Introduction to Quantum Statistics

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Plan of first lecture

- Historical remarks and motivation
- The basic notions: states, measurements, channels
- Current topics in Quantum Statistics
- State estimation; Quantum Cramér-Rao

Plan of second lecture

- Local asymptotic normality for i.i.d. states
- Quantum Cramér-Rao revisited
- [Local asymptotic normality for quantum Markov chains]

Plan of third lecture

- Quantum learning, sparsity
- Bell inequalities, quantum non-locality
- The measurement problem: the new eventum mechanics

Plan of third lecture

- Quantum learning, sparsity
- Bell inequalities, quantum non-locality
- The measurement problem: the new eventum mechanics

Don't worry...

There will be no third lecture!

Quantum mechanics up to the 60's

- Q.M. predicts probability distributions of measurement outcomes
- Perform measurements on huge ensembles
- Observed frequencies = probabilities

Old Paradigm

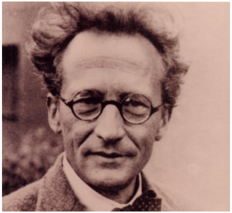
It makes no sense to talk about individual quantum systems

E. Schrödinger

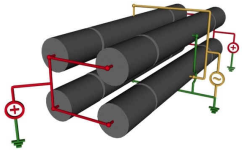
["Are there quantum jumps ?", British J.Phil. Science 1952]

"We are not experimenting with single particles, any more than we can raise Ichtyosauria in the zoo.

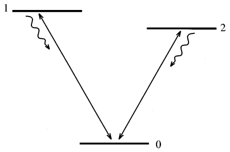
We are scrutinizing records of events long after they have happened."



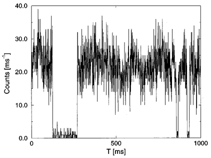
Are there quantum jumps ?



Paul ion trap



3 level atom driven by 2 lasers



Recorded fluorescence signal from 1 ion

- First experiments with individual quantum systems
- Measurements with stochastic outcomes
- Stochastic Schrödinger equations

Influx of mathematical ideas in the 70's

E. B. Davies

V. P. Belavkin

A. S. Holevo

C. W. Helstrom

- Probability
  what is the nature of quantum noise ?
- Filtering Theory
  what happens to the quantum system during measurement ?
- Information Theory
  how to encode, transmit and decode quantum information ?
- Statistics
  what do we learn from measurement outcomes ?

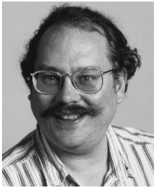
Quantum Information and Technology

New Paradigm

Individual quantum systems are carriers of a new type of information

Emerging fields:

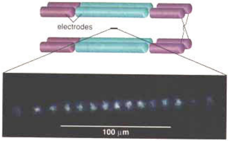
- Quantum Information Processing
- Quantum Computation and Cryptography
- Quantum Probability and Statistics
- Quantum Filtering and Control
- Quantum Engineering and Metrology



P. Shor

State estimation in quantum engineering

Multiparticle entanglement of trapped ions



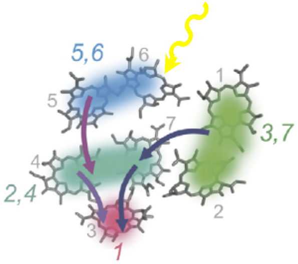
[Häfner *et al*, Nature 2005]

Experiment validation: statistical 'reconstruction' of the quantum state

- $4^8 - 1 = 65\,535$ parameters to estimate (8 ions)
- 10 hours measurement time
- weeks of computer time ('maximum likelihood')

System identification for complex dynamics

Photosynthesis: energy from light is transferred to a reaction center

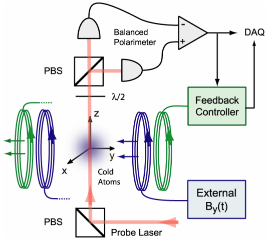


$$H = \begin{pmatrix} 215 & -104.1 & 5.1 & -4.3 & 4.7 & -15.1 & -7.8 \\ -104.1 & 220.0 & 32.6 & 7.1 & 5.4 & 8.3 & 0.8 \\ 5.1 & 32.6 & 0.0 & -46.8 & 1.0 & -8.1 & 5.1 \\ -4.3 & 7.1 & -46.8 & 125.0 & -70.7 & -14.7 & -61.5 \\ 4.7 & 5.4 & 1.0 & -70.7 & 450.0 & 89.7 & -2.5 \\ -15.1 & 8.3 & -8.1 & -14.7 & 89.7 & 330.0 & 32.7 \\ -7.8 & 0.8 & 5.1 & -61.5 & -2.5 & 32.7 & 280.0 \end{pmatrix}$$

