Bands in Pervasive Pre-Riesz Spaces

O. van Gaans and A. Kalauch

Abstract

Pre-Riesz spaces are partially ordered vector spaces which are order dense subspaces of vector lattices. A band in a pre-Riesz space can be extended to a band in the ambient vector lattice. The corresponding restriction property does not hold in general. We provide sufficient conditions on the underlying space such that the restriction property for bands holds. As an application, we consider the space $L^r(l_0^\infty)$ of all regular operators on the space l_0^∞ of all finally constant sequences. We establish that $L^r(l_0^\infty)$ is pre-Riesz and that its subspace of all order continuous operators is a band in $L^r(l_0^\infty)$.

Subject Classification (MSC2000): 46A40, 06F20

Key words: band, disjointness, order dense subspace, partially ordered vector space, pre-Riesz space, regular operator, Riesz completion, vector lattice

1 Introduction

The notions of ideal and band in a partially ordered vector space have been investigated in [6]. The method used there is embedding of the space as an order dense subspace of a vector lattice. The spaces that allow such an embedding are the pre-Riesz spaces. It has been studied whether ideals or bands in a pre-Riesz space can be extended to ideals or bands in the ambient vector lattice and whether the restriction to the pre-Riesz space of an ideal or band in the vector lattice is an ideal or band in the pre-Riesz space. An appropriate generalization of ideal has been established such that both extension and restriction works in arbitrary pre-Riesz spaces. For bands extension always works, whereas restriction may fail. In this paper we give a condition on the pre-Riesz space such that both extension and restriction hold for bands. We show that the condition is satisfied by a space of operators.

We begin by fixing our notations and terminology. Let X be a real vector space and let K be a cone in X, that is, K is a wedge ($x, y \in K$, $\lambda, \mu \geq 0$ imply $\lambda x + \mu y \in K$) and $K \cap (-K) = \{0\}$. In X a partial order is introduced by defining $y \geq x$ if and only if $y - x \in K$. A set $M \subset X$ is called *order bounded* if there are $y, z \in X$ such that $y \leq x \leq z$ for all $x \in M$. Denote for a subset $M \subseteq X$ the set of all upper bounds by

$$M^u = \{x \in X\colon\ x \geq m \ \text{ for all } \ m \in M\}\,.$$

We denote the natural numbers by $\mathbb{N}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The space (X, K) is called *Archimedean* if for every $x, y \in X$ with $nx \leq y$ for all $n \in \mathbb{N}_0$ one has $x \leq 0$. A set $M \subseteq X$ is called *directed* if for every $x, y \in M$ there is an element $z \in M$ such that $z \geq x$ and $z \geq y$. X is directed if and only if the cone K is *generating* in X, that is, $X = K - K$. X has the *Riesz decomposition property* if for every $y, x_1, x_2 \in K$ with $y \leq x_1 + x_2$ there exist $y_1, y_2 \in K$ such that $y = y_1 + y_2$ and $y_1 \leq x_1$, $y_2 \leq x_2$.

For standard notations in the case that X is a vector lattice see [3]. Recall that a vector lattice is *Dedekind complete* whenever every non-empty subset that is bounded from above has a supremum, and *σ-Dedekind complete* whenever every countable subset that is bounded from above has a supremum.

A partially ordered vector space X is called *pre-Riesz* if for every $x, y, z \in X$ the inclusion $\{x + y, x + z\}^u \subseteq \{y, z\}^u$ implies $x \in K$ [7, Definition 1.1(viii), Theorem 4.15]. Every pre-Riesz space is directed and every directed Archimedean partially ordered vector space is pre-Riesz [7]. Clearly, each vector lattice is pre-Riesz.

By a subspace of a partially ordered vector space or a vector lattice we mean an arbitrary linear subspace with the inherited order. We do not require it to be a lattice or a sublattice. We say that a subspace X of a vector lattice Y *generates* Y as a vector lattice if for every $y \in Y$ there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in X$ such that $y = \bigvee_{i=1}^m a_i - \bigvee_{i=1}^n b_i$.

We call a linear subspace D of a partially ordered vector space X *order dense* in X if for every $x \in X$ we have $x = \inf\{y \in D\colon\ y \geq x\}$, that is, each x is the greatest lower bound of the set $\{y \in D\colon\ y \geq x\}$ in X [4, p. 360].

Recall that a linear map $i\colon X \to Y$, where X and Y are partially ordered vector spaces, is called *bipositive* if for every $x \in X$ one has $i(x) \geq 0$ if and only if $x \geq 0$. An embedding map is required to be linear and bipositive, which implies injectivity.

Let X be a partially ordered vector space. The following statements are equivalent [7, Corollaries 4.9–11 and Theorems 3.5, 3.7, 4.13]:

(i) X is pre-Riesz.

(ii) There exist a vector lattice Y and a bipositive linear map $i\colon\ X \to Y$ such that $i(X)$ is order dense in Y.

(iii) There exist a vector lattice Y and a bipositive linear map $i\colon\ X \to Y$ such that $i(X)$ is order dense in Y and generates Y as a vector lattice.

All spaces Y as in (iii) are isomorphic as vector lattices. A pair (Y, i) as in (iii) is called a *Riesz completion* of X. As it is unique up to isomorphism we will speak of *the* Riesz completion of X and denote it by X^ρ.

Let X be a partially ordered vector space. The elements $x, y \in X$ are called *disjoint*, in symbols $x \perp y$, if $\{x + y, -x - y\}^u = \{x - y, -x + y\}^u$, cf. [5]. If X is a vector lattice, then this notion of disjointness coincides with the usual one, cf. [3, Theorem 1.4(4)]. The *disjoint complement* of a subset $M \subseteq X$ is the set $M^d = \{y \in X\colon\ y \perp x \text{ for all } x \in M\}$.

Proposition 1.1. [5, Proposition 2.1] *Let X and Y be partially ordered vector spaces and let $x, y \in X$.*

(i) *If X is a subspace of Y, then $x \perp y$ in Y implies $x \perp y$ in X.*

(ii) *If X is an order dense subspace of Y, then $x \perp y$ in Y if and only if $x \perp y$ in X.*

A linear subspace M of a partially ordered vector space X is called a *band* in X if

$$(M^d)^d = M\,,$$

cf. [5, Definition 5.4]. If X is an Archimedean vector lattice, then this notion of a band coincides with the usual one. For every subset M of X, the disjoint complemet M^d is a band [5, Proposition 5.5].

Let Y be a vector lattice and X an order dense subspace of Y. In [6] the following restriction property (R) and extension property (E) for bands are considered:

(R) If J is a band in Y, then $J \cap X$ is a band in X.

(E) If I is a band in X, then there exists a band J in Y such that $I = J \cap X$.

The extension property (E) for bands is shown in [6, Proposition 3.15]. The restriction property (R) for bands is not true in general [6, Example 4.16]. In the present paper, we provide sufficient conditions on X such that (R) is satisfied. We verify (R) for several spaces, in particular for a space of operators.

