Gradient flows in measure spaces

(Topics in Analysis 2011)

Assignment 2

A map $F\colon H \to H$ on a Hilbert space $(H, \langle , \rangle)$ is called *monotone* if

$$\langle F(x) - F(y), x - y\rangle \geq 0 \text{ for all } x, y \in H.$$

1. Let $(H, \langle , \rangle)$ be a Hilbert space over $\mathbb{R}$ and let $F\colon H \to H$ be a map such that its graph $\{(x, F(x))\colon x \in H\}$ is a c-monotone subset of $H \times H$, where

$$c(x, y) = \|x - y\|_H^2, \quad x, y \in H.$$

   Show that F is monotone.

2. Let $(H, \langle , \rangle)$ be a Hilbert space over $\mathbb{R}$ and let $\varphi\colon H \to \mathbb{R}$ be convex and differentiable. Show that $\{(x, \nabla\varphi(x))\colon x \in H\}$ is a c-monotone subset of $H \times H$, where

$$c(x, y) = \|x - y\|_H^2, \quad x, y \in H.$$

3. Let $F\colon \mathbb{R}^d \to \mathbb{R}^d$ be defined by

$$F(x) = \|x\|^2 x, \quad x \in \mathbb{R}^d,$$

   where $\|\cdot\|$ is the usual Euclidean norm. Show that F is monotone (with respect to the standard inner product on $\mathbb{R}^d$).

4. Let $F\colon \mathbb{R}^d \to \mathbb{R}^d$ be monotone (with respect to the standard inner product on $\mathbb{R}^d$). Suppose that for every $u \in \mathbb{R}^d$ there exists a differentiable function $y_u\colon [0, \infty) \to \mathbb{R}^d$ such that

$$y_u'(t) = -F(y_u(t)) \text{ for all } t \geq 0 \text{ and } y_u(0) = u.$$

   Show that

$$\|y_u(t) - y_v(t)\| \leq \|u - v\| \text{ for all } t \geq 0,$$

   for each $u, v \in \mathbb{R}^d$.

5. Consider $X = [0, 1]$ with the metric $d(x, y) = |x - y|$, $x, y \in [0, 1]$. Let $f, g\colon [0, 1] \to (0, \infty)$ be continuous and such that

$$\int_0^1 f(x)\, dx = \int_0^1 g(x)\, dx = 1$$

   and let μ and ν be the Borel probability measures on $[0, 1]$ with density functions f and g, respectively, that is,

$$\mu([a, b]) = \int_a^b f(x)\, dx \text{ and } \nu([a, b]) = \int_a^b g(x)\, dx$$

   for any $0 \leq a < b \leq 1$. Consider the cost function

$$c(x, y) = (x - y)^2, \quad x, y \in [0, 1].$$

(a) Show that there exists an optimal transport map $r\colon X \to X$ which transports μ to ν.

(b) Show that an optimal transport map r must satisfy

$$\int_a^x f(s)\,ds = \int_0^{r(x)} g(s)\,ds$$

for all $x \in [0,1]$.
(Hint: applying 1. might be useful.)

(c) Show that the map r defined by

$$r(x) = G^{-1}(F(x)), \quad x \in [0,1],$$

is an optimal transport map which transports μ to ν, where

$$F(x) = \int_0^x f(s)\,ds \text{ and } G(x) = \int_0^x g(s)\,ds$$

for $x \in [0,1]$.

— Due: April 20, 2011 —

Onno van Gaans
Spring 2011