J. Adolphs and T. Renger, Biophys. J. **91**, 2778 (2006)

- Complex system in noisy environment
- Theoretical modelling in parallel with statistical 'system identification'
- Find appropriate preparation and measurement designs

Quantum Filtering and Control



[Quantum Magnetometer, Mabuchi Lab]

- Observe and control quantum systems in real time
- Dynamics governed by Quantum Stochastic Differential equations
- Need for effective low dimensional dynamical models (e.g. 'Gaussian approx.')

Quantum mechanics as a probability theory

- States
- Observables

Quantum mechanics as a probabilistic theory

- States
- Observables
- Measurements
- Channels
- Instruments

Quantum states

- Complex Hilbert space of 'wave functions' $\mathcal{H} = \mathbb{C}^d, L^2(\mathbb{R})$...

- State = preparation: 'density matrix' ρ on $\mathcal{H}$
  - $\rho = \rho^*$ (selfadjoint)
  - $\rho \geq 0$ (positive)
  - $\mathrm{Tr}(\rho) = 1$ (normalised)

- Pure state: one dimensional projection $\Pi_\psi = |\psi\rangle\langle\psi|$ with $\|\psi\| = 1$

- Mixed state: convex combination of pure states $\rho = \sum_i q_i \Pi_{\psi_i}$

- Natural distances: $\tau := \rho_1 - \rho_2$

$$\|\tau\|_1 := \mathrm{Tr}(|\tau|) \quad \|\tau\|_2^2 := \mathrm{Tr}(\tau^2), \quad h(\rho_1, \rho_2) := 1 - \mathrm{Tr}\left(\sqrt{\rho_1^{1/2}\, \rho_2 \rho_1^{1/2}}\right)$$

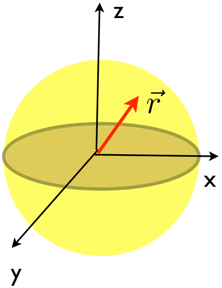
Example: spin (qubit) states

- Any density matrix ρ on $\mathbb{C}^2$ is of the form

$$\rho(\vec{r}) := \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - i r_y \\ r_x + i r_y & 1 - r_z \end{pmatrix} = \frac{1}{2}\left(\mathbf{1} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\right), \quad \|\vec{r}\| \leq 1$$

- $\rho(\vec{r})$ is pure if and only if $\|\vec{r}\| = 1$

- Bloch sphere representation



Quantum observables

- Observable: selfadjoint operator A on $\mathcal{H}$

- Spectral Theorem (diagonalisation):

$$A = \int_{\sigma(A)\subset\mathbb{R}} a\,\Pi(\mathrm{d}a) \qquad\qquad (A = \sum_j a_j \Pi_j)$$

- Probabilistic interpretation: measuring A gives random outcome $\mathbf{A} \in \{a_j\}$

$$\mathbb{P}_\rho[\mathbf{A} = a_j] = p_j = \mathrm{Tr}(\rho \Pi_j)$$

- Quantum and classical expectations

$$\mathrm{Tr}(\rho f(A)) \;\; = \;\; \sum_j f(a_j)\,\mathrm{Tr}(\rho\Pi_j) = \sum_j f(a_j) p_j = \mathbb{E}_\rho(f(\mathbf{A}))$$

Example: spin components

- Components of spin in x, y, z directions are given by the Pauli matrices

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

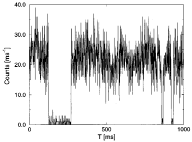
- Let $\rho = \frac{1}{2}\left(\mathbf{1} + \vec{r}\vec{\sigma}\right)$ then

$$\mathbb{P}_\rho[\sigma_i = \pm 1] = (1 \pm r_i)/2$$

- Different spin components are incompatible: $\sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z$