We conclude the section by collecting standard definitions concerning spaces of operators. Let X be a partially ordered vector space and $L(X)$ the set of all linear operators on X. As usual, an operator $T \in L(X)$ is called *positive* whenever $T(K) \subseteq K$; we write $S \geq T$ if $S - T$ is positive. An operator $T \in L(X)$ is called *order bounded* if T maps order bounded subsets into order bounded subsets, and *regular* whenever T can be written as a difference of two positive operators. The set of all order bounded operators is denoted by $L^b(X)$, whereas the set of all regular operators is denoted by $L^r(X)$.

A net $\{x_\alpha\} \subseteq X$ is said to be decreasing (in symbols, $x_\alpha \downarrow$), whenever $\alpha \geq \beta$ implies $x_\alpha \leq x_\beta$. For $x \in X$ the notation $x_\alpha \downarrow x$ means that $x_\alpha \downarrow$ and $\inf_\alpha\{\, x_\alpha\} = x$ both hold. We say that a net $\{x_\alpha\}_\alpha \subset X$ *(o)-converges* to $x \in X$ (in symbols, $x_\alpha \xrightarrow{(o)} x$), if there is a net $\{y_\alpha\}_\alpha \subset X$ such that $y_\alpha \downarrow 0$ and for all α one has $\pm(x_\alpha - x) \leq y_\alpha$. The equivalence of $x_\alpha \xrightarrow{(o)} x$ and $x_\alpha - x \xrightarrow{(o)} 0$ is obvious. If a net (o)-converges, then the limit is unique.

An operator $T \in L^b(X)$ is called *(o)-continuous* if for each net $\{x_\alpha\}_{\alpha\in A} \subset X$ with $x_\alpha \xrightarrow{(o)} 0$ follows $T(x_\alpha) \xrightarrow{(o)} 0$. Clearly, the (o)-continuity of T implies $T(x_\alpha) \xrightarrow{(o)} T(x)$ for every net $\{x_\alpha\} \subset X$ with $x_\alpha \xrightarrow{(o)} x \in X$. Moreover, if T is positive, then T is (o)-continuous if and only if $x_\alpha \downarrow 0$ implies $T(x_\alpha) \downarrow 0$. Denote the set of all (o)-continuous operators in $L^b(X)$ by $L^n(X)$.

If X is a Dedekind complete vector lattice, then $L^b(X)$ is a Dedekind complete vector lattice as well, which implies $L^b(X) = L^r(X)$ [3, Theorem 1.13]. Moreover, $L^n(X)$ is a band in $L^b(X)$ (Ogasawara's theorem, see [3, Theorem 4.4]). We are interested in this statement if X is not Dedekind complete. We consider an example of a vector lattice X, where X is not Dedekind complete, $L^b(X)$ is pre-Riesz, and where $L^n(X)$ turns out to be a band in $L^b(X)$. For the proof we calculate the Riesz completion of $L^b(X)$ and show that $L^n(X)$ can be obtained as the restriction of a band in the Riesz completion.

2 The restriction property for bands

In the present section, we study (R) for a pre-Riesz space X and its Riesz completion $Y = X^\rho$. In certain pre-Riesz spaces the restriction property (R) is trivially satisfied.

Example 2.1. We refer to [6, Example 3.4], where subspaces of the space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$ are considered, ordered by the natural cone

$$\{x \in C(\mathbb{R})\colon\ x(t) \geq 0 \ \text{ for all } \ t \in \mathbb{R}\}\,.$$

Let $X = P_2(\mathbb{R})$ be the ordered vector space of all real polynomial functions on $\mathbb{R}$ of at most degree 2. X is pre-Riesz, and its Riesz completion is given by

$$\begin{aligned} Y = \{y \in C(\mathbb{R})\colon\ & y \text{ is a piecewise polynomial function of at most degree 2} \\ & \text{with } L(y) = R(y)\,\}\,, \end{aligned}$$

where

$$L(y) = \lim_{t\to-\infty} y(t)/t^2 \text{ and } R(y) = \lim_{t\to\infty} y(t)/t^2.$$

Let B be a non-trivial band in Y and let $x \in B \cap X$. Since $B^d \neq \{0\}$, there is $y \in Y$, $y \neq 0$, such that $x \perp y$. This implies $x = 0$. Consequently, $B \cap X = \{0\}$, which is a band in X.

In general, the restriction property (R) for bands does not hold. We continue with an example of a sequence space which turns out to be pre-Riesz, calculate its Riesz completion, and define a band in the Riesz completion that does not restrict to a band in the pre-Riesz space.

Example 2.2. Consider the space $X = c$ of all convergent sequences and denote $c^+ = \{(x_n)_{n\in\mathbb{N}} \in c\colon\ x_n \geq 0 \ \text{ for all } n \in \mathbb{N}\}$. Consider the linear functional f on X given by

$$f(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^n} - \lim_{n\to\infty} x_n \ \text{ for each } x = (x_n)_{n\in\mathbb{N}} \in X\,.$$

Let X be ordered by the cone $K = \{x \in c^+\colon\ f(x) \geq 0\}$. X is not a lattice, since e. g. for the elements $u = (u_n)_{n\in\mathbb{N}}$ with $u_n = 1$ for all $n \in \mathbb{N}$ and $v = (v_n)_{n\in\mathbb{N}}$ with $v_1 = 1$ and $v_n = 0$ for all $n \geq 2$ the infimum does not exist. Indeed, assume that w is the infimum of u and v. On one hand, w is coordinatewise less or equal v, which yields $w_1 \leq 1$ and $w_n \leq 0$ for all $n \geq 2$. On the other hand, w is coordinatewise equal or greater than each lower bound of u and v. Since obviously $f(u) = 0$, $f(v) = \frac{1}{2}$ and $f(0) = 0$, one gets $0 \leq u, v$. For the sequence $z = (z_n)_{n\in\mathbb{N}}$ with $z_1 = 1$, $z_2 = -2$ and $z_n = 0$ for all $n \geq 3$ one has $f(z) = 0$, so $z \leq u, v$. Since w is coordinatewise greater than 0 and z, one gets $w_1 = 1$ and $w_n = 0$ for all $n \geq 2$, thus $w = v$. From $f(v) \not\leq f(u)$ follows $w \not\leq u$, a contradiction.

The space $Y = c \times \mathbb{R}$, ordered by the cone $Y^+ = \{(x, r) \in Y : x \in c^+, r \geq 0\}$, is a vector lattice, and the mapping $i\colon X \to Y$, $i(x) = (x, f(x))$ for each $x \in X$, is linear and bipositive. We show that $i(X)$ is order dense in Y and that $i(X)$ generates Y as a vector lattice, which implies that X is a pre-Riesz space and Y is its Riesz completion.

Let $y = (x, r) \in Y$, i. e. $x = (x_n)_{n\in\mathbb{N}} \in X$. Consider the first case $f(x) \leq r$. Put

$$v_n = \begin{cases} x_1 + 2(r - f(x)) & \text{if } n = 1 \\ x_n & \text{otherwise} \end{cases}, \qquad w_n = \begin{cases} x_2 + 4(r - f(x)) & \text{if } n = 2 \\ x_n & \text{otherwise} \end{cases}.$$

The sequences $v = (v_n)_{n\in\mathbb{N}}$ and $w = (w_n)_{n\in\mathbb{N}}$ are elements of X. Furthermore, $f(v) = f(x) + (r - f(x)) = r$ and, analogously, $f(w) = r$. So, $y \leq i(v)$, $y \leq i(w)$, and moreover, $y = i(v) \wedge i(w) = -(i(-v) \vee i(-w))$.

