J. Since every element of X^ρ is a bounded function, we obtain $J = X^\rho$. Now, e. g. the function $s \mapsto 1$ $(s \in S)$ is an element of $J \cap X = X$, but does not belong to I. Due to Proposition 5.6, I is not solvex.

(c) I is (o)-closed.

Let $\{x_\alpha\}_\alpha$ be a net in I and $x \in X$ such that $x_\alpha \xrightarrow{(o)} x$ in X, i. e. there is a net $\{y_\alpha\}_\alpha$ in X with $y_\alpha \downarrow 0$ and $\pm(x_\alpha - x) \leq y_\alpha$ for all α. There are $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ such that $x(\xi,\eta) = \mu_1\xi + \mu_2\eta + \mu_3$. We have to show that $x \in I$, which means $\mu_3 = 0$. We will show that y_α converges pointwise to zero and then conclude that x_α converges pointwise to x. It then follows that $\mu_3 = 0$. For each $s \in S$ the net $\{y_\alpha(s)\}_\alpha$ is decreasing in $[0,\infty)$, and hence convergent. Denote

$$y(s) = \lim_\alpha y_\alpha(s)\,.$$

We claim that y is the restriction on S of an affine function on $\mathbb{R}^2$. The affine functions are determined by their values at three distinct points on the circle S. We choose the points $(1,0)$, $(0,1)$, and $(-1,0)$. Every point of S is a linear combination of these points with the sum of the coefficients equal to 1. That is, for $(\xi,\eta) \in S$ there are $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and

$$(\xi,\eta) = \lambda_1(1,0) + \lambda_2(0,1) + \lambda_3(-1,0)\,.$$

Denote

$$\begin{aligned}
\nu_1 &= \tfrac{1}{2}[y(1,0) - y(-1,0)] \\
\nu_2 &= y(0,1) - \tfrac{1}{2}[y(1,0) + y(-1,0)] \\
\nu_3 &= \tfrac{1}{2}[y(1,0) + y(-1,0)]\,.
\end{aligned}$$

Then

$$\begin{aligned}
y(\xi,\eta) &= \lim_\alpha y_\alpha(\xi,\eta) \\
&= \lim_\alpha y_\alpha(\lambda_1(1,0) + \lambda_2(0,1) + \lambda_3(-1,0)) \\
&= \lim_\alpha(\lambda_1 y_\alpha(1,0) + \lambda_2 y_\alpha(0,1) + \lambda_3 y_\alpha(-1,0)) \\
&= \lambda_1 y(1,0) + \lambda_2 y(0,1) + \lambda_3 y(-1,0) \\
&= \lambda_1(\nu_1 + \nu_3) + \lambda_2(\nu_2 + \nu_3) + \lambda_3(\nu_3 - \nu_1) \\
&= \nu_1(\lambda_1 - \lambda_3) + \nu_2\lambda_2 + \nu_3(\lambda_1 + \lambda_2 + \lambda_3) \\
&= \nu_1\xi + \nu_2\eta + \nu_3\,.
\end{aligned}$$

So, y is an affine function, i. e. $y \in X$. One has $0 \leq y$ and $y \leq y_\alpha$ for all α. Since $\inf_\alpha y_\alpha = 0$, it follows that $y = 0$. Hence $\lim_\alpha y_\alpha(\xi,\eta) = 0$ for all $(\xi,\eta) \in S$. So, as $\pm(x(\xi,\eta) - x_\alpha(\xi,\eta)) \leq y_\alpha(\xi,\eta)$, one has $x(\xi,\eta) = \lim_\alpha x_\alpha(\xi,\eta)$. From

$x(1,0) = \mu_1 + \mu_3$ and $x(-1,0) = -\mu_1 + \mu_3$ follows

$$2\mu_3 = x(1,0) + x(-1,0) = \lim_\alpha (x_\alpha(1,0) + x_\alpha(-1,0)) .$$

Since $x_\alpha \in I$ one has $x_\alpha(1,0) = -x_\alpha(-1,0)$, so the last limit is equal to 0. We conclude $x \in I$.

Since in a vector lattice a band is (o)-closed and solvex, we obtain the following consequence of Proposition 5.1 (iii) and Proposition 5.5 (i).

