Local asymptotic normality in Quantum Statistics

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Plan

- Estimation of quantum Gaussian states
- Local asymptotic normality in 'classical' statistics
- Convergence of quantum models
- Local asymptotic normality for i.i.d. quantum models
- Local asymptotic normality for quantum Markov chains

The quantum probabilistic framework

- State: positive operator ρ of trace one
- Measurement: $M : \rho \mapsto p_\rho \in L^1(\Omega, \Sigma, \mathbb{P})$

$$\mathbb{P}_\rho(E) = \mathrm{Tr}(\rho\, m(E)), \qquad E \in \Sigma$$

- Channel: $\rho \mapsto C(\rho)$

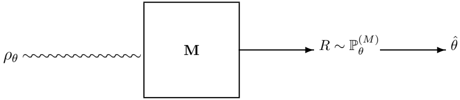
$$C(\rho) = \sum_i V_i \rho V_i^*, \qquad \sum_i V_i^* V_i = \mathbf{1}$$

State estimation

- Quantum statistical model over Θ:

$$\mathcal{Q} = \{\rho_\theta \ : \ \theta \in \Theta\}$$

- Estimation procedure: measure state ρ_θ and devise estimator $\hat{\theta} = \hat{\theta}(R)$

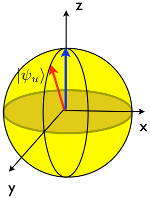


- Measurement design:
  - which classical model $\mathcal{P}^{(M)} = \{\mathbb{P}_\theta^{(M)} : \theta \in \Theta\}$ is 'best' ?
  - trade-off between incompatible observables
  - optimal measurement depends on statistical problem

Two examples

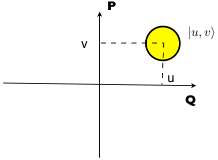
- Two parameter model in $\mathbb{C}^2$

  $|\psi_{u,v}\rangle = \exp(i(v\sigma_x - u\sigma_y))|\uparrow\rangle$



- Coherent (laser) state

  $|u, v\rangle = D(u, v)|0\rangle$



Quantum Gaussian states

- Quantum 'particle' with canonical observables Q, P on $\mathcal{H} = L^2(\mathbb{R})$

$$QP - PQ = i\mathbf{1} \qquad \text{(Heisenberg's commutation relations)}$$

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- Centred Gaussian state Φ

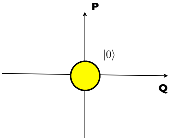
$$\mathrm{Tr}\left(\Phi \exp(-ivQ - iuP)\right) = \exp\left(-\frac{1}{2}\begin{pmatrix} u & v \end{pmatrix} V \begin{pmatrix} u \\ v \end{pmatrix}\right)$$

with 'covariance matrix' V satisfying the uncertainty principle

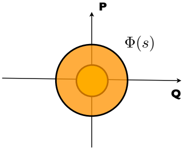
$$\mathrm{Det}(V) = \begin{vmatrix} \mathrm{Tr}(\Phi Q^2) & \mathrm{Tr}(\Phi Q \circ P) \\ \mathrm{Tr}(\Phi Q \circ P) & \mathrm{Tr}(\Phi P^2) \end{vmatrix} \geq \frac{1}{4}$$

Examples

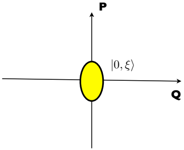
- Vaccum state $|0\rangle$
  $V = \mathrm{Diag}(\frac{1}{2},\ \frac{1}{2})$



- Thermal equilibrium state $\Phi(s)$
  $V = \mathrm{Diag}(\frac{s}{2},\ \frac{s}{2})$



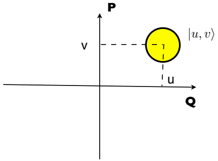
- Squeezed state $|0, \xi\rangle$
  $V = \mathrm{Diag}(\frac{e^{-\xi}}{2},\ \frac{e^{\xi}}{2})$



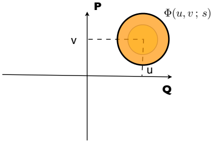
Quantum Gaussian shift model(s)

Displacement operator $D(u,v) := \exp(ivQ - iuP)$

- Coherent (laser) state
  $|u,v\rangle := D(u,v)|0\rangle$



- Displaced thermal state
  $\Phi(u,v\,;\,s) = D(u,v)\Phi(s)D(u,v)^*$



Optimal measurement for Gaussian shift

- Oscillator (Q, P) in state $|u, v\rangle$
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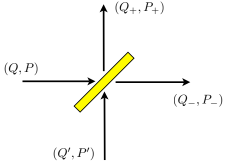
$$Q_\pm \quad := \quad Q \pm Q'$$
$$P_\pm \quad := \quad P \pm P'$$



