Teleportation into quantum information

Or: elements of quantum information

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Lecture hour 2 of my short course
on Quantum Information and Statistical Science.
Lecture/hour 1 was the theory part of a crash course on the Hilbert space stuff…

Don't worry, we stick to finite dimensions and finite number of outcomes.
The Hilbert space is C^d. Often, $d = 2$.
Or $d = 2^N$ (N cubits)

The qubit and the Bloch sphere (actually: the Block ball)

- If the Hilbert space has dimension 2 we speak of a **qubit**. The "natural" basis of $\mathbb{C}^2$, consists of the transposes of the row vectors (1 0), and (0 1). These usually are identified in bra-ket notation using descriptive names or labels:

  - |0>, |1> ("computational basis")
  - |+½>, |–½> ("spin 1 system")
  - |"ground state">, |"first excited state">
  - |"horizontal">, |"vertical"> (polarization)
  - |"up">, |"down"> ("spin of one electron")

Spin (Bloch sphere)
Polarisation (Poincaré sphere)

- **Spin** might be in any direction in real 3d space
- “up” and “down” are opposite
- **Polarisation** might be in any “orientation” in space. (An orientation is an undirected direction)
- “horizontal” and “vertical” are opposite
- There is also “circular” (clockwise or anti-clockwise) versus “linear” polarization

Unit vector in $\mathbb{C}^2$

- Every unit vector in $\mathbb{C}^2$ can be written as α |0> + β |1> where $|\alpha|$^2 + $|\beta|$^2 = 1

- Since we can multiply by any complex number of absolute value 1 and still be describing the same state, we may as well take α to be real

Given an orthonormal basis, any pure state $|\psi\rangle$ of a two-level quantum system can be written as a superposition of the basis vectors $|0\rangle$ and $|1\rangle$, where the coefficient or amount of each basis vector is a complex number. Since only the relative phase between the coefficients of the two basis vectors has any physical meaning, we can take the coefficient of $|0\rangle$ to be real and non-negative.

We also know from quantum mechanics that the total probability of the system has to be one: $\langle\psi|\psi\rangle = 1$, or equivalently $\| \, |\psi\rangle \, \|^2 = 1$. Given this constraint, we can write $|\psi\rangle$ using the following representation:

$$|\psi\rangle = \cos(\theta/2)|0\rangle \, + \, e^{i\phi}\sin(\theta/2)|1\rangle = \cos(\theta/2)|0\rangle \, + \, (\cos\phi + i\sin\phi)\, \sin(\theta/2)|1\rangle \text{ , where } 0 \leq \theta \leq \pi \text{ and } 0 \leq \phi < 2\pi.$$

Except in the case where $|\psi\rangle$ is one of the ket vectors (see Bra-ket notation) $|0\rangle$ or $|1\rangle$, the representation is unique. The parameters θ and ϕ , re-interpreted in spherical coordinates as respectively the colatitude with respect to the *z*-axis and the longitude with respect to the *x*-axis, specify a point

$$\vec{a} = (\sin\theta\cos\phi, \; \sin\theta\sin\phi, \; \cos\theta) = (u, v, w)$$

on the unit sphere in $\mathbb{R}^3$.

For mixed states, one considers the density operator. Any two-dimensional density operator ρ can be expanded using the identity I and the Hermitian, traceless Pauli matrices $\vec{\sigma}$:

$$\rho = \frac{1}{2}\left(I + \vec{a}\cdot\vec{\sigma}\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a_x}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{a_y}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{a_z}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