Proposition 5.8. *Let Y be a vector lattice, X an order dense subspace of Y and J a band in Y. Then $J \cap X$ is an (o)-closed solvex ideal in X.*

We recall a preliminary statement concerning disjoint elements.

Proposition 5.9. [4, Proposition 2.1] *Let X and Y be partially ordered vector spaces and let $x, y \in X$.*

(i) If X is a subspace of Y, then $x \perp y$ in Y implies $x \perp y$ in X.

(ii) If X is an order dense subspace of Y, then $x \perp y$ in Y if and only if $x \perp y$ in X.

Let Y be a partially ordered vector space, X an order dense subspace of Y and $M \subseteq X$. Let I and J be the disjoint complements of M in X and Y, respectively. Proposition 5.9 (ii) implies that

$$I = J \cap X . \tag{6}$$

If Y, in addition, is a vector lattice, then J is a band in Y, so, in particular, J is a linear subspace, which implies that I is a linear subspace as well (cf. also [4, Corollary 2.2]). Moreover, J is solvex due to Lemma 3.3. Applying Proposition 5.5 (i), one gets that I is solvex. Finally, since J is (o)-closed, I is (o)-closed as well, according to Proposition 5.1 (iii). For the assertion (v) in the subsequent theorem see [4, Proposition 5.5].

Theorem 5.10. *Let X be a pre-Riesz space, X^ρ its Riesz completion with the according embedding map i, and $M \subseteq X$.*

(i) One has $i(M^d) = i(M)^d \cap i(X)$ in X^ρ.

(ii) The disjoint complement M^d in X is a linear subspace of X.

(iii) M^d is solvex, and hence solid.

(iv) M^d is (o)-closed.

(v) M^d is a band.

For the statement in (ii) the condition on X being pre-Riesz is sufficient, but not necessary, as the next example illustrates.

Example 5.11. Let $X = \mathbb{R}^2$ and $K = \{(x_1, x_2)^T\colon\ 0 < x_1, x_2\} \cup \{(0,0)^T\}$. For each $x \in X \setminus \{(0,0)^T\}$ one has $\{x\}^d = \{(0,0)^T\}$. So, for each set $M \subseteq X$ the disjoint complement M^d is a linear subspace of X. On the other hand, for $y = (1,0)^T$ and $x = z = (0,1)^T$ one has

$$\{y+x, z+x\}^u = \{(x_1, x_2)^T\colon\ 1 < x_1, 2 < x_2\} \subseteq \{(x_1, x_2)^T\colon\ 1 < x_1, x_2\} = \{y, z\}^u ,$$

but $x \notin K$. So, (X, K) is not pre-Riesz.

We justify the extension property for bands in a pre-Riesz space.

Proposition 5.12. *Let Y be a pre-Riesz space, X an order dense subspace of Y and I a band in X. Then there is a band J in Y such that $I = J \cap X$.*

Proof. Let $D = I^d$ in X, so $D^d = (I^d)^d = I$, since I is a band in X. So, I is the disjoint complement of D in X. Let J be the disjoint complement of D in Y. J is a band in Y due to Theorem 5.10 (v), and we observed in (6) that $I = J \cap X$. $\square$

If Y is a vector lattice, X an order dense subspace of Y and B a band in Y, then $B \cap X$ need not be a band in X, i. e. the restriction property for bands does not hold, in general. Moreover, bands need not be directed.

Example 5.13. We continue Example 4.8. In [4, Example 4.6] it is shown that the only directed bands in $(\mathbb{R}^3, K)$ are the subspaces

$$\{0\}\,,\ X\,,\ \operatorname{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}\,,\ \operatorname{span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}\,,\ \operatorname{span}\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}\,,\ \operatorname{span}\left\{\begin{pmatrix}0\\-1\\1\end{pmatrix}\right\}\,.$$

Moreover, there are two non-directed bands, namely

$$\operatorname{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\} \text{ and } \operatorname{span}\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\}\,.$$