Consider the second case $f(x) > r$. Then, due to the first case, there are $\hat{v}, \hat{w} \in X$ such that $-y = -(i(\hat{v}) \vee i(\hat{w}))$, i. e. $y = i(\hat{v}) \vee i(\hat{w})$. So, $i(X)$ generates Y as a vector lattice.

In the case $f(x) > r$ it remains to show that the element $y = (x, r)$ is the infimum of elements of $i(X)$. For each $m \in \mathbb{N}$ consider the sequence $z^{(m)} = (z_n^{(m)})_{n\in\mathbb{N}}$ with $z_n^{(m)} = 0$ for all $n < m$ and $z_n^{(m)} = 1$ otherwise. Clearly, $z^{(m)} \in X$, and since $\lim_{n\to\infty} z_n^{(m)} = 1$, one has

$$f(z^{(m)}) = \sum_{n=m}^{\infty} \frac{1}{2^n} - 1 = \frac{1}{2^{m-1}} - 1\,.$$

For each $m \in \mathbb{N}$ put $u^{(m)} = x + (f(x) - r)z^{(m)}$, then $u^{(m)} \in X$, $u^{(m)} \geq x$ and

$$f(u^{(m)}) = f(x) + (f(x) - r)f(z^{(m)}) = r + \frac{1}{2^{m-1}}(f(x) - r) \geq r\,,$$

so for each $m \in \mathbb{N}$ one has $(u^{(m)}, f(u^{(m)})) \geq (x, r) = y$. Let $(v, s) \in Y$ be an arbitrary lower bound of the set $\{(u^{(m)}, f(u^{(m)}))\colon\, m \in \mathbb{N}\}$. Then v is elementwise less or equal x, moreover $s \leq f(u^{(m)})$ for all $m \in \mathbb{N}$, so $s \leq r$, which implies $(v, s) \leq (x, r) = y$. So,

$$y = \inf\{i(u^{(m)})\colon\, m \in \mathbb{N}\}\,.$$

Consequently, $i(X)$ is order dense in Y.

The set $B = \{(x, r) \in Y : r = 0\}$ is a band in Y. Let $B' = B \cap X = \{x \in c\colon f(x) = 0\}$. As $\mathbb{1} = (1, 1, 1, \ldots) \in B'$, due to Proposition 1.1 one gets

$$(B')^d = \{v \in c\colon (v, f(v)) \perp (b, f(b)) \text{ in } Y \text{ for all } b \in B'\} = \{0\}\,.$$

Hence, $(B')^{dd} = c \neq B'$, so B' is not a band in X. Thus, the restriction property (R) for bands is not satisfied.

We provide conditions on a pre-Riesz space X which ensure that the restriction property (R) for bands holds.

Definition 2.3. A pre-Riesz space X is called *pervasive*, if for each element $y \in X^\rho$, $y \geq 0$, $y \neq 0$, there is $x \in X$, $x \neq 0$, such that $0 \leq x \leq y$.

Lemma 2.4. *If X is a pervasive pre-Riesz space, then for every $y \in X^\rho$, $y \geq 0$, there is $S \subseteq X$ such that $\{y\}^d = S^d$ in X^ρ.*

Proof. Let $y \in X^\rho$, $y \geq 0$. We assume $y \neq 0$. Let $S = \{x \in X : 0 \leq x \leq y\}$. On one hand, if $z \in X^\rho$ is such that $|z| \wedge y = 0$, then $0 \leq |z| \wedge x \leq |z| \wedge y = 0$ for all $x \in S$, hence $\{y\}^d \subseteq S^d$ in X^ρ.

On the other hand, assume that there exists $z \in S^d \setminus \{y\}^d$. Put $w = |z| \wedge y$, then $w \in X^\rho$, $w \geq 0$, $w \neq 0$. Since X is pervasive, there is $x \in X$, $x \neq 0$, such that $0 \leq x \leq w$. As $w \leq y$ we obtain $x \in S$ and therefore $x \wedge |z| = 0$. Hence,

$$x = x \wedge x \leq x \wedge w \leq x \wedge |z| = 0\,,$$

which is a contradiction. We conclude $\{y\}^d = S^d$ in X^ρ. □

Proposition 2.5. *Let X be a pre-Riesz space such that for every $y \in X^\rho$, $y \geq 0$, there is $S \subseteq X$ such that $\{y\}^d = S^d$. Then the restriction property (R) for bands holds.*

Proof. Let B be a band in X^ρ and $B' = B \cap X$. We have to show $(B')^{dd} = B'$ in X. Since $B' \subseteq (B')^{dd}$ is obvious, it suffices to establish $(B')^{dd} \subseteq B'$. Let $x \in (B')^{dd}$ in X. We show $x \in B^{dd}$ in X^ρ. Indeed, let $y \in B^d$ in X^ρ. Then $|y| \in X^\rho$, $|y| \geq 0$. So there is a set $S \subseteq E$ such that $\{|y|\}^d = S^d$ in X^ρ. Let $z \in B' \subseteq B$. Then $y \perp z$ in X^ρ, so $z \in \{|y|\}^d = S^d$ in X^ρ. So, for any $s \in S$ one has $z \perp s$ in X^ρ, which implies $z \perp s$ in X due to Proposition 1.1. Hence $z \in S^d$ in X, and one gets $B' \subseteq S^d$ in X. Therefore $(B')^{dd} \subseteq S^{ddd} = S^d$ in X. So $x \in S^d$ in X. Due to Proposition 1.1, one has $x \in S^d$ in X^ρ, and, hence, $x \in \{|y|\}^d$ in X^ρ. Consequently, $x \perp y$ in X^ρ. As y was an arbitrary element of B^d in X^ρ, we obtain $x \in B^{dd} = B$ in X^ρ. Thus, $x \in B \cap E = B'$. We conclude $(B')^{dd} \subseteq B'$ in X. □

Observe that the condition in Proposition 2.5 is not necessary. Indeed, referring to Example 2.1, for $S \subseteq X$ one has either $S^d = \{0\}$ or $S^d = Y$. On the other hand, there are $y \in Y$ with $\{y\}^d$ non-trivial. Nevertheless, (R) is satisfied.

We combine Lemma 2.4 and Proposition 2.5.

Theorem 2.6. *In a pervasive pre-Riesz space the restriction property (R) for bands holds.*

Observe that the pre-Riesz space in Example 2.2 is not pervasive.

Example 2.7. Consider the space $X = C^k[0,1]$ of k times continuously differentiable functions on $[0,1]$, equipped with the pointwise ordering. X is order dense in $C[0,1]$, and the Riesz completion of X is given by

$$Y = \{y \in C[0,1] \colon\ y \text{ is a piecewise } k \text{ times continuously differentiable function}\}\,.$$

(A detailed argumentation can be given analogously to [6, Example 3.4].) X is pervasive, so the restriction property (R) for bands holds. An analogous statement is satisfied for the space $C^\infty[0,1]$.