Indirect measurements

- Most real measurements are
  - indirect
  - extended in time



Recorded fluorescence signal from 1 ion

- 3 steps
  - couple state ρ with 'environment' in state σ: $\rho \Longrightarrow \rho \otimes \sigma$
  - interaction entangles systems 1 & 2: $\rho \otimes \sigma \Longrightarrow U(\rho \otimes \sigma)U^*$
  - measure environment observable $A = \sum_i a_i \Pi_i$

$$\mathbb{P}_\rho[\mathbf{A} = a_i] = \mathrm{Tr}_{1\&2}(U(\rho \otimes \sigma)U^* \, \mathbf{1} \otimes \Pi_i)$$

$$= \mathrm{Tr}_{1\&2}(\rho \otimes \sigma \, U^*(\mathbf{1} \otimes \Pi_i)U) = \mathrm{Tr}(\rho M_i)$$

- Positive, normalised linear map: $\rho \mapsto \{p_i = \mathrm{Tr}(\rho M_i)\}$

General measurements

Definition

A measurement on $\mathcal{H}$ with outcomes in (Ω, Σ) is a linear map

$$M : \mathcal{T}_1(\mathcal{H}) \to L^1(\Omega, \Sigma, \mu)$$

such that $p_\rho := M(\rho)$ is a probability density w.r.t. μ for each state ρ.

Any M is of the form

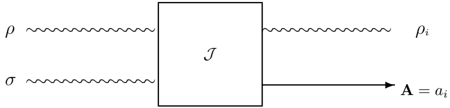
$$\mathbb{P}_\rho(E) = \int_E p_\rho \mathrm{d}\mu = \mathrm{Tr}(\rho\, m(E))$$

for some Positive Operator valued Measure (POVM) $\{m(E) : E \in \Sigma\}$.

- **Naimark's Theorem**

  Any measurement can be realised indirectly by 'usual' projection measurement on the environment

Quantum Instrument



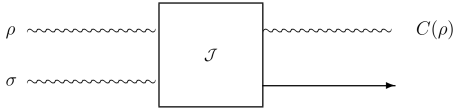
- Measure $B = \sum_j b_j Q_j$ and $A = \sum_i a_i P_j$

$$
\begin{aligned}
\mathbb{P}[\mathbf{B} = b_j \,\&\mathbf{A} = a_i] &= \mathrm{Tr}(U(\rho \otimes \sigma)U^* \; Q_j \otimes P_i) \\
&= \mathrm{Tr}(U(\rho \otimes |\psi\rangle\langle\psi|)U^* \; Q_j \otimes |e_i\rangle\langle e_i|) \\
&= \mathrm{Tr}(V_i \rho V_i^* Q_j) = p_i \mathrm{Tr}(\rho_i Q_j)
\end{aligned}
$$

where $V_i := \langle e_i, U\psi \rangle$ are the Kraus operators with $\sum_i V_i^* V_i = \mathbf{1}$

- Conditional state $\rho_i = V_i \rho V_i^* / p_i$

Quantum Channels



Definition

A quantum channel is a completely positive, normalised linear map

$$C : \mathcal{T}_1(\mathcal{H}) \to \mathcal{T}_1(\mathcal{H})$$

Stinespring-Kraus Theorem

Any channel C is of the form

$$C(\rho) = \sum_i V_i \rho V_i^*,$$

for some operators V_i satisfying $\sum_i V_i^* V_i = \mathbf{1}$.

Any channel can be realised indirectly by 'usual' product construction with some environment

Summary of quantum probability

- States are the analogue of probability distributions
- Observables are the analogue of random variables
- Dualities: $\mathcal{B}(\mathcal{H}) = \mathcal{T}_1(\mathcal{H})^*$ and $L^\infty(\Omega, \Sigma, \mu) = L^1(\Omega, \Sigma, \mu)^*$
- Measurements are quantum-to-classical randomisations $\rho \mapsto \mathbb{P}_\rho$
- Channels are quantum-to-quantum randomisations $\rho \mapsto C(\rho)$
- Instruments are quantum-to-mixed (classical and quantum) randomisations

Quantum Statistics

- The 70's
- Some current topics
- State estimation

Quantum Statistics in the 70's

- Helstrom, Holevo, Belavkin, Yuen, Kennedy...