Employing the embedding map i given in (5), the linear subspaces

$$\operatorname{span}\left\{\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\right\}\,,\ \operatorname{span}\left\{\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right\}\,,\ \operatorname{span}\left\{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}\,,\ \operatorname{span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\}$$

and

$$\operatorname{span}\left\{\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}\right\}\,,\ \operatorname{span}\left\{\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right\}$$

of $\mathbb{R}^4$ are the non-trivial bands in $i(\mathbb{R}^3)$. Even if $i(\mathbb{R}^3)$ is order dense in $\mathbb{R}^4$, the intersection of a band in $\mathbb{R}^4$ with $i(\mathbb{R}^3)$ need not be a band in $i(\mathbb{R}^3)$. Indeed, consider the band

$$B = \operatorname{span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}$$

in $\mathbb{R}^4$, then

$$B \cap i(\mathbb{R}^3) = \operatorname{span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}\right\}\,,$$

which is not a band in $i(\mathbb{R}^3)$. Due to Proposition 5.8, $B \cap i(\mathbb{R}^3)$ is an (o)-closed solvex ideal in $i(\mathbb{R}^3)$.

As a consequence of Theorem 5.10 one gets the following properties of a band in a pre-Riesz space.

Theorem 5.14. *If X is a pre-Riesz space, then every band in X is an (o)-closed solvex ideal.*

In general, an (o)-closed solvex ideal in a pre-Riesz space need not be a band, see Example 5.13.

If X is an arbitrary pre-Riesz space, then the extension and restriction property both hold for solvex ideals. Hence, a subspace of a pre-Riesz space is the restriction of an ideal in the Riesz completion if and only if it is a solvex ideal. For bands, the extension property is valid, but in general the restriction property fails. For pre-Riesz spaces satisfying an additional condition it can be shown that both the extension and the restriction property for bands hold [5].

6. A note on disjointness

We discuss another notion of disjointness for positive elements in a partially ordered vector space (X, K) which is introduced in [7, Definition 8]. Two elements $x, y \in K$ are called D-disjoint (in symbols, xDy) if the condition

$$[0, x] \cap [0, y] = \{0\}$$

holds. If X is pre-Riesz and $x, y \in K$ with $x \perp y$, then $x \perp y$ in X^ρ, so $[0, x] \cap [0, y] = \{0\}$ in X^ρ, which yields xDy. If X has, in addition, the Riesz decomposition property, we obtain the following.

Proposition 6.1. *Let (X, K) be a pre-Riesz space which has the Riesz decomposition property, and let $x, y \in K$. Then $x \perp y$ if and only if xDy.*

Proof. Let X have the Riesz decomposition property and let $x, y \in K$ be such that xDy holds. Since x and y are positive, one has

$$\{x + y, -x - y\}^u = \{x + y\}^u \subseteq \{x - y, -x + y\}^u .$$

We show the converse inclusion. Let $z \in \{x - y, -x + y\}^u$, then $0 \leq x \leq z + y$, so there are elements $x_1, x_2 \in K$ with $x = x_1 + x_2$, $0 \leq x_1 \leq z$ and $0 \leq x_2 \leq y$. Now $0 \leq x_2 \leq x$ in combination with the D-disjointness of x and y implies $x_2 = 0$, i. e. $z \geq x$. Analogously, we get $z \geq y$. Now, $0 \leq x \leq z = (z - y) + y$, so $x = x_3 + x_4$ with $0 \leq x_3 \leq z - y$ and $0 \leq x_4 \leq y$; from $0 \leq x_4 \leq x$ we get $x_4 = 0$ and so $x \leq z - y$, which shows that $z \in \{x + y\}^u$. So, $x \perp y$. □

We remark that in general the disjointness and the D-disjointness for two positive elements differ. In view of Example 4.5, for $x = (1, 0, 1)^T$ the only positive element which is disjoint to x is 0, whereas the elements $y = \lambda(y_1, y_2, 1)^T$ with $y_1^2 + y_2^2 = 1$, $y_1 \neq 1$, $\lambda \geq 0$, are all the (positive) elements D-disjoint to x. Observe that the set of all such elements y is not convex, so fundamental properties as in Theorem 5.10 are in general not obtained for D-disjointness. Anyway, in pre-Riesz spaces with the Riesz decomposition property disjointness and D-disjointness for positive elements coincide and the advantages of both notions are on hand.

Acknowledgment. The authors thank the referee for pointing out some improvements. O. van Gaans acknowledges the support by a 'VIDI subsidie' (639.032.510) of the Netherlands Organisation for Scientific Research (NWO).

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