Remark 2.8. For a linear subspace D of a partially ordered vector space X we compare order denseness and the property

(p) $\forall x \in X$, $x \geq 0$, $x \neq 0$ $\exists y \in D$: $0 \leq y \leq x$, $y \neq 0$,

which appears in Definition 2.3.

(i) Example 2.2 shows that in general order denseness does not imply (p). Vice versa, (p) does in general not imply order denseness. Indeed, let $X = \mathbb{R}^2$ be ordered by the cone

$$K = \{(x_1, x_2)\colon\ x_1 > 0, \text{ or } x_1 = 0 \text{ and } x_2 \geq 0\}.$$

Let $D = \{(0, x_2)\colon\ x_2 \in \mathbb{R}\}$. Then $D \cap K = \{(0, x_2)\colon\ x_2 \geq 0\}$. On one hand, if $x = (x_1, x_2) \in X$ is such that $x \geq 0$ and $x \neq 0$, then either $x_1 > 0$ so that $(0,0) \leq (0,1) \leq (x_1, x_2)$ with $(0,1) \in D$, or $x_2 = 0$ and $x_1 > 0$ so that $x \in D$. Hence (p) holds. On the other hand, D is not order dense, since for $x = (1,0)$ we have $y \leq x$, $y \neq x$ for all $y \in D$.

(ii) If X is a vector lattice and D is an order dense sublattice of X then (p) holds. Indeed, let $x \in X$, $x \geq 0$, $x \neq 0$. Then $-x = \inf\{y \in D\colon\ y \geq -x\}$, so that there exists a $y \in D$ with $y \geq -x$ and $y \not\geq 0$, so $y^- \neq 0$. Then $0 \wedge y \geq 0 \wedge (-x) = -x$ and since $y^- = -(y \wedge 0)$ we have $-y^- \geq -x$ and therefore $0 \leq y^- \leq x$. So (p) holds.

(iii) If X is an Archimedean vector lattice and D is a majorizing subspace of X, then (p) implies that D is order dense in X. For a proof, suppose that D is not order dense in X. Then there exists an $x \in X$ such that $x \neq \inf\{y \in D\colon\ y \geq x\}$. That is, there exists a $v \in X$ such that $v \leq y$ for all $y \in D$ with $y \geq x$, but $v \not\leq x$. Put $w = v \wedge x \in X$. Then $w \leq y$ for all $y \in D$ with $y \geq x$, and we have $0 \leq w - x$ and $w - x \neq 0$. Since (p) holds, there exists a $u \in D$, $u \neq 0$, with $0 \leq u \leq w - x$, hence $w - u \geq x$. Since D is majorizing, there exists a $y_0 \in D$ with $y_0 \geq w$. Then $y_1 = y_0 - u \geq w - u \geq x$, where $y_1 \in D$ and $y_1 \geq w$. Inductively, we define $y_n = y_{n-1} - u = y_0 - nu \in D$ with $y_n \geq w - u \geq x$ and therefore $y_n \geq w$, where $n \in \mathbb{N}$. Then $y_0 - w \geq nu$ for all $n \in \mathbb{N}$. As X is Archimedean we obtain that $u \leq 0$, which is a contradiction.

(iv) If X is an Archimedean vector lattice and D is a sublattice, then D is order dense in X if and only if (p) holds (cf. [3, Theorem 3.1]).

3 A space of operators

In [2] the space l_0^∞ of all real sequences which are constant except for a finite number of terms is investigated. This vector lattice is not Dedekind complete. In [2, Theorem 4.1] it is established that $L^r(l_0^\infty) = L^b(l_0^\infty)$. The space $L^r(l_0^\infty)$ does not satisfy the Riesz decomposition property [2, Theorem 5.1], so $L^r(l_0^\infty)$ is not a vector lattice. Since every (o)-continuous operator is automatically order bounded (see [1, Theorem 2.1]), the space of (o)-continuous operators $L^n(l_0^\infty)$ is a subspace of $L^r(l_0^\infty)$. We show that $L^n(l_0^\infty)$ is a band in $L^r(l_0^\infty)$. Indeed, we establish that $L^r(l_0^\infty)$ is a pre-Riesz space, calculate its Riesz completion, show that $L^r(l_0^\infty)$ is pervasive and apply the restriction property for bands.

In view of the subsequent example, we fix some notations. The space of all real sequences which are zero except for a finite number of terms is denoted by c_{00}. Let X be a vector space with a countable algebraic basis $\mathcal{B} = (b^{(i)})_{i\in\mathbb{N}_0}$, i. e. for every $x \in X$ there is a unique sequence $\xi = (\xi_i)_{i\in\mathbb{N}_0} \in c_{00}$ such that

$$x = \sum_{i=0}^{\infty} \xi_i b^{(i)} .$$

For $i \in \mathbb{N}$ define $f^{(i)}\colon X \to \mathbb{R}$ by $f^{(i)}(x) = \xi_i$, so one can write

$$x = \sum_{i=0}^{\infty} f^{(i)}(x) b^{(i)} .$$

Let $A\colon X \to X$ be a linear operator. We denote the matrix representation of A with respect to $\mathcal{B}$ by $\hat{A}$, i. e.

$$\hat{A} = (a_{ij})_{i,j\in\mathbb{N}_0} \quad \text{with} \quad a_{ij} = f^{(i)}(Ab^{(j)}) . \tag{1}$$

By definition, for each $j \in \mathbb{N}_0$ one has $(a_{ij})_{i\in\mathbb{N}_0} \in c_{00}$. Conversely, every matrix $(a_{ij})_{i,j\in\mathbb{N}_0}$ with $(a_{ij})_{i\in\mathbb{N}_0} \in c_{00}$ for all $j \in \mathbb{N}_0$ corresponds to a linear operator $A\colon X \to X$. We view $\hat{A}$ as a linear operator on c_{00}, where

$$\hat{A}\colon c_{00} \to c_{00} .$$

Example 3.1. Let $X = l_0^\infty$ be the vector lattice of all eventually constant sequences, i. e.

$$l_0^\infty = \{(x_i)_{i\in\mathbb{N}}\colon \text{ there is } \beta \in \mathbb{R} \text{ and } k \in \mathbb{N} \text{ such that } x_i = \beta \text{ for all } i > k\} ,$$

equipped with the coordinatewise order. The cone in X is denoted by X_+. For every $j \in \mathbb{N}$ denote

$$e^{(j)} = (x_i)_{i\in\mathbb{N}} \quad \text{with} \quad x_j = 1 \quad \text{and} \quad x_i = 0 \quad \text{for all} \quad i \neq j .$$

Moreover, denote $\mathbb{1} = (1)_{i\in\mathbb{N}}$. The set

$$\mathcal{B} = \{\mathbb{1}, e^{(1)}, e^{(2)}, \ldots\}$$

is an algebraic basis of X, where for $x = (x_i)_{i\in\mathbb{N}} \in X$ one has

$$\begin{aligned} f^{(0)}(x) &= \lim_{i\to\infty} x_i \\ \text{and} \quad f^{(j)}(x) &= x_j - \lim_{i\to\infty} x_i \quad \text{for all } j \geq 1 . \end{aligned} \tag{2}$$

For a linear operator $A\colon X \to X$ we have the matrix representation $\hat{A}$ according to (1).