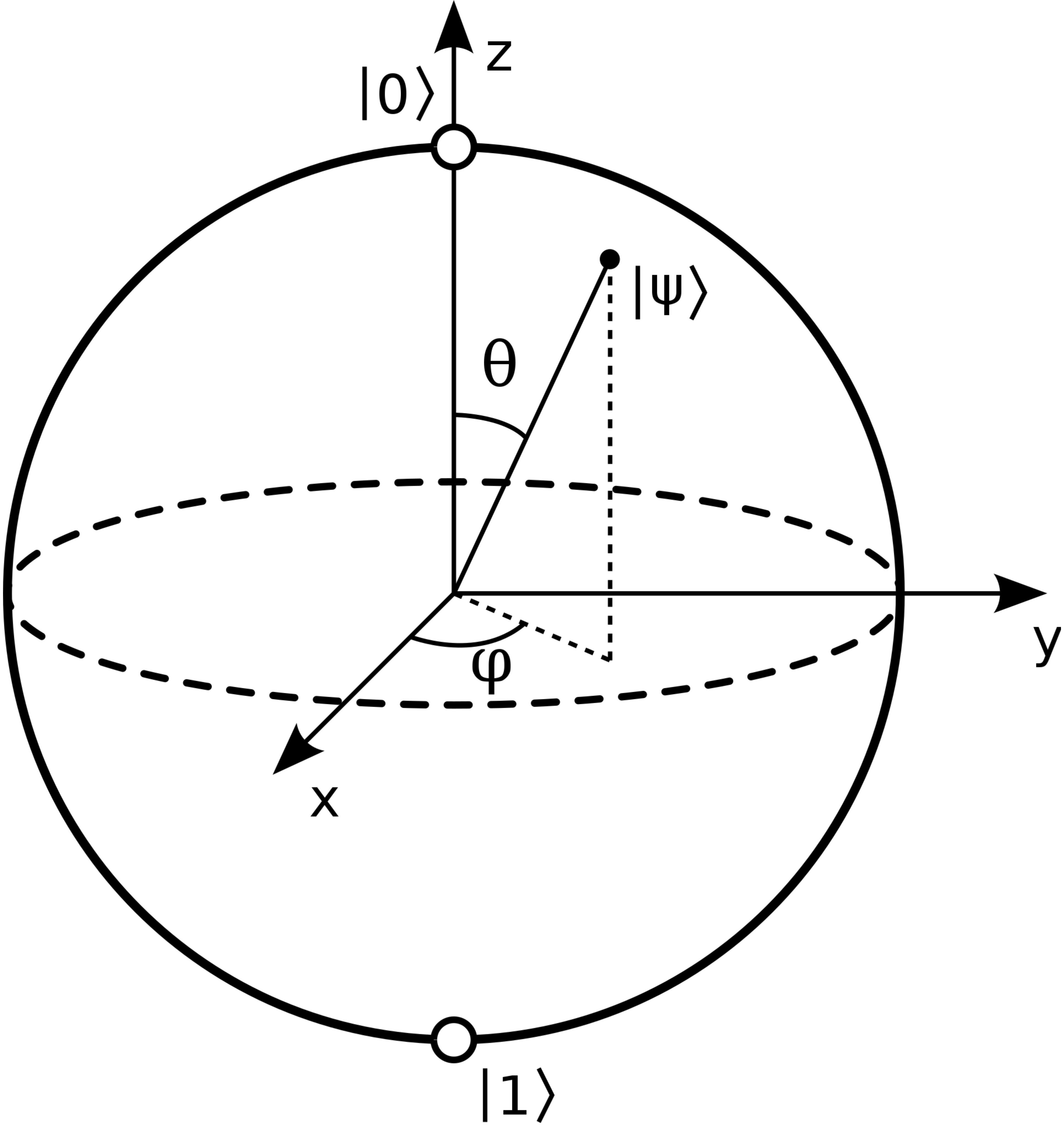
where $\vec{a} \in \mathbb{R}^3$ is called the **Bloch vector**.

It is this vector that indicates the point within the sphere that corresponds to a given mixed state. Specifically, as a basic feature of the Pauli vector, the eigenvalues of ρ are $\frac{1}{2}\left(1 \pm |\vec{a}|\right)$. Density operators must be positive-semidefinite, so we conclude that $|\vec{a}| \leq 1$. For pure states, we must then have

$$\operatorname{tr}(\rho^2) = \frac{1}{2}\left(1 + |\vec{a}|^2\right) = 1 \quad \Leftrightarrow \quad |\vec{a}| = 1 \, ,$$

in accordance with the above.

