Teleportation into quantum information

Or: elements of quantum information

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Lecture hour 1 my short course
on Quantum Information and Statistical Science.
Lecture/hour 0 was the “warm up” on the Delft Bell experiment
Now we do a crash course on the Hilbert space stuff…
In **Lecture hour 2** we’ll do examples.

Don’t worry, we stick to finite dimensions and finite number of outcomes.
The Hilbert space is $\mathbb{C}^d$. Often, $d = 2$.
Or $d = 2^N$ (N qubits)

<u>Baby</u> quantum information

- Pure states and state vectors
- Projective (simple, projector-valued) measurements
- Entanglement
- Unitary evolution

Toy quantum information

- Mixed states and density matrices
- POVM’s (generalised measurements)
- Quantum instruments
- Kraus representation and the Kraus theorems

Special case: the qubit

- Two-dimensional Hilbert space, and tensor products of many copies! All (nearly all) of quantum computing, quantum cryptography, quantum information …

Pure state

- A *d*-dimensional quantum state is represented by a unit length complex vector (thought of as a column vector

- We may write ψ, or $|\psi>$

- Denote complex conjugate and transpose with a star (physicists use a dagger)

- We may write ψ^* or $<\psi|$

- $<\psi|\psi> = 1$

- $|\psi><\psi|$ is a *d*x*d* matrix, and it's the orthogonal projector onto the one-dimensional space spanned by ψ

Observables

- Suppose A is a self-adjoint matrix, i.e., $A^* = A$
- A has real eigenvalues and one can find an orthonormal basis of eigenvectors.
- One can write $A = \Sigma_i\, a_i\, |\phi_i\rangle\langle\phi_i|$ where the a_i are the eigenvalues, real, (may not be unique), and the ϕ_i are the eigenvectors (may not be unique)
- Perhaps better to write $A = \Sigma_a$ Proj[$A = a$] where the a are the distinct eigenvalues, [$A = a$] is the eigenspace belonging to eigenvalue a, and Proj[$A = a$] projects onto that eigenspace.

Measurement: Born’s law

- When the system is in state ψ and we measure the observable *A*, we observe one of the eigenvalues *a*. The state “collapses” to the projection of ψ onto the eigenspace corresponding to that eigenvalue. The probability of seeing value *a* is $||\mathrm{Proj}[A = a]\ \psi||^2$

- By Pythagoras, $\mathrm{sum}_a\ ||\mathrm{Proj}[A = a]\ \psi||^2 = 1$

- This generalisation of the Born law is called the von Neumann-Lüders projection postulate

- One can call the measurement itself a “simple measurement” or a “projector-valued measurement”

Unitary evolution

- Undisturbed, the state evolves according to Schrödinger's equation
- d/dt ψ = i H t for some "Hamiltonian" H

  [Take units s.t. "reduced Planck's constant" $\hbar = h/2\pi = 1$]

- The Hamiltonian is a self-adjoint operator
- The solution of Schrödinger's equation is ψ(t) = exp(i H t) ψ(0)
- U = exp(i H t) is a unitary operator, i.e. $U U^* = U^* U$ = Id

Interaction between several systems

- If two systems of dimension d and d are interacting then they form a joint system of dimension $d \times d'$
- The Hilbert space of the joint system is the tensor product of the Hilbert spaces of the component systems
- This means that if ϕ_i and ψ_i are state vectors of the two subsystems, and c_i are complex numbers, not all zero, then $\Sigma_i \, c_i \, \phi_i \otimes \psi_i$, normalised to have length one, is a (possible) state vector of the joint system

Entanglement

- Initially completely separate component systems can evolve into entangled systems of the joint state through time evolution with a Hamiltonian (or equivalently, a unitary) which is not itself of tensor product form.

Randomisation

- We already saw that quantum measurement generates randomness
- We can think of *classical* randomness as a pure ingredient of quantum mechanics in itself – toss a coin, toss dice, shuffle cards…

Building blocks for a general picture

- *Measurement* according to projection postulate, bringing the system of interest into *interaction* with an "*ancillary*" (auxiliary) system in fixed state, *unitary evolution*, and classical *randomisation*, generate a vast range of ways in which a quantum system can be transformed while in the process generating classical information ("measurement results") which aren't necessarily "observed" at all.

Mixed states, density matrices

- A **density matrix** is a non-negative self-adjoint matrix ρ of trace 1
- Such a matrix can be written (not necessarily uniquely) as $\rho = \Sigma_i \, p_i \, |\psi_i\rangle\langle\psi_i|$
- The p_i are nonnegative and add to 1
- Suppose we **prepare** a quantum system by creating it in pure state $|\psi_i\rangle$ with **probability** p_i
- We call this a **"system in a mixed state"**

Theorem: density matrix $\rho = \Sigma_i \, p_i \, |\psi_i\rangle\langle\psi_i|$ is "the state" of the mixture

- The "state" of a physical system is the catalogue of all joint probability distributions of measurement results, given all collections of "generalised" measurements which can be performed on it

- In our case, a generalised measurement is the operation of combining any number of times: entanglement with ancillas, unitary evolution, randomization, projective measurements …

Partial trace, subsystems

- Theorem: the state of a component of a larger system in a general entangled, mixed, state, is the partial trace of the density matrix of the joint system

Generalised measurements

- A generalised measurement is determined by a collection of self-adjoint non-negative matrices M_i which add to the identity; and an associated distinct outcome value x_i for each component

- The probability of getting outcome x_i is trace(ρ M_i)

Kraus matrices; instrument

- Suppose we are given “Kraus matrices’ A_{ij} and distinct outcome values x_i, satisfying $\Sigma_{ij}\, A_{ij}^*\, A_{ij}$ = Id.

- Consider a *transformation with observation* of a quantum system initially in state ρ: the system is transformed into the state $\Sigma_j\, A_{ij}\, \rho\, A_{ij}^*$ / trace($\Sigma_j\, A_{ij}\, \rho\, A_{ij}^*$) (depending on i) and one observes outcome x_i, with probability Σ_j trace($A_{ij}\, \rho\, A_{ij}^*$)

- The associated <u>instrument</u> is the mapping from density matrices ρ to the combined quantum-classical state ($\Sigma_\mathrm{j}\, A_{ij}\, \rho\, A_{ij}^*$: i ∈ $\mathscr{I}$), with classical outcome space (x_i : $i \in \mathscr{I}$)

Theorem: Kraus representation

- Every <u>*totally*</u> *positive*, *normalised, linear* transformation ($\rho \mapsto (\tau_i : i \in \mathscr{J})$) along with an outcome space ($x_i : i \in \mathscr{J}$) defines an instrument with a Kraus representation

- Every combination of entanglement with ancillary systems, unitary evolution, measurement by simple measurements on component sub-systems, classical randomisation using random measurement outcomes of earlier measurement … results in a <u>*totally*</u> *positive*, *normalised, linear* transformation of the density matrix

Church of the larger Hilbert space

- Every instrument, every measurement, every state transformation can be represented by entanglement with an ancillary Hilbert space, a unitary evolution of the joint system, and then discarding the ancilla.

Some mysteries

- If you create a mixed state and “lose” the information of how you did it, it can never be determined again, *how* you created the state.

- For instance, the completely mixed state Identity/dimension, can be created by picking *any* orthonormal basis and then picking one of the elements of the basis as state vector completely at random. Yet there is no way to detect, how it was created