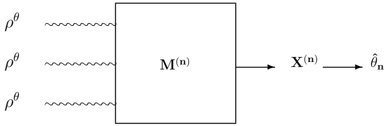
- Formulated and solved first quantum statistical decision problems

  - quantum statistical model $\mathcal{Q} = \{\rho_\theta \ : \ \theta \in \Theta\}$
  - decision problem (estimation, testing)
  - find optimal measurement (and estimator)

- Quantum Gaussian states, covariant families, state discrimination...

- Elements of a (purely) quantum statistical theory
  - Quantum Fisher Information
  - Quantum Cramér-Rao bound(s)
  - Holevo bound (*now* known to be *the* asymptotic quantum Cramér-Rao bound)
  - ...

Asymptotics in state estimation



- (Asymptotically) optimal measurements and rates for $d = 2$

  [Gill and Massar, P.R.A. 2002] [Bagan et al. (incl. Gill), P.R.A. 2006]

  [Hayashi and Matsumoto 2004] [Gill, 2005]

- Local asymptotic normality for $d < \infty$

  [Guta and Kahn, C.M.P. 2009]

Quantum Homodyne Tomography

- I.I.D. samples from Radon transform of the Wigner function
  [Vogel and H. Risken., P.R.A. 1989]

[Breitenbach *et al*, Nature 1997]

- Estimation of infinite dimensional states (non-parametric)
  [Artiles Guta and Gill, J.R.S.S. B, 2005] [Butucea, Guta and Artiles, Ann.Stat. 2007]

State estimation and compressed sensing

- $\{A_0 := \mathbf{1}, A_1, \ldots, A_{d^2-1}\}$ basis in $M(\mathbb{C}^d)$
- State ρ is characterised by Fourier coefficients $a_i := \mathrm{Tr}(\rho A_i)$
- Often ρ is known to be 'sparse' $(\mathrm{Rank}(\rho) = r \ll d)$
- How many (and which) observables are sufficient to estimate ρ ?
- Similar to the matrix completion problem (Netflix)

  [Candes and Recht, Found. Comp. Math. 2008]

State estimation and compressed sensing

Theorem [Gross 2009]

- $\{A_0 = \mathbf{1}, A_1, \ldots, A_{d^2-1}\}$ 'incoherent basis'
- Choose $i_1, \ldots, i_m \in \{1, \ldots, d^2 - 1\}$ randomly with $m = cdr(\log d)^2$
- Measure A_{i_k} and obtain the averages $a_k = \mathrm{Tr}(\rho A_{i_k})$

Then with high probability ρ is the unique solution of the s.d.o. problem

L_1-minimisation: minimise $\|\tau\|_1$ with constraints

$$\mathrm{Tr}(\tau) = 1 \quad \text{and} \quad \mathrm{Tr}(\tau A_{i_k}) = a_k$$

- The proof uses a Bernstein inequality for matrix valued r.v.

  [Ahlswede and Winter, IEEE Trans.Inf.Th. 2002]

$$\mathbb{P}\left[\|\sum_{i=1}^{m} X_i\| > t\right] \leq 2de^{-t^2/4m\sigma^2}, \qquad \sigma^2 = \|\mathbb{E}(X^2)\|$$

Asymptotics in state discrimination

- Two hypotheses $\rho^{\otimes n}$ and $\sigma^{\otimes n}$
- Test $M_n = \{A_{1,n}, A_{2,n} = \mathbf{1} - A_{1,n}\}$
- Error probabilities

$$\alpha(M_n) = \mathrm{Tr}(\rho^{\otimes n}(\mathbf{1} - A_{1,n})), \qquad \beta(M_n) = \mathrm{Tr}(\sigma^{\otimes n} A_{1,n})$$

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Quantum Stein Lemma

Let $\beta_n(\epsilon) = \inf\{\beta(M_n) \; : \; \alpha(M_n) \leq \epsilon\}$. Then

$$\lim_{n\to\infty} \frac{1}{n} \log \beta_n(\epsilon) = -S(\rho|\sigma) := -\mathrm{Tr}(\rho(\log\rho - \log\sigma))$$