As a consequence, the surface of the Bloch sphere represents all the pure states of a two-dimensional quantum system, whereas the interior corresponds to all the mixed states.



z
|0⟩
|ψ⟩
θ
y
φ
x
|1⟩

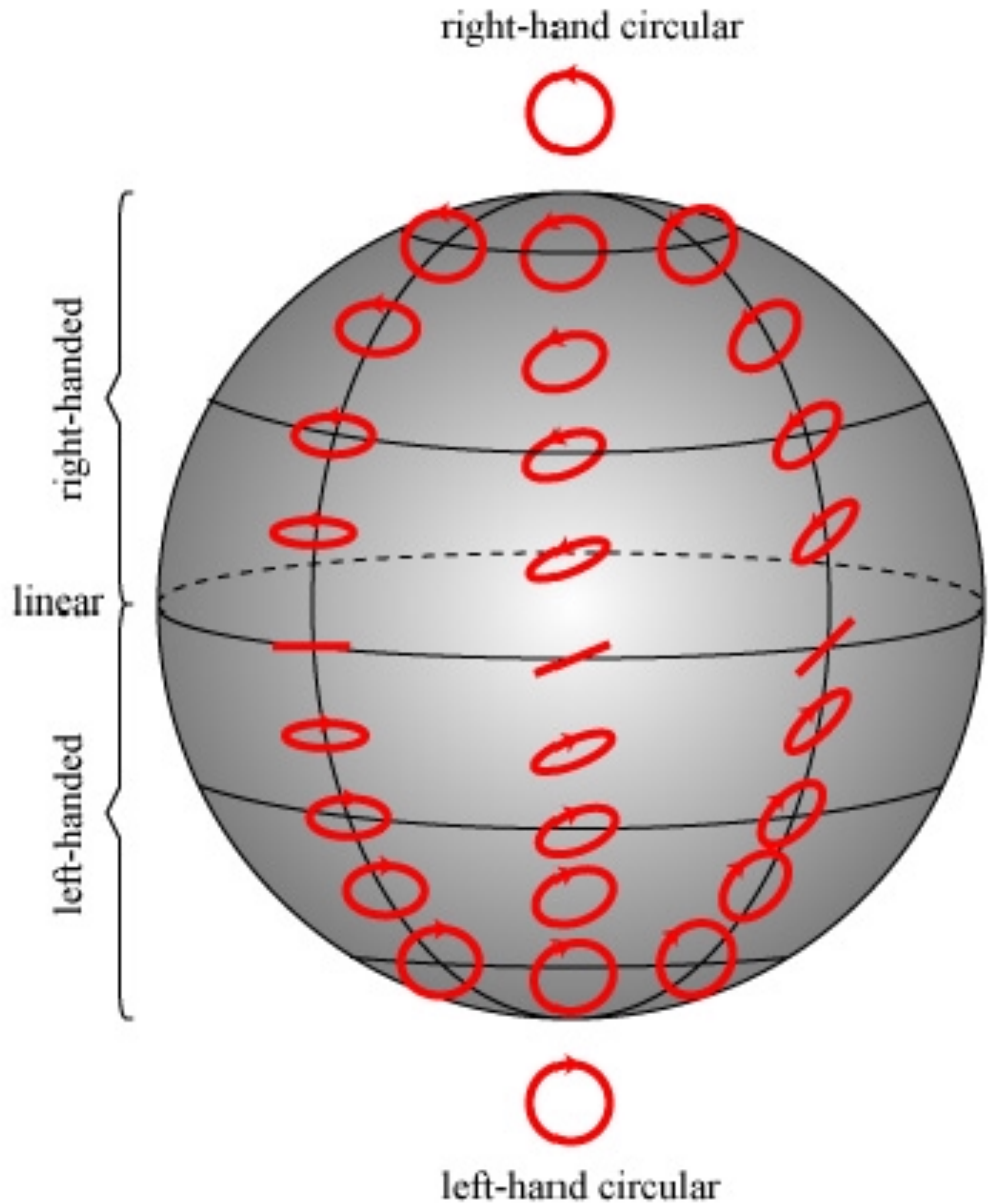
Bloch sphere

[should be called “Bloch ball”]

- The density matrix is a point *in* the ball
- Pure states are points on the surface of the ball
- Unitary evolutions are rotations of the ball or a reflection of the ball about a plane through the origin
- Two opposite points on the sphere correspond to orthogonal state vectors, hence to a basis of $\mathbb{C}^2$
- Every observable corresponds to a direction through the sphere
- Projective measurement involves projection onto a direction through the centre of the sphere
- The probabilities of the outcomes are proportional to the lengths of the two line segments (order switched!) into which the diameter is broken by the projection of the state onto the diameter.