Next we address the issue of order. Define

$$K = \{\xi = (\xi)_{i\in\mathbb{N}_0} \in c_{00}\colon \ \xi_0 \geq 0, \ \xi_i + \xi_0 \geq 0 \quad \text{for all} \quad i \in \mathbb{N}\} ,$$

then one has

$$x \in X_+ \quad \text{if and only if} \quad (f^{(i)}(x))_{i\in\mathbb{N}_0} \in K .$$

So, A is positive if and only if $\hat{A}$ is positive in the space (c_{00}, K). We characterize the positivity of $\hat{A}$.

(a) $\hat{A} = (a_{ij})_{i,j\in\mathbb{N}_0}$ is positive in (c_{00}, K) if and only if for all $i \in \mathbb{N}$ the both conditions

(i) $a_{0j} + a_{ij} \geq 0$ for all $j \in \mathbb{N}$, and

(ii) $a_{00} + a_{i0} \geq \sum_{j=1}^{\infty}(a_{0j} + a_{ij})$

are satisfied.

For a proof, assume first that $\hat{A}$ is positive in (c_{00}, K). Denote $\epsilon^{(j)} = (\epsilon_i)_{i\in\mathbb{N}_0}$ with $\epsilon_j = 1$ and $\epsilon_i = 0$ for all $i \neq j$. Since $\epsilon^{(j)} \in K$, we have $\hat{A}\epsilon^{(j)} \in K$ for all $j \in \mathbb{N}_0$, which means that (i) holds and that

(iii) $a_{0j} \geq 0$ for all $j \in \mathbb{N}_0$.

Further, let $N \in \mathbb{N}$ and consider $\xi = (\xi_i)_{i\in\mathbb{N}_0} \in K$ defined by $\xi_0 = 1$, $\xi_i = -1$ for $1 \leq i \leq N$, and $\xi_i = 0$ for $i > N$. As $\hat{A}\xi \in K$ we obtain $a_{00} \geq \sum_{j=1}^{N} a_{0j}$ and

$$a_{00} + a_{i0} \geq \sum_{j=1}^{N}(a_{0j} + a_{ij}) \quad \text{for all } i \in \mathbb{N}.$$

We infer (ii) and

(iv) $a_{00} \geq \sum_{j=1}^{\infty} a_{0j}$.

Conversely, assume that (i) and (ii) hold and let $\xi \in K$. For every $i \in \mathbb{N}$ we have

$$\begin{aligned} (\hat{A}\xi)_i + (\hat{A}\xi)_0 &= (a_{i0} + a_{00})\xi_0 + \sum_{j=1}^{\infty}(a_{ij} + a_{0j})\xi_j \\ &\geq (a_{i0} + a_{00})\xi_0 - \sum_{j=1}^{\infty}(a_{ij} + a_{0j})\xi_0 \geq 0. \end{aligned}$$

Since the columns of $\hat{A}$ are eventually zero, we see that (iii) and (iv) follow from (i) and (ii). Hence

$$(\hat{A}\xi)_0 = a_{00}\xi_0 + \sum_{j=1}^{\infty} a_{0j}\xi_j \geq a_{00}\xi_0 - \sum_{j=1}^{\infty} a_{0j}\xi_0 \geq 0.$$

Thus $\hat{A}$ is positive.

(b) $\hat{A} = (a_{ij})_{i,j\in\mathbb{N}_0}$ is regular with respect to (c_{00}, K) if and only if the sequence of absolute row sums $(\sum_{j=1}^{\infty} |a_{ij}|)_{i\in\mathbb{N}_0}$ is bounded.

Indeed, assume that $\hat{A}$ is regular. Then there exist positive operators G and H on (c_{00}, K) with $G = (g_{ij})_{i,j\in\mathbb{N}_0}$ and $H = (h_{ij})_{i,j\in\mathbb{N}_0}$ such that $\hat{A} = G - H$. Define $B = (b_{ij})_{i,j\in\mathbb{N}_0}$ by $B = G + H$. Then B is positive and $B - \hat{A} = 2H$ is positive, so that for all $i \in \mathbb{N}$,

$$\begin{aligned}
&b_{0j} \geq 0, \quad b_{0j} + b_{ij} \geq 0 \text{ for all } j \in \mathbb{N}, \\
&b_{00} \geq \sum_{j=1}^{\infty} b_{0j}, \quad b_{00} + b_{i0} \geq \sum_{j=1}^{\infty} (b_{0j} + b_{ij}), \\
&b_{0j} - a_{0j} \geq 0, \quad b_{0j} - a_{0j} + b_{ij} - a_{ij} \geq 0 \text{ for all } j \in \mathbb{N}, \text{ and} \\
&b_{00} - a_{00} \geq \sum_{j=1}^{\infty} (b_{0j} - a_{0j}).
\end{aligned}$$

Hence

$$a_{ij} \leq b_{0j} + b_{ij} - a_{0j} \leq (b_{0j} + b_{ij}) + (b_{0j} - a_{0j}) \quad \text{for all } i, j \in \mathbb{N}.$$

Both terms at the right hand side are positive, so taking positive parts and summing over j yields

$$\begin{aligned}
\sum_{j=1}^{\infty} a_{ij}^{+} &\leq \sum_{j=1}^{\infty} \Big((b_{ij} + b_{0j}) + (b_{0j} - a_{0j}) \Big) \\
&\leq b_{00} + b_{i0} + b_{00} - a_{00} \quad \text{for all } i \in \mathbb{N}.
\end{aligned}$$

Similarly, if we consider $-\hat{A}$ instead of $\hat{A}$ we obtain $b_{0j} + a_{0j} \geq 0$ for all $j \in \mathbb{N}_0$ and

$$\sum_{j=1}^{\infty} (-a_{ij})^{+} \leq b_{00} + b_{i0} + b_{00} + a_{00} \quad \text{for all } i \in \mathbb{N}.$$

Since $(b_{i0})_{i\in\mathbb{N}_0} \in c_{00}$, the maximum $b = \max\{b_{i0} \colon i \in \mathbb{N}_0\}$ exists, and $\sum_{j=1}^{\infty} |a_{ij}| \leq 4b_{00} + 2b$ for all $i \in \mathbb{N}$. It also follows that $|a_{0j}| \leq b_{0j}$ for all $j \in \mathbb{N}_0$ and hence $\sum_{j=1}^{\infty} |a_{0j}| \leq b_{00}$. Thus, $(\sum_{j=1}^{\infty} |a_{ij}|)_{i\in\mathbb{N}_0}$ is bounded by $4b_{00} + 2b$.

Next assume that $(\sum_{j=1}^{\infty} |a_{ij}|)_{i\in\mathbb{N}_0}$ is bounded. Let $b \geq 4 \sum_{j=1}^{\infty} |a_{ij}|$ for all $i \in \mathbb{N}_0$. Define $B = (b_{ij})_{i,j\in\mathbb{N}_0}$ by

$$b_{00} = b + |a_{00}| \quad \text{and} \quad b_{ij} = |a_{ij}| \text{ for } i, j \in \mathbb{N}_0 \text{ with } (i, j) \neq (0, 0).$$

With the aid of (a) it follows that B and $B - \hat{A}$ are positive. Hence $\hat{A} = B - (B - \hat{A})$ is regular.