[Hiai and Petz, C.M.P. 1991] [Ogawa and Nagaoka IEEE Trans. Inform. 2000]

Asymptotics in state discrimination

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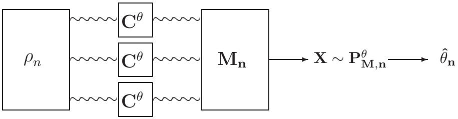
Quantum Chernoff bound

Let $p_n = \inf\{\pi_1 \alpha(M_n) + \pi_2 \beta(M_n) \ : \ M_n\}$ for prior (π_1, π_2). Then

$$\lim_{n\to\infty} \frac{1}{n} \log p_n = -\log\left(\inf_{s\in[0,1]} \mathrm{Tr}(\rho^s \sigma^{1-s})\right)$$

[Nussbaum and Szkola, Ann. Stat. 2009] [Audenaert *et al* C.M.P. 2008]

Estimation of Quantum Channels



- Fast(er) estimation rates (n^{-2}) for entangled input states

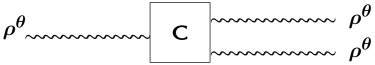
  [Kahn, P.R.A. 2007]

- Applications in Quantum Metrology

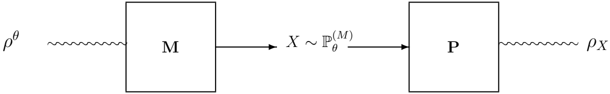
  [Giovanetti *et al*, Science 2004]

Quantum Cloning and Quantum Benchmarks

- Quantum no-cloning Theorem
  [381 papers on arXiv.org]



- Measure and prepare scheme vs teleportation
  [Hammerer *et al* P.R.L. 2005] [Owari *et al* N.J.P. 2008]



Quantum state estimation

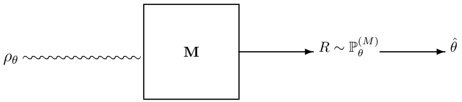
- Set-up
- Example: spin rotation model
- Quantum Cramér-Rao bound
- Quantum Gaussian states

Set-up of quantum estimation problems

- Quantum statistical model over Θ:

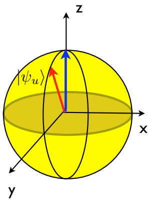
$$\mathcal{Q} = \{\rho_\theta \ : \ \theta \in \Theta\}$$

- Estimation procedure: measure state ρ_θ and devise estimator $\hat{\theta} = \hat{\theta}(R)$



- Measurement design:
  - which classical model $\mathcal{P}^{(M)} = \{\mathbb{P}_\theta^{(M)} : \theta \in \Theta\}$ is 'best' ?
  - trade-off between incompatible observables
  - optimal measurement depends on statistical problem

Example: estimating the direction of the spin vector



- One-dim. model: (small) rotation of $|\uparrow\rangle$

$$|\psi_u\rangle := \exp(iu\sigma_x)|\uparrow\rangle = \cos(u)|\uparrow\rangle + \sin(u)|\downarrow\rangle$$

- 'Most informative' spin observable is σ_y

$$\mathbb{E}(\sigma_y) = \sin(2u) \approx 2u$$

- Two parameter model $|\psi_{u_x,u_y}\rangle = \exp(i(u_y\sigma_x - u_x\sigma_y))|\uparrow\rangle$

- Optimal measurements for u_x and u_y are incompatible: $[\sigma_x, \sigma_y] \neq 0$

Quantum Cramér-Rao bound(s)*

Theorem [Helstrom, Holevo, Belavkin]

Let $\mathcal{Q} = \{\rho^\theta : \theta \in \mathbb{R}^k\}$ be a 'smooth' quantum model.