Poincaré sphere

(Polarization: Stokes parameters, …)



Teleportation

- Suppose we have a state α 0 + β 1 and an entangled state 00 + 11
- Altogether, we have α 000 + α 011 + β 100 + β 111
- The first component is the target.
- Rewrite (ignoring normalisation constants) as 00 α0 + 01 α1 + 10 β0 + 11 β1

$$= (00 + 11)\ \alpha 0 + (01 + 10)\ \alpha 1 + (10 + 01)\ \beta 0 + (11 + 00)\ \beta 1$$

$$+ (00 - 11)\ \alpha 0 + (01 - 10)\ \alpha 1 + (10 - 01)\ \beta 0 + (11 - 00)\ \beta 1$$

$$= (00 + 11)\ (\alpha 0 + \beta 1) + (00 - 11)\ (\alpha 0 - \beta 1) + (01 + 10)\ (\alpha 1 + \beta 0) + (01 - 10)\ (\alpha 1 - \beta 0)$$

Now measure first two qubits in the "Bell basis" 00 +/– 11, 01 +/– 10 (four orthogonal vectors, same length, in $\mathbb{C}^2 \otimes \mathbb{C}^2$

- The third qubit, completely at random, becomes one of the four states α 0 +/– β 1 and α 1 +/– β 0

- Alice transmits *which* of her four she found, to Bob (two *classical bits* of communication)

- He now knows which of his four states he has, and knows which unitary brings it back to what he wants

- Nobody learns anything about α or β.

Computation of negative cosine for Bell state

Quantum mechanical predictions violate CHSH inequalities [edit]

The measurements performed by Alice and Bob are spin measurements on electrons. Alice can choose between two detector settings labeled a and a'; these settings correspond to measurement of spin along the z or the x axis. Bob can choose between two detector settings labeled b and b'; these correspond to measurement of spin along the z' or x' axis, where the $x' - z'$ coordinate system is rotated 135° relative to the $x - z$ coordinate system. The spin observables are represented by the 2 × 2 self-adjoint matrices:

$$S_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These are the Pauli spin matrices, which are known to have eigenvalues equal to ± 1. As is customary, we will use bra–ket notation to denote the eigenvectors of S_z as $|0\rangle, |1\rangle$, where

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consider now the single state $|\Phi^-\rangle$ defined as

$$|\Phi^-\rangle \equiv \frac{1}{\sqrt{2}} \left(|0,1\rangle - |1,0\rangle\right),$$

where we used the shortened notation $|0,1\rangle \equiv |0\rangle \otimes |1\rangle, |1,0\rangle \equiv |1\rangle \otimes |0\rangle$.

According to quantum mechanics, the choice of measurements is encoded into the choice of Hermitian operators applied to this state. In particular, consider the following operators:

$$\begin{aligned} A(a) &= S_z \otimes I \\ A(a') &= S_x \otimes I \\ B(b) &= \frac{-1}{\sqrt{2}}\, I \otimes (S_z + S_x) \\ B(b') &= \frac{1}{\sqrt{2}}\, I \otimes (S_z - S_x), \end{aligned}$$

where $A(a), A(a')$ represent two measurement choices of Alice, and $B(b), B(b')$ two measurement choices of Bob.

EXERCISE:
Find the measurements corresponding to *a*, *a´*, *b*, *b´* in the Bloch sphere

EXERCISE:
Suppose that we perform the simple measurement corresponding to the observable *A* on a system in the pure state $|\psi\rangle$.

Prove that the expectation value of the random outcome value is $\langle\psi|A|\psi\rangle = \mathrm{trace}(\rho A)$, $\rho = |\psi\rangle\langle\psi|$.