(c) The space $L^r(c_{00}, K)$ of regular operators on (c_{00}, K) is not a lattice.

For example, consider $A = (a_{ij})_{i,j\in\mathbb{N}_0}$ with $a_{ij} = -1/i$ if $i = j \geq 1$ and $a_{ij} = 0$ otherwise. Due to (b), A is a regular operator on (c_{00}, K). We show that $A \vee 0$ does not exist. Define for $n \in N_0$ the operator $B^{(n)} = (b_{ij}^{(n)})_{i,j\in\mathbb{N}_0}$ by

$$b_{ij}^{(n)} = \begin{cases} \frac{1}{n+1} & \text{if } i = j = 0, \\ \frac{1}{i} & \text{if } j = 0 \text{ and } 1 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily checked that $B^{(n)}$ and $B^{(n)} - A$ are positive for all $n \in \mathbb{N}_0$. Further, let $G = (g_{ij})_{i,j\in\mathbb{N}_0}$ be an operator on (c_{00}, K) such that $0 \le G \le B^{(n)}$ for all $n \in \mathbb{N}_0$. If we show that $G \ge A$ cannot hold, then we know that there is no least upper bound of $\{0, A\}$. For $i \in \mathbb{N}$ and $n \in \mathbb{N}_0$, the positivity of G and $B^{(n)} - G$ yields $g_{00} + g_{i0} \ge 0$ and $b_{00}^{(n)} - g_{00} + b_{i0}^{(n)} - g_{i0} \ge 0$. Since the columns of $B^{(n)}$ and G are eventually zero, it follows that $0 \le g_{00} \le b_{00}^{(n)}$ for all n and therefore $g_{00} = 0$. If we now suppose that $G \ge A$, then

$$g_{00} - a_{00} + g_{i0} - a_{i0} \ge \sum_{j=1}^{\infty} (g_{0j} - a_{0j} + g_{ij} - a_{ij}) \quad \text{for all } i \in \mathbb{N},$$

so $g_{i0} \ge 1/i$, which contradicts the fact that the columns of G are eventually zero. Thus $A \vee 0$ does not exist in $L^r(c_{00}, K)$.

Recall that the space $L^r(c_{00}, K)$ consists of the matrix representations of the regular operators on l_0^∞. Due to (b), this space of matrix representations is given by

$$\mathcal{R} = \{(a_{ij})_{i,j\in\mathbb{N}_0} \colon\ (a_{ij})_i \in c_{00}\ \forall j \in \mathbb{N}_0 \text{ and } \left(\textstyle\sum_{j=1}^{\infty} |a_{ij}|\right)_{i\in\mathbb{N}_0} \text{ is bounded}\}.$$

The space $\mathcal{R}$ is equipped with the cone of operators that are positive with respect to (c_{00}, K), that is, the representations satisfying (i) and (ii) of (a). It is our aim to embed $\mathcal{R}$ into an appropriate vector lattice. Define a space Y as the space of all matrices $(b_{ij})_{i\in\mathbb{N},j\in\mathbb{N}_0}$ that satisfy the following four conditions:

$$(b_{ij})_{i\in\mathbb{N}} \text{ is eventually constant for all } j \ge 1, \tag{3}$$

$$\sum_{j=1}^{\infty} |\beta_j| < \infty, \text{ where } \beta_j = \lim_{i\to\infty} b_{ij}, \tag{4}$$

$$(b_{i0})_{i\in\mathbb{N}} \text{ is bounded}, \tag{5}$$

$$\left(\sum_{j=1}^{\infty} |b_{ij}|\right)_{i\in\mathbb{N}} \text{ is bounded.} \tag{6}$$

We endow Y with the entrywise order. We define a map F on $\mathcal{R}$ by

$$F(A) = (f_{ij}(A))_{i\in\mathbb{N},j\in\mathbb{N}_0},$$

for $A = (a_{ij})_{i,j\in\mathbb{N}_0} \in \mathcal{R}$, where

$$f_{ij}(A) = \begin{cases} a_{0j} + a_{ij} & \text{for } i \in \mathbb{N},\ j \ge 1, \\ a_{00} + a_{i0} - \sum_{\ell=1}^{\infty} (a_{0\ell} + a_{i\ell}) & \text{for } i \in \mathbb{N},\ j = 0. \end{cases}$$

(d) The space Y is a vector lattice and $F\colon\ \mathcal{R} \to Y$ is a bipositive linear map.

It is straightforward that Y is a vector lattice. We next show that F maps into Y. Let $A \in \mathcal{R}$ and put $b_{ij} = f_{ij}(A)$ for $i \in \mathbb{N}$ and $j \in \mathbb{N}_0$. For $j \ge 1$ we have $b_{ij} = a_{0j}$ for i large, so (3) holds. Further, $\beta_j := \lim_{i\to\infty} b_{ij} = a_{0j}$ and $\sum_{j=1}^{\infty} |\beta_j| = \sum_{j=1}^{\infty} |a_{0j}| < \infty$, so we have (4). To infer (5) and (6), use that $\left(\sum_{j=1}^{\infty} |a_{ij}|\right)_i$ is bounded and $(a_{i0})_i \in c_{00}$. It is clear that the map F is linear and that $F(A)$ is positive if and only if A is a positive element of $\mathcal{R}$.

As each bipositive map is injective, it follows that $\mathcal{R}$ is embedded in Y by the map F.

(e) The subspace $F(\mathcal{R})$ is order dense in Y.

Indeed, let $B = (b_{ij})_{i\in\mathbb{N},j\in\mathbb{N}_0} \in Y$. We construct a sequence $(A^N)_{N\in\mathbb{N}}$ in $\mathcal{R}$ such that for each $N \in \mathbb{N}$ we have

$$\begin{array}{ll} F(A^N)_{ij} = b_{ij} & \text{for } i \in \mathbb{N},\ j \in \mathbb{N}, \text{ and for } i = 1, \ldots, N,\ j = 0, \\ F(A^N)_{ij} \geq b_{ij} & \text{for } j = 0,\ i > N. \end{array}$$

Then it follows that $B = \inf\{F(A)\colon\ A \in \mathcal{R},\ F(A) \geq B\}$. Fix $N \in \mathbb{N}$. The construction of A^N is as follows. Denote $\beta_j = \lim_{i\to\infty} b_{ij}$ for $j \geq 1$ and choose $\beta_0 \geq 0$ such that

$$\beta_0 \geq \sup_{i\in\mathbb{N}} |b_{i0}| + \sup_{i\in\mathbb{N}} \sum_{j=1}^{\infty} |b_{ij}|,$$

which is possible due to (5) and (6). Define

$$a_{ij}^N = \begin{cases} \beta_j & \text{for } i = 0,\ j \in \mathbb{N}_0, \\ b_{ij} - \beta_j & \text{for } i \in \mathbb{N},\ j \in \mathbb{N}, \\ b_{i0} - \beta_0 + \sum_{\ell=1}^{\infty} b_{i\ell} & \text{for } i \in \mathbb{N} \text{ with } i \leq N,\ j = 0, \\ 0 & \text{for } i \in \mathbb{N} \text{ with } i > N,\ j = 0, \end{cases}$$

and put $A^N = (a_{ij}^N)_{i,j\in\mathbb{N}_0}$. It is straightforward that $A^N \in \mathcal{R}$ and that $F(A^N)$ is as desired.