For any unbiased measurement M with outcome $\hat{\theta} \in \mathbb{R}^k$

$$\mathrm{Var}(\hat{\theta}) \geq I^{(M)}(\theta)^{-1} \geq H(\theta)^{-1}$$

- Helstrom's Quantum Fisher information matrix

$$H(\theta)_{i,j} := \mathrm{Tr}(\rho_\theta \mathcal{L}_{\theta,i} \circ \mathcal{L}_{\theta,j})$$

- Symmetric logarithmic derivatives: $\frac{\partial \rho_\theta}{\partial \theta_j} = \rho_\theta \circ \mathcal{L}_{\theta,j}$

*= several inequivalent C.R. bounds exist depending on symmetrisation

Proof (projection valued measurements)

Measure observable X and get result $\mathbf{X} \equiv \hat{\theta} \sim \mathbb{P}_\theta$

- Hilbert spaces $L^2(\rho_\theta)$ and $L^2(\mathbb{R}, \mathbb{P}_\theta)$

$$\langle A, B\rangle_\theta := \mathrm{Tr}(\rho_\theta A \circ B) \qquad \langle f(\mathbf{X}), g(\mathbf{X})\rangle_\theta = \mathbb{E}_\theta(f(\mathbf{X})g(\mathbf{X}))$$

- Fisher informations

$$H(\theta) := \mathrm{Tr}(\rho_\theta \mathcal{L}_\theta^2) = \|\mathcal{L}_\theta\|_\theta^2$$

$$I^{(M)}(\theta) := \mathbb{E}_\theta(\ell_\theta^2(\mathbf{X})) = \|\ell_\theta(\mathbf{X})\|_\theta^2$$

- Isometry

$$\begin{aligned} I : L^2(\mathbb{R}, \mathbb{P}_\theta) &\to L^2(\rho_\theta) \\ f(\mathbf{X}) &\mapsto f(X) \end{aligned}$$

$$\mathbb{E}_\theta(f(\mathbf{X})g(\mathbf{X})) = \mathrm{Tr}(\rho_\theta f(X) \circ g(X)) = \langle f(X), g(X)\rangle_\theta$$

Proof for projection valued measurements

- The projection of $\mathcal{L}_\theta$ onto $L^2(\mathbb{R}, \mathbb{P}_\theta)$ is $\ell_\theta(X)$

  Indeed for every $f \in L^2(p_\theta)$

$$\langle f(\mathbf{X}), \ell_\theta(\mathbf{X})\rangle_\theta = \frac{d\mathbb{E}_\theta(f)}{d\theta} = \mathrm{Tr}\left(\frac{d\rho_\theta}{d\theta} f(X)\right) = \langle f(X), \mathcal{L}_\theta\rangle_\theta$$

  $\Longrightarrow \|\ell_\theta\|_\theta^2 \leq \|\mathcal{L}_\theta\|_\theta^2$

$\square$

Bound achieved (locally) at θ_0 by

$$X = \theta_0 \mathbf{1} + \frac{\mathcal{L}_{\theta_0}}{H(\theta_0)}$$

Trade-off between parameters

- One-dimensional model: C.R. bound can be achieved asymptotically
  1. measure fraction $\tilde{n} \ll n$ of systems to obtain rough estimator θ_0
  2. measure $\mathcal{L}_{\theta_0}^{(n)} := \mathcal{L}_{\theta_0} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathcal{L}_{\theta_0}$
  3. set $\hat{\theta}_n := \theta_0 + \mathbf{L}_{\theta_0}^{(n)}/H(\theta_0)$

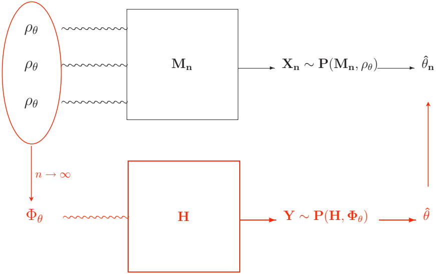
- Multi-dimensional model: $H(\theta)$ is achievable iff

$$\mathrm{Tr}(\rho_\theta[\mathcal{L}_{\theta,j}, \mathcal{L}_{\theta,i}]) = 0, \qquad \forall 1 \leq i,j \leq k$$

- Trade-off between estimation of different coordinates

- Optimal measurement depends on loss function (Holevo bound) and asymptotic risk is not simply expressible in terms of some quantum information matrix

Optimal estimation using local asymptotic normality



- Sequence of I.I.D. quantum statistical models $Q_n = \{\rho_\theta^{\otimes n} : \theta \in \Theta\}$
- $\mathcal{Q}_n$ converges (locally) to simpler Gaussian shift model $\mathcal{Q}$
- Optimal measurement for limit $\mathcal{Q}$ can be pulled back to $\mathcal{Q}_n$

