Topics in Analysis 1 – Real Functions

Assignment 1

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1. **Dini derivatives.** (The right upper, right lower, left upper, and left lower derivatives of a function are sometimes called its four *Dini derivatives.*) Let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = 2^{-n}$ on $(2^{-n}, 2^{-n+1}]$, $n \in \mathbb{N}$, and $f(0) = 0$.

   (a) Show that f is increasing.

   (b) Compute the right upper derivative $D^r f(0)$ and the right lower derivative $D_r f(0)$ of f at 0. (In Royden's book these are denoted by $D^+ f(0)$ and $D_+ f(0)$, respectively.)

   (c) Does there exist a *continuous* increasing function f such that $D^r f(0) \neq D_r f(0)$?

   (d) Construct an increasing function $f : [-1,1] \to \mathbb{R}$ such that none of the four Dini derivatives at 0 are equal.

2. **Bounded variation.**

   (a) Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation. Let $c \in (a,b)$. Prove that $\lim_{x \downarrow c} f(x)$ and $\lim_{x \uparrow c} f(x)$ exist.
   (Hint: try first for an increasing function.)

   (b) Compute for $n \in \mathbb{N}$ the total variation of $f : x \mapsto \sin(1/x)$ on the interval $[\frac{1}{n\pi}, \frac{1}{\pi}]$. Is f of bounded variation on $[0,1]$?

   (c) Let $f : [a,b] \to \mathbb{R}$ and let $V_{[x,y]}(f)$ be the total variation of f on $[x,y]$, where $a \leq x < y \leq b$. Let $x \in (a,b)$. Prove or disprove each of the following two statements.

      (i) $\lim_{n\to\infty} V_{[x-1/n,x+1/n]}(f) = 0 \implies f$ is continuous at x;

      (ii) f is continuous at $x \implies \lim_{n\to\infty} V_{[x-1/n,x+1/n]}(f) = 0$.

3. **Derivative of the total variation function.** Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation and let $V(x) := V_{[a,x]}(f)$ be its total variation on $[a,x]$, for $x \in [a,b]$. Follow the steps below to show that

$$V' = |f'| \text{ a.e.}$$

   (a) Let $a = x_0 < x_1 < \cdots < x_m = b$ be a partition of $[a,b]$. Denote $y_k := f(x_k)$, $k = 0, \ldots, m$. Define a function $g : [a,b] \to \mathbb{R}$ by

$$\text{for } x \in [x_0, x_1]: \quad g(x) := \begin{cases} f(x) - y_0 & \text{if } y_0 \leq y_1, \\ y_0 - f(x) & \text{if } y_0 > y_1, \end{cases}$$

   and, inductively,

$$\text{for } x \in (x_k, x_{k+1}]: \quad g(x) := \begin{cases} g(x_k) + (f(x) - y_k) & \text{if } y_k \leq y_{k+1}, \\ g(x_k) + (y_k - f(x)) & \text{if } y_k > y_{k+1}. \end{cases}$$

   Show that

(i) $|g'| = |f'|$ a.e.;
(ii) on each $(x_k, x_{k+1}]$, either $g - f$ or $g + f$ is constant;
(iii) the total variation of g on $[a, x]$ equals $V_{[a,x]}(g) = V(x)$ for all $x \in [a, b]$;
(iv) $V - g$ is positive and increasing;
(v) $g(b) = \sum_{k=0}^{m-1} |f(x_{k+1}) - f(x_k)|$.

(b) Let $\varepsilon > 0$. Show that there exists a partition of $[a, b]$ such that the corresponding function g of (a) satisfies $0 \le V(x) - g(x) \le \varepsilon$ for all $x \in [a, b]$.

(c) Show that there exists a sequence (g_n) of functions on $[a, b]$ such that $0 \le V(x) - g_n(x) \le 2^{-n}$, $x \mapsto V(x) - g_n(x)$ is increasing, and $|g_n'| = |f'|$ a.e. for every $n \in \mathbb{N}$.

Consider the sequence (g_n) of (c) and let $h_n := V - g_n$, $n \in \mathbb{N}$.

(d) Show that the sequence (h_n) satisfies:
(i) $s(x) := \sum_{n=1}^{\infty} h_n(x)$ exists for all $x \in [a, b]$;
(ii) $x \mapsto s_N(x) := \sum_{n=1}^{N} h_n(x)$ is increasing and $s_{N+1} - s_N$ is increasing for each $N \in \mathbb{N}$;
(iii) $s_1'(x) \le s_2'(x) \le \cdots \le s'(x)$ for almost every $x \in [a, b]$;
(iv) $\lim_{N\to\infty} s_N'(x)$ exists for almost every $x \in [a, b]$.

(e) Show that $V' = |f'|$ a.e.

4. **Another application of Vitali's lemma.** Follow the steps below to prove the following theorem of Lusin: Let $f : [a, b] \to \mathbb{R}$ be an arbitrary function. Define

$$D := \{x \in (a, b) \colon\ f \text{ is differentiable at } x \text{ and } f'(x) = 0\}.$$

Then $f(D)$ is a null set.

(a) Let $y \in f(D)$, $\varepsilon > 0$, and $\alpha > 0$. Show that there exist an $x \in D$ and a $\delta > 0$ such that
- $f(u) \in [y - \delta\varepsilon, y + \delta\varepsilon]$ for all $u \in (x - \delta, x + \delta)$
- the length of $[y - \delta\varepsilon, y + \delta\varepsilon]$ is less than α.

(b) Find a suitable collection of closed intervals that is a Vitali cover of $f(D)$ and apply Vitali's lemma to show that $m^*(f(D)) = 0$.
(Hint: it may help to consider $E = f(D) \cap [-M, M]$, suppose that $m^*(E) > 0$, set $\varepsilon := \frac{1}{2} m^*(E)/(1 + b - a)$ and obtain a contradiction.)

— *Please hand in before March 4, 2008* —

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