Computation of negative cosine for Bell state

For example, the expectation value $\langle A(a)B(b)\rangle$ corresponding to Alice choosing the measurement setting a and Bob choosing the measurement setting b is computed as

$$\langle A(a)B(b)\rangle \equiv \langle\Phi^-|\left(\frac{-1}{\sqrt{2}}S_z\otimes(S_x+S_z)\right)|\Phi^-\rangle = -\frac{1}{2}\langle\Phi^-|\Big[|0\rangle\otimes(|0\rangle-|1\rangle)+|1\rangle\otimes(|1\rangle+|0\rangle)\Big] = \frac{1}{\sqrt{2}}.$$

Similar computations are used to obtain

$$\begin{aligned}\langle A(a)B(b)\rangle = \langle A\left(a'\right)B(b)\rangle = \langle A\left(a'\right)B\left(b'\right)\rangle &= \frac{1}{\sqrt{2}}\\ \langle A(a)B\left(b'\right)\rangle &= -\frac{1}{\sqrt{2}}.\end{aligned}$$

It follows that the value of S given by this particular experimental arrangement is

$$\langle A(a)B(b)\rangle + \langle A\left(a'\right)B\left(b'\right)\rangle + \langle A\left(a'\right)B(b)\rangle - \langle A(a)B\left(b'\right)\rangle = \frac{4}{\sqrt{2}} = 2\sqrt{2} > 2.$$

Bell's Theorem: If the quantum mechanical formalism is correct, then the system consisting of a pair of entangled electrons cannot satisfy the principle of local realism. Note that $2\sqrt{2}$ is indeed the upper bound for quantum mechanics called Tsirelson's bound. The operators giving this maximal value are always isomorphic to the Pauli matrices.[26]

Optimal state and measurement for CHSH inequality

Tsirelson bound for the CHSH inequality [edit]

The first Tsirelson bound was derived as an upper bound on the correlations measured in the CHSH inequality. It states that if we have four (Hermitian) dichotomic observables A_0, A_1, B_0, B_1 (i.e., two observables for Alice and two for Bob) with outcomes $+1, -1$ such that $[A_i, B_j] = 0$ for all i, j, then

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2\sqrt{2}$$

For comparison, in the classical (or local realistic case) the upper bound is 2, whereas if any arbitrary assignment of $+1, -1$ is allowed it is 4. The Tsirelson bound is attained already if Alice and Bob each makes measurements on a qubit, the simplest non-trivial quantum system.

Lots of proofs have been developed for this bound, but perhaps the most enlightening one is based on the Khalfin-Tsirelson-Landau identity. If we define an observable

$$\mathcal{B} = A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1$$

and $A_i^2 = B_j^2 = \mathbb{I}$, i.e., if the outcomes of the observables are associated to projective measurements, then

$$\mathcal{B}^2 = 4\mathbb{I} - [A_0, A_1][B_0, B_1]$$

If $[A_0, A_1] = 0$ or $[B_0, B_1] = 0$, which can be regarded as the classical case, it already follows that $\langle \mathcal{B} \rangle \leq 2$. In the quantum case, we need only notice that $\|[A_0, A_1]\| \leq 2\|A_0\|\|A_1\| \leq 2$ and the Tsirelson bound $\langle \mathcal{B} \rangle \leq 2\sqrt{2}$ follows.

And what do <u>grown-ups</u> use?

- Some mathematicians swear by C^* algebras
- This allows us to work smoothly with "quantum ⊗ classical" spaces, and more
- Physicists use heuristics and intuition and heavy-duty functional analysis
- Obviously, Hilbert spaces have to become *infinite dimensional* and measurements may take values in *arbitrary measure spaces*
- Quantum field theory is what we use when there are *continuously* many "observables" everywhere in space
- There is no theory of *measurement* in quantum field theory … so it does not predict anything!
- The problem of quantising gravity (or even of deciding if that is a wise thing to do) is wide, wide open …

https://backreaction.blogspot.com/2019/09/the-five-most-promising-ways-to.html