- Heterodyne measurement (Q_+, P_-) gives estimator $(\hat{u}, \hat{v}) \sim N((u, v), \mathbf{1})$

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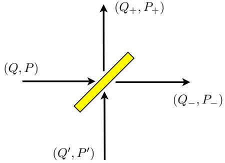
$$\begin{aligned} Q_\pm &:= Q \pm Q' \\ P_\pm &:= P \pm P' \end{aligned}$$



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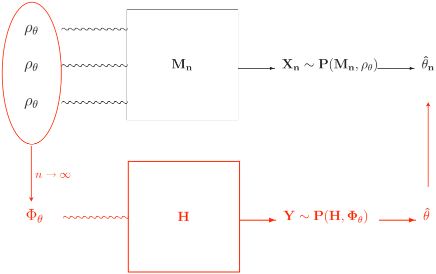
Theorem

The heterodyne measurement is optimal among covariant measurements and achieves the minimax risk for the loss function $|u - \hat{u}|^2 + |v - \hat{v}|^2$.

Optimal estimation using local asymptotic normality



- Sequence of I.I.D. quantum statistical models $Q_n = \{\rho_\theta^{\otimes n} : \theta \in \Theta\}$
- $\mathcal{Q}_n$ converges (locally) to simpler Gaussian shift model $\mathcal{Q}$
- Optimal measurement for limit $\mathcal{Q}$ can be pulled back to $\mathcal{Q}_n$

Convergence of quantum statistical models

- Sequence of quantum statistical models $\mathcal{Q}_n := \{\rho_{\theta,n} \ : \ \theta \in \Theta\}$
- Statistical decision problem for $\mathcal{Q}_n$ (estimation, testing...)

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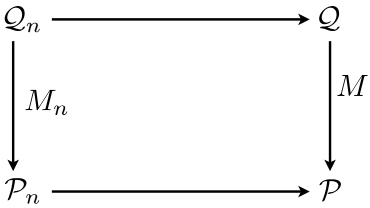
Guiding Principle

If $\mathcal{Q}_n$ 'converges' to $\mathcal{Q} := \{\rho_\theta \ : \ \theta \in \Theta\}$

then the optimal measurements M_n (and risks) 'converge' as well



Local asymptotic normality for coin toss

- $X_1, \ldots X_n$ i.i.d. with $\mathbb{P}[X_i = 1] = \theta$ and $\mathbb{P}[X_i = 0] = 1 - \theta$
- Estimator $\hat{\theta}_n := \frac{1}{n} \sum_{i=1}^{n} X_i$

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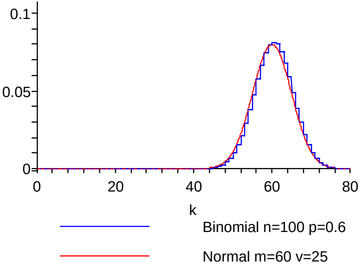

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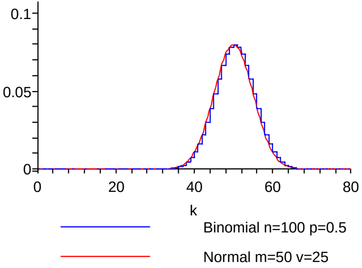

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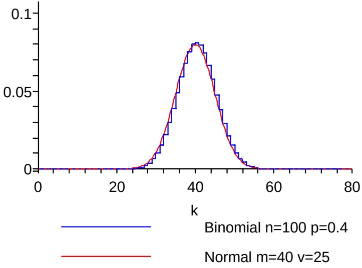

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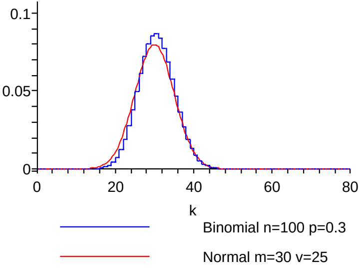

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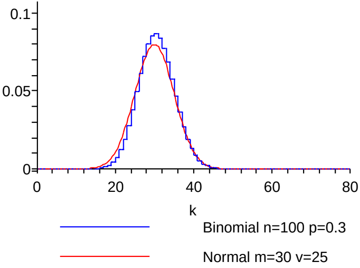
- Estimator $\hat{\theta}_n := \frac{1}{n} \sum_{i=1}^{n} X_i$
- Central Limit Theorem $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} N(0, \theta(1-\theta))$
- Local parameter: $\theta = \theta_0 + u/\sqrt{n}$

$$\hat{u}_n := \sqrt{n}(\hat{\theta}_n - \theta_0) \approx N(u, \theta_0(1-\theta_0))$$



