Gradient flows in measure spaces

(Topics in Analysis 2011)

Assignment 1

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1. (Convergence in distribution) For a Borel probability measure μ on $\mathbb{R}$ its distribution function is defined by
$$F(x) := \mu((-\infty, x]), \quad x \in \mathbb{R}.$$
Let $\mu, \mu_1, \mu_2, \dots$ be Borel probability measures on $\mathbb{R}$ and let $F, F_1, F_2, \dots$ be their distribution functions. Show that $\mu_n \to \mu$ narrowly if and only if $F_n(x) \to F(x)$ for every $x \in \mathbb{R}$ at which F is continuous.

2. (Embedding of X into $\mathcal{P}(X)$) Let (X, d) be a metric space. Consider a modification of the bounded Lipschitz metric:
$$d_{bl}(\mu, \nu) := \sup\{\left|\int f\, d\mu - \int f\, d\nu\right| : \ f \in BL(X), \ \max\{\|f\|_\infty, Lip(f)\} \le 1\},$$
$\mu, \nu \in \mathcal{P}(X)$.
   (a) Show that d_{bl} is a metric on $\mathcal{P}(X)$ and that there exist $c, C \in (0, \infty)$ such that
$$c d_{BL}(\mu, \nu) \le d_{bl}(\mu, \nu) \le C d_{BL}(\mu, \nu) \text{ for all } \mu, \nu \in \mathcal{P}(X).$$
   (b) Assume that $d(x, y) \le 1$ for all $x, y \in X$. Show that $x \mapsto \delta_x \colon (X, d) \to (\mathcal{P}(X), d_{bl})$ is an isometry. Here δ_x denotes the point measure at x: $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ if $x \notin A$, $A \subset X$ Borel.
   (If (X, d) is an arbitrary metric space, then the metric $\tilde{d}(x, y) = \min\{d(x, y), 1\}$ induces the same topology and same Cauchy sequences as d.)

3. (Krylov-Bogoliubov theorem) Consider a non-empty compact metric space (X, d) and a continuous map $T \colon X \to X$. We think of T as a *transition map*: a particle which is at the point x at time n will be at the point $T(x)$ at time $n + 1$. After n time steps a particle starting at x will be at $T^n(x)$. If we start with particles distributed according to a distribution $\mu \in \mathcal{P}(X)$, then at time n they will be distributed according to the distribution $(T^n)_\#\mu$.
   (a) Show that there exists an *invariant distribution* for T, that is, show that there exists a $\nu \in \mathcal{P}(X)$ such that $(T^n)_\#\nu = \nu$ for all n.
   (b) Explicit case: let X be the unit circle around the origin in $\mathbb{R}^2$ (or $\mathbb{C}$) and let T be rotation over an angle $\alpha \in \mathbb{R}$. Give an explicit invariant distribution for T. Give an example where there are at least two distinct invariant distributions.

4. (Sufficient conditions for tightness) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

   (a) Let $\mathcal{A}$ be a set of measurable real valued functions on $(\Omega, \mathcal{F}, \mathbb{P})$ such that there exists an $M \in \mathbb{R}$ such that

   $$\int_\Omega |f(\omega)| \, d\mathbb{P}(\omega) \leq M \text{ for all } f \in \mathcal{A}.$$

   Show that $\{f_\# \mathbb{P} \colon f \in \mathcal{A}\}$ is tight.
   (Hint: $\int_\Omega |f(\omega)| \, d\mathbb{P}(\omega) \geq \int_\Omega n 1_{\{\omega\colon |f(\omega)| > n\}} d\mathbb{P}(\omega)$.)

   (b) Let $(E, \|\cdot\|)$ be a Banch space and let F_t, $t \in [a,b]$, be a family of $\mathcal{F}$-Borel-measurable functions from Ω to E such that

   $$\int_\Omega \|F_t(\omega)\|_E \, d\mathbb{P}(\omega) \text{ is finite for all } t \in [a,b].$$

   Assume that $t \mapsto F_t$ is continuous in the sense that

   $$\lim_{s \to t} \int_\Omega \|F_s(\omega) - F_t(\omega)\|_E \, d\mathbb{P}(\omega) = 0 \text{ for all } t \in [a,b].$$

   Show that $\{(F_t)_\# \mathbb{P} \colon t \in [a,b]\}$ is tight.
   (Hint: *not* as in (a)!)

5. (Disintegration with densities)

   (a) Let γ be a Borel probability measure on $\mathbb{R}^2$ and μ a Borel probability measure on $\mathbb{R}$ such that $\gamma(A \times \mathbb{R}) = \mu(A)$ for every Borel set $A \subseteq X$. Assume that γ has a density h with respect to the Lebesgue measure on $\mathbb{R}^2$ and that μ has a density f with respect to the Lebesgue measure on $\mathbb{R}$. Find a formula for Borel probability measures ν_x on $\mathbb{R}$, $x \in \mathbb{R}$, in terms of h and f, such that

   $$\int_{\mathbb{R}^2} u(x,y) \, d\gamma(x,y) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} u(x,y) \, d\nu_x(y) \right) d\mu(x)$$

   for all bounded Borel functions $u \colon \mathbb{R}^2 \to \mathbb{R}$.

   (b) Let γ_{12} and γ_{13} be Borel probability measures on $\mathbb{R}^2$ with densities h_{12} and h_{13}, respectively, with respect to the Lebesgue measure on $\mathbb{R}^2$. Assume that $\gamma_{12}(A \times \mathbb{R}) = \gamma_{13}(A \times \mathbb{R})$ for every Borel set $A \subseteq \mathbb{R}$. Find a Borel measure μ on $\mathbb{R}^3$ such that

   $$\gamma_{12}(A \times B) = \mu(A \times B \times \mathbb{R})$$

   and

   $$\gamma_{13}(A \times B) = \mu(A \times \mathbb{R} \times B)$$

   for all Borel sets $A, B \subseteq \mathbb{R}$.

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