Thus, we have shown that the space $L^r(c_{00}, K)$ is pre-Riesz and that its Riesz completion is the vector lattice generated by $F(\mathcal{R})$ in Y.

(f) The space Y is not σ-Dedekind complete.

For a proof, define $B^{(n)} = (b_{ij}^{(n)})_{i\in\mathbb{N},j\in\mathbb{N}_0}$ for $n \in \mathbb{N}$ by

$$b_{ij}^{(n)} = \begin{cases} 1/i & \text{for } i = 1, \ldots, n,\ j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $B^{(n)} \in Y$ for all $n \in \mathbb{N}$ and $\{B^{(n)}\colon\ n \in \mathbb{N}\}$ does not have a supremum in Y.

(g) The space $L^r(c_{00}, K)$ is pervasive.

We show that for any $B \in Y$, $B \geq 0$, $B \neq 0$, there is $A \in \mathcal{R}$ such that $F(A) \geq 0$, $F(A) \neq 0$, $F(A) \leq B$. Let $B = (b_{ij})_{i\in\mathbb{N},j\in\mathbb{N}_0}$, $b_{ij} \geq 0$, $B \neq 0$. We examine two cases.

(i) There is $i \in \mathbb{N}$ such that $b_{i0} > 0$.

Put $a_{i0} = b_{i0}$ and $a_{kl} = 0$ otherwise. Then $A = (a_{kl})_{k,l\in\mathbb{N}_0} \in \mathcal{R}$ and

$$F(A)_{kl} = \begin{cases} b_{i0} & \text{for } k = i,\ l = 0 \\ 0 & \text{otherwise} \end{cases}.$$

(ii) There are $i, j \in \mathbb{N}$ such that $b_{ij} > 0$.

Put $a_{i0} = a_{ij} = b_{ij}$ and $a_{kl} = 0$ otherwise. Then $A = (a_{kl})_{k,l\in\mathbb{N}_0} \in \mathcal{R}$ and

$$F(A)_{kl} := \begin{cases} b_{ij} & \text{for } k = i,\ l = j \\ 0 & \text{otherwise} \end{cases}.$$

In both cases A satisfies $F(A) \geq 0$, $F(A) \neq 0$, and $F(A) \leq B$.

We conclude from Theorem 2.6 that the space $L^r(c_{00}, K)$ (and, hence, $L^r(l_0^\infty)$) has the restriction property (R) for bands.

Let us now consider the space $L^n(c_{00}, K)$ of (o)-continuous operators on (c_{00}, K), which is a subspace of $L^r(c_{00})$. We will show that $L^n(c_{00}, K)$ is a directed band in $L^r(c_{00}, K)$. (Note that, in general, a band need not be directed, cf. [6, Example 5.13].) First, we characterize the positive (o)-continuous operators and then we consider the arbitrary case. In order to prove the characterizations, we need two statements concerning (o)-convergence of sequences in (c_{00}, K).

(h) (i) A net $(x^\alpha)_\alpha$ in (c_{00}, K) satisfies $x^\alpha \downarrow 0$ if and only if $x_0^\alpha \downarrow$ and $x_0^\alpha + x_i^\alpha \downarrow 0$ for all $i \in \mathbb{N}$.

(ii) The sequence $(x^n)_{n\in\mathbb{N}}$ in (c_{00}, K) defined by $x_0^n := 1$, $x_i^n := -1$ for $i = 1, \ldots, n$, and $x_i^n := 0$ for $i > n$, $n \in \mathbb{N}$, satisfies $x^n \downarrow 0$.

To show (i), we first assume that $x^\alpha \downarrow 0$. Then $x_0^\alpha \geq 0$ and $x_0^\alpha + x_i^\alpha \geq 0$, and both x_0^α and $x_0^\alpha + x_i^\alpha$ are decreasing in α for all $i \in \mathbb{N}$. Suppose that $x_0^\alpha + x_k^\alpha \geq \delta$ for all α for some $k \in \mathbb{N}$ and $\delta > 0$. If we define $y \in c_{00}$ by $y_k := \delta$ and $y_i := 0$ for all $i \neq k$, then $y \leq x^\alpha$ for all α. However, $y \not\leq 0$, since $y_0 + y_k = \delta \not\leq 0$, which contradicts $x^\alpha \downarrow 0$.

Conversely, we assume that $x_0^\alpha \downarrow$ and $x_0^\alpha + x_i^\alpha \downarrow 0$ for all $i \in \mathbb{N}$. Then $(x^\alpha)_\alpha$ is a decreasing net of positive elements. If $y \in c_{00}$ satisfies $y \leq x^\alpha$ for all α, then $y_0 + y_i \leq x_0^\alpha + x_i^\alpha$ for all $i \in \mathbb{N}$ and all α. Hence $y_0 + y_i \leq 0$ for all $i \in \mathbb{N}$ and then also $y_0 \leq 0$ as $y \in c_{00}$. So $y \leq 0$ and therefore $x^\alpha \downarrow 0$.

For a proof of assertion (ii), observe that for $n > m$,

$$(x^m - x^n)_0 = 0 \quad \text{and} \quad (x^m - x^n)_i + (x^m - x^n)_0 = \begin{cases} 1 & \text{if } m < i \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

so $x^m \geq x^n$. Further, $x^n \geq 0$ for all n. Finally, if $y \leq x^n$ for all n, then for every $i \in \mathbb{N}$ we have $y_i + y_0 \leq x_i^n + x_0^n$ for all n, so $y_i + y_0 \leq 0$. Since $y \in c_{00}$, we also obtain $y_0 \leq 0$ and $y \leq 0$. Hence $x^n \downarrow 0$.

(j) A regular operator $A = (a_{ij})_{i,j\in\mathbb{N}_0}$ on (c_{00}, K) is (o)-continuous if and only if

$$\sum_{j=1}^{\infty} (a_{ij} + a_{0j}) = a_{i0} + a_{00} \quad \text{for all } i \in \mathbb{N}.$$

Moreover, the space $L^n(c_{00}, K)$ is directed.