LAN for general parametric model

- $(Y_1, \ldots, Y_n)$ i.i.d. with $\mathbb{P}^{\theta_0 + u/\sqrt{n}}$ a 'smooth' family with $u \in \mathbb{R}^k$. Then

$$\left\{\mathbb{P}^n_{\theta_0 + u/\sqrt{n}} : u \in \mathbb{R}^k\right\} \rightsquigarrow \left\{N(u, I_{\theta_0}^{-1}) : u \in \mathbb{R}^k\right\}$$

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- Weak convergence:

$$\left\{ \frac{\mathrm{d}\mathbb{P}^n_{\theta_0 + u/\sqrt{n}}}{\mathrm{d}\mathbb{P}^n_{\theta_0}} : u \in \mathbb{R}^k \right\} \xrightarrow{\mathcal{D}} \left\{ \frac{\mathrm{d}N(u, I_{\theta_0}^{-1})}{\mathrm{d}N(0, I_{\theta_0}^{-1})} : u \in \mathbb{R}^k \right\}$$

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- Strong convergence (Le Cam):

  there exist randomizations T_n, S_n such that

$$\lim_{n\to\infty} \sup_{\|u\|<a} \left\| T_n \, \mathbb{P}^n_{\theta_0+u/\sqrt{n}} - N(u, I_{\theta_0}^{-1}) \right\|_{\mathrm{tv}} = 0$$

  and

$$\lim_{n\to\infty} \sup_{\|u\|<a} \left\| \mathbb{P}^n_{\theta_0+u/\sqrt{n}} - S_n \, N(u, I_{\theta_0}^{-1}) \right\|_{\mathrm{tv}} = 0$$

Strong convergence of quantum models

Definition

Let $\mathcal{Q}_n := \{\rho_{\theta,n} \ : \ \theta \in \Theta\}$ and $\mathcal{Q} := \{\rho_\theta \ : \ \theta \in \Theta\}$.

Then $\mathcal{Q}_n$ converges strongly to $\mathcal{Q}$ if there exist quantum channels T_n, S_n s.t.

$$\lim_{n\to\infty} \sup_{\theta\in\Theta} \| T_n (\rho_{\theta,n}) - \rho_\theta \|_1 = 0$$

$$\lim_{n\to\infty} \sup_{\theta\in\Theta} \|\rho_{\theta,n} - S_n (\rho_\theta) \|_1 = 0$$

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Theorem

- If $\rho_{\theta,n} = |\psi_{\theta,n}\rangle\langle\psi_{\theta,n}|$ and $\rho_\theta = |\psi_\theta\rangle\langle\psi_\theta|$ then strong convergence implies

$$\lim_{n\to\infty} \langle \psi_{\theta_1,n} \,|\, \psi_{\theta_2,n}\rangle = \langle \psi_{\theta_1} \,|\, \psi_{\theta_2}\rangle, \qquad \text{(for some choice of phases!)}$$

- If Θ is finite the converse holds as well

Local asymptotic normality for i.i.d. spin states

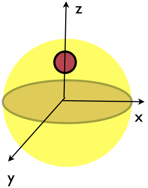
- Local spin model and its Gaussian limit
- Quantum C.L.T. and the big ball picture
- Coupling through isometry

Local spin model and the Gaussian limit

- $\left\{\rho_{\mathbf{u}/\sqrt{n}} \ : \ \mathbf{u} = (u_x, u_y, u_z)\right\}$ neighbourhood of $\rho_0 := \mathrm{Diag}(\mu, 1-\mu)$

$$\rho_{\mathbf{u}/\sqrt{n}} := U_n\left(u_x, u_y\right) \begin{bmatrix} \mu + \frac{u_z}{\sqrt{n}} & 0 \\ 0 & 1 - \mu - \frac{u_z}{\sqrt{n}} \end{bmatrix} U_n\left(u_x, u_y\right)^*$$

$$U_n(u_x, u_y) := \exp(i(u_y\sigma_x - u_y\sigma_y)/\sqrt{n})$$



Local spin model and the Gaussian limit

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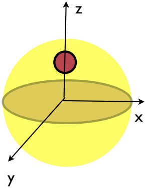