Quantum Gaussian states

- Quantum 'particle' with canonical observables Q, P on $\mathcal{H} = L^2(\mathbb{R})$

$$QP - PQ = i\mathbf{1} \qquad \text{(Heisenberg's commutation relations)}$$

- Centred Gaussian state Φ

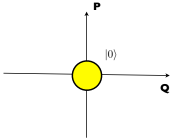
$$\mathrm{Tr}\left(\Phi \exp(-ivQ - iuP)\right) = \exp\left(-\frac{1}{2}\begin{pmatrix} u & v \end{pmatrix} V \begin{pmatrix} u \\ v \end{pmatrix}\right)$$

with 'covariance matrix' V satisfying the uncertainty principle

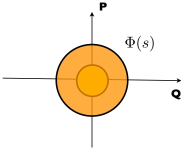
$$\mathrm{Det}(V) = \begin{vmatrix} \mathrm{Tr}(\Phi Q^2) & \mathrm{Tr}(\Phi Q \circ P) \\ \mathrm{Tr}(\Phi Q \circ P) & \mathrm{Tr}(\Phi P^2) \end{vmatrix} \geq \frac{1}{4}$$

Examples

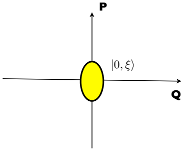
- Vaccum state $|0\rangle$
  $V = \mathrm{Diag}(\frac{1}{2},\ \frac{1}{2})$



- Thermal equilibrium state $\Phi(s)$
  $V = \mathrm{Diag}(\frac{s}{2},\ \frac{s}{2})$

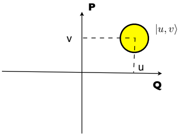


- Squeezed state $|0, \xi\rangle$
  $V = \mathrm{Diag}(\frac{e^{-\xi}}{2},\ \frac{e^{\xi}}{2})$

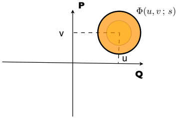


Quantum Gaussian shift model(s)

Displacement operator $D(u, v) := \exp(ivQ - iuP)$



- Coherent (laser) state
  $|u, v\rangle := D(u, v)|0\rangle$



- Displaced thermal state
  $\Phi(u, v\,;\,s) = D(u, v)\Phi(s)D(u, v)^*$

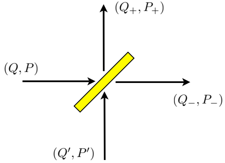
Optimal measurement for Gaussian shift

- Oscillator (Q, P) in state $|u, v\rangle$

- Oscillator (Q', P') in vacuum state $|0\rangle$

- Noisy coordinates commute: $[Q_+, P_-] = 0$

$$
\begin{aligned}
Q_\pm &:= Q \pm Q' \\
P_\pm &:= P \pm P'
\end{aligned}
$$



- Heterodyne measurement (Q_+, P_-) gives estimator $(\hat{u}, \hat{v}) \sim N((u, v), \mathbf{1})$

Theorem

The heterodyne measurement is optimal among covariant measurements and
achieves the minimax risk for the loss function $|u - \hat{u}|^2 + |v - \hat{v}|^2$.

Outlook

- Statistics is more and more employed in quantum experiments
- Remarkable coherence between quantum and classical statistics
- Trade-off between estimation of different parameters
- Optimal measurement depends on decision problem

More information:

Madalin Guta's Quantum Statistics course

http://maths.dept.shef.ac.uk/magic/course.php?id=64

Madalin Guta's Lunteren lectures

http://www.maths.nottingham.ac.uk/personal/pmzmig/Lunteren.pdf

References

R.D. Gill (2001)
Teleportation into quantum statistics
J. Korean Statistical Society 30, 291–325
arXiv:math.ST/0405572

R.D. Gill (2001)
Asymptotics in quantum statistics
IMS Monographs 36, 255–285
arXiv:math.ST/0405571

R.D. Gill (2005)
Asymptotic information bounds in quantum statistics.
QP and PQ
arXiv:math.ST/0512443.