Gradient flows in measure spaces

(Topics in Analysis 2011)

Assignment 3

Let (X, d) be a separable complete metric space, let $1 \leq p < \infty$ and consider

$$W_p(\mu, \nu) := \inf \left\{ \int_{X \times X} d(x, y)^p \, d\eta(x, y) \colon \ \eta \in \Gamma(\mu, \nu) \right\}^{1/p}, \ \mu, \nu \in \mathcal{P}_p(X).$$

The aim of this assignment is to provide an alternative proof that W_p satisfies the triangle inequality, by more elementary means than the disintegration theorem (see P. Clément and W. Desch, An elementary proof of the triangle inequality for the Wasserstein metric, *Proc. Amer. Math. Soc.* **136** (2008), no. 1, 333–339).

1. Let $\mu_1, \mu_2, \mu_3 \in \mathcal{P}_p(X)$ be such that there exists a countable set $V = \{v_1, v_2, \ldots\} \subseteq X$ such that $\mu_i(X \setminus V) = 0$, $i = 1, 2, 3$. Let $\gamma_{1,2} \in \Gamma(\mu_1, \mu_2)$ and $\gamma_{2,3} \in \Gamma(\mu_2, \mu_3)$. Define for $k, m, n \in \mathbb{N}$,

$$\alpha_{k,m,n} := \left(\begin{array}{ll} \frac{\gamma_{1,2}(\{(v_k, v_m)\})\gamma_{2,3}(\{(v_m, v_n)\})}{\mu_2(\{v_m\})} & \text{if } \mu_2(\{v_m\}) \neq 0, \\ 0 & \text{if } \mu_2(\{v_m\}) = 0 \end{array} \right.$$

   and define a measure γ on $X \times X \times X$ by

$$\gamma := \sum_{k,m,n \in \mathbb{N}} \alpha_{k,m,n} (\delta_{v_k} \otimes \delta_{v_m} \otimes \delta_{v_n}),$$

   where $\delta_v \in \mathcal{P}(X)$ denotes the point measure (Dirac measure) at $v \in X$.

   (a) Show that $\pi^{1,2}_{\#}\gamma = \gamma_{1,2}$ and that $\pi^{2,3}_{\#}\gamma = \gamma_{2,3}$, where $\pi^{1,2}(x, y, z) = (x, y)$ and $\pi^{2,3}(x, y, z) = (y, z)$, $(x, y, z) \in X \times X \times X$.

   (b) Show that $\gamma \in \mathcal{P}(X \times X \times X)$.

   (c) Show that

$$W_p(\mu_1, \mu_2) \leq \left(\int_{X \times X} d(x, y)^p \, d\gamma_{1,2}(x, y) \right)^{1/p} + \left(\int_{X \times X} d(y, z)^p \, d\gamma_{2,3}(y, z) \right)^{1/p}.$$

   (Hint: define a suitable measure $\gamma_{1,3} \in \Gamma(\mu_1, \mu_3)$.)

   (d) Conclude from (a)–(c) that

$$W_p(\mu_1, \mu_3) \leq W_p(\mu_1, \mu_2) + W_p(\mu_2, \mu_3).$$

2. Let $\varepsilon > 0$.

   (a) There exists a countable subset $V = \{v_1, v_2, \ldots\} \subseteq X$ and there exist mutually disjoint Borel sets $S_1, S_2, \ldots \subseteq X$ such that $v_i \in S_i$ and $S_i \subseteq \{x \in X \colon \ d(x, v_i) < \varepsilon\}$. for all $i \in \mathbb{N}$.

(b) Let $\mu \in \mathcal{P}_p(X)$ Show that there is a $\tilde{\mu} \in \mathcal{P}_p(X)$ such that $\tilde{\mu}(V \setminus X) = 0$ and

$$\tilde{\mu}(\{v_i\}) = \mu(S_i) \text{ for all } i \in \mathbb{N}.$$

(c) Let $\mu, \nu \in \mathcal{P}_p(X)$ and $\gamma \in \Gamma(\mu, \nu)$. Show that there exists a $\tilde{\gamma} \in \Gamma(\tilde{\mu}, \tilde{\nu})$ such that

$$\left| \left(\int_{X \times X} d(x,y)^p \, d\gamma(x,y) \right)^{1/p} - \left(\int_{X \times X} d(x,y)^p \, d\tilde{\gamma}(x,y) \right)^{1/p} \right| < \varepsilon,$$

where $\tilde{\mu}$ and $\tilde{\nu}$ are as in (b) (for μ and μ replaced by ν, respectively).

(d) Let $\tilde{\eta} \in \Gamma(\tilde{\mu}, \tilde{\nu})$. Show that there exists an $\eta \in \Gamma(\mu, \nu)$ such that

$$\left(\int_{X \times X} d(x,z)^p \, d\eta(x,z) \right)^{1/p} \leq \left(\int_{X \times X} d(x,z)^p \, d\tilde{\eta}(x,y) \right)^{1/p} + \varepsilon,$$

where $\mu, \nu, \tilde{\mu}, \tilde{\nu}$ are as in (c).
(Hint: $\eta(U) = \sum_{(k,l) \in I} \frac{\tilde{\eta}(\{(v_k, v_l)\})}{\mu(\{v_k\})\nu(\{v_l\})} (\mu \times \nu)(U \cap (S_k \times S_l))$, $U \subseteq X \times X$ Borel, where $I = \{(k,l) \in \mathbb{N}^2 \colon \mu(\{v_k\})\nu(\{v_l\}) \neq 0\}$.

3. Use 1 and 2 to show that

$$W_p(\mu_1, \mu_3) \leq W_p(\mu_1, \mu_2) + W_p(\mu_2, \mu_3) \text{ for all } \mu_1, \mu_2, \mu_3 \in \mathcal{P}_p(X).$$

— *Due: June 10, 2011* —

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Spring 2011