We first assume that A is an (o)-continuous operator on (c_{00}, K) and show that the algebraic condition holds. For a net $x^\alpha \downarrow 0$ we have $Ax^\alpha \xrightarrow{(o)} 0$, so that there exists a net $(v^\alpha)_\alpha$ in (c_{00}, K) with $\pm Ax^\alpha \leq v^\alpha$ for all α and $v_0^\alpha \downarrow$ and $v_i^\alpha + v_0^\alpha \downarrow 0$ for all $i \in \mathbb{N}$. We then have $\pm[(Ax^\alpha)_i + (Ax^\alpha)_0] \leq v_i^\alpha + v_0^\alpha$, so $|\sum_{j=0}^\infty (a_{ij} + a_{0j})x_j^\alpha| \leq v_i^\alpha + v_0^\alpha \downarrow 0$ for all $i \in \mathbb{N}$. Further, for each $i \in \mathbb{N}$,

$$\left|\sum_{j=1}^\infty (a_{ij} + a_{0j})(x_j^\alpha + x_0^\alpha)\right| \leq \sum_{j=1}^\infty (|a_{ij}| + |a_{0j}|)(x_j^\alpha + x_0^\alpha),$$

which converges to 0 by the Monotone Convergence Theorem, since $\sum_{j=1}^\infty |a_{ij}| < \infty$ for all $i \in \mathbb{N}_0$ and $x_j^\alpha + x_0^\alpha \downarrow 0$ for all $j \in \mathbb{N}$. Hence for each $i \in \mathbb{N}$, the right hand side of the identity

$$\sum_{j=1}^\infty (a_{ij} + a_{0j})x_0^\alpha - (a_{i0} + a_{00})x_0^\alpha = \sum_{j=1}^\infty (a_{ij} + a_{0j})(x_j^\alpha + x_0^\alpha) - \sum_{j=0}^\infty (a_{ij} + a_{0j})x_j^\alpha$$

converges to 0. If we consider the sequence $(x^n)_{n\in\mathbb{N}}$ of (ii) of (h), we have $x_0^n = 1$ for all n, and we infer that $\sum_{j=1}^\infty (a_{ij} + a_{0j}) = a_{i0} + a_{00}$ for all $i \in \mathbb{N}$.

Next, we show that any regular operator $A = (a_{ij})_{i,j\in\mathbb{N}_0}$ on (c_{00}, K) that satisfies the algebraic condition is (o)-continuous. We assume as a first step that A is positive. Let $(x^\alpha)_\alpha$ be a net in (c_{00}, K) with $x^\alpha \downarrow 0$. Then $Ax^\alpha \downarrow$ and in particular $(Ax^\alpha)_0 \downarrow$. Due to the property of A we have for every $i \in \mathbb{N}$,

$$(Ax^\alpha)_0 + (Ax^\alpha)_i = \sum_{j=1}^\infty (a_{0j} + a_{ij})(x_0^\alpha + x_j^\alpha).$$

Since $\sum_{j=1}^\infty (a_{0j} + a_{ij}) < \infty$ and $x_0^\alpha + x_j^\alpha \downarrow 0$ for all i, it follows that $(Ax^\alpha)_0 + (Ax^\alpha)_i \downarrow 0$. By (i) of (h) we infer that $Ax^\alpha \downarrow 0$. Hence A is (o)-continuous.

Now, let $A = (a_{ij})_{i,j\in\mathbb{N}_0}$ be an arbitrary regular operator on (c_{00}, K) that satisfies the algebraic condition. We show that there exists a positive (o)-continuous operator B on (c_{00}, K) such that $B - A$ is positive and (o)-continuous. Let

$$d = \sup_{i\in\mathbb{N}} \left(\sum_{j=1}^\infty (|a_{0j}| + |a_{ij}|) - (|a_{00}| + |a_{i0}|) \right)^+,$$

which is a finite number, as A is regular. Define $d_{ij} = |a_{ij}|$ for $i, j \in \mathbb{N}_0$ with $(i, j) \neq (0, 0)$ and $d_{00} = |a_{00}| + d$. Then

$$\beta_i = d_{00} + d_{i0} - \sum_{j=1}^\infty (d_{0j} + d_{ij}) \geq 0 \quad \text{for all } i \in \mathbb{N}.$$

Now define $B = (b_{ij})_{i,j\in\mathbb{N}_0}$ by

$$b_{ij} = \begin{cases} d_{00} & \text{for } i = j = 0, \\ d_{ij} & \text{for } i, j \in \mathbb{N} \text{ with } i \neq j, \\ d_{ii} + \beta_i & \text{for } i, j \in \mathbb{N} \text{ with } i = j. \end{cases}$$

Then the columns of B are eventually zero, so B is an operator on (c_{00}, K). Further, B satisfies the algebraic condition, and so does $B - A$. Also, B and $B - A$ are positive. Therefore, both B and $B-A$ are (o)-continuous. Thus, $A = B-(B-A)$ is (o)-continuous and the proof is complete.

(k) The space $L^n(c_{00}, K)$ is a band in $L^r(c_{00}, K)$.

We show the equivalent assertion, namely that the set $\mathcal{N}$ of matrix representations of elements of $L^n(c_{00}, K)$ is a band in $\mathcal{R}$. We use the embedding in Y and the fact that disjointness in Y is the entrywise disjointness. According to (d) and (e), the map F embeds $\mathcal{R}$ order densely into Y. Due to (j), an element $A \in \mathcal{R}$ is in $\mathcal{N}$ if and only if $F(A)_{i0} = 0$ for all $i \in \mathbb{N}$. The set $\{B = (b_{ij})_{i\in\mathbb{N}, j\in\mathbb{N}_0} \in Y : b_{i0} = 0 \text{ for all } i \in \mathbb{N}\}$ is a band in Y. Since the restriction property (R) for bands holds, $F(\mathcal{R}) \cap B$ is a band in $F(\mathcal{R})$. Accordingly, $\mathcal{N}$ is a band in $\mathcal{R}$.

We conclude that $L^n(l_0^\infty)$ is a band in the pre-Riesz space $L^r(l_0^\infty)$.

In the previous example the Riesz completion shows to be an appropriate tool to deal with spaces of operators on a vector lattice that is not Dedekind complete. If X is a Dedekind complete vector lattice, Ogasawara's theorem states that $L^n(X)$ is a band in $L^r(X)$. The above example may be seen as an instance of this theorem in the more general setting of pre-Riesz spaces. Thus we arrive at the following question for a vector lattice X:

> If $L^b(X)$ is pre-Riesz (e. g., Archimedean and directed), which conditions on X ensure that $L^n(X)$ is a band in $L^b(X)$?

Acknowledgement. The authors thank Martin R. Weber for inspiring questions and advice. O. van Gaans acknowledges the support by a 'VIDI subsidie' (639.032.510) of the Netherlands Organisation for Scientific Research (NWO).

References

[1] Y.A. Abramovich and G. Sirotkin, *On order convergence of nets*, Positivity **9** (3) (2005), 287–292.

[2] Y.A. Abramovich and A.W. Wickstead, *Regular operators from and into a small Riesz space*, Indag. Mathem., N.S. **2** (3) (1991), 257–274.

[3] C.D. Aliprantis and O. Burkinshaw, *Positive operators,* Academic Press Inc., London, 1985.

[4] G. Buskes and A.C.M. van Rooij, *The vector lattice cover of certain partially ordered groups*, J. Austral. Math. Soc. (Series A) **54** (1993), 352–367.

[5] O. van Gaans and A. Kalauch, *Disjointness in partially ordered vector spaces*, Positivity **10** (3) (2006), 573–589.

[6] O. van Gaans and A. Kalauch, *Ideals and bands in pre-Riesz spaces*, Positivity, to appear.

[7] M. van Haandel, *Completions in Riesz space theory,* Ph.D. thesis, University of Nijmegen, 1993.

Onno van Gaans
Mathematical Institute
Leiden University
P.O. Box 9512
2300 RA Leiden
The Netherlands
vangaans@math.leidenuniv.nl

Anke Kalauch
Institut für Analysis
Fachrichtung Mathematik
TU Dresden
01062 Dresden
Germany