- Gaussian shift model: $N_{\mathbf{u}} \otimes \Phi_{\mathbf{u}}$
  - Classical part: $N_{\mathbf{u}} := N(u_z, \mu(1-\mu))$
  - Quantum part: $\Phi_{\mathbf{u}} := \Phi\left(u_x\sqrt{2(2\mu-1)},\ u_y\sqrt{2(2\mu-1)};\ (2\mu-1)^{-1}\right)$

Local asymptotic normality for mixed spin states

Theorem

Let $\rho_{\mathbf{u},n} := \left(\rho_{\mathbf{u}/\sqrt{n}}\right)^{\otimes n}$ be the state of n i.i.d. spins with $1/2 < \mu < 1$.

Then there exist quantum channels T_n, S_n such that for any $\eta < 1/4$

$$\lim_{n\to\infty} \sup_{\|\mathbf{u}\|<n^\eta} \left\| T_n\left(\rho_{\mathbf{u},n}\right) - N_{\mathbf{u}} \otimes \Phi_{\mathbf{u}} \right\|_1 = 0,$$

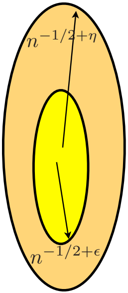
and

$$\lim_{n\to\infty} \sup_{\|\mathbf{u}\|<n^\eta} \left\| \rho_{\mathbf{u},n} - S_n\left(N_{\mathbf{u}} \otimes \Phi_{\mathbf{u}}\right) \right\|_1 = 0.$$

[Guta, Janssens and Kahn, C.M.P. 2008]

Asymptotically optimal (adaptive) measurement procedure

Given n i.i.d. spins prepared in state ρ_θ



1. Use $n^{1-\epsilon}$ copies to produce a rough estimator ρ_0
2. Map remaining $\tilde{n} = n - n^{1-\epsilon}$ states through $T_{\tilde{n}}$
3. Perform optimal Gaussian measurement and produce estimator

$$\hat{\theta}_n = \theta_0 + \hat{\mathbf{u}}/\sqrt{\tilde{n}}$$

L.A.N.: the big ball picture

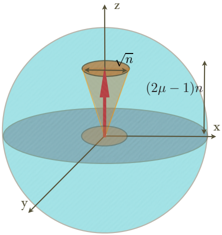
- Collective observables $L_{x,y,z} := \sum_{i=1}^{n} \sigma_{x,y,z}^{(i)}$

- Quantum Central Limit Theorem

$$\frac{L_{x,y}}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1)$$

$$\frac{L_z - n(2\mu-1)}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,\mu(1-\mu))$$

$$\left[\frac{L_x}{\sqrt{n}}, \frac{L_y}{\sqrt{n}}\right] = \frac{2i}{n} L_z \xrightarrow{l.l.n.} 2(2\mu-1)i\mathbf{1}$$



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$$\frac{L_{x,y}}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(2(2\mu-1)u_{x,y}, 1)$$

$$\frac{L_z - n(2\mu-1)}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(u_z, \mu(1-\mu))$$

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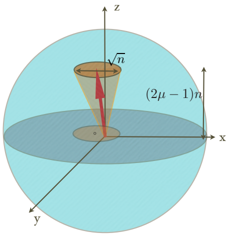

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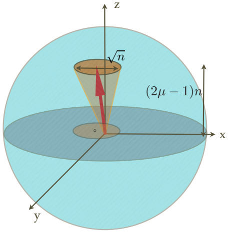


- Gaussian shift limit

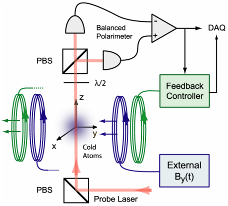
$$L_x/\sqrt{2n(2\mu-1)} \implies Q$$

$$L_y/\sqrt{2n(2\mu-1)} \implies P$$

$$\rho_{\mathbf{u},n} \implies N_{\mathbf{u}} \otimes \phi_{\mathbf{u}}$$

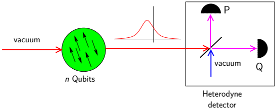
L.A.N. in practice

- L. A. N. is the proper statistical framework for “Gaussian approximation”



[Quantum Magnetometer, Mabuchi Lab]

- Proposal for experimental implementation of optimal estimation

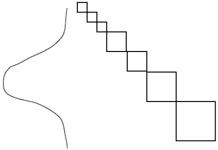


Idea of the proof: coupling

- Block diagonal form (Weyl Theorem)

$$\left(\mathbb{C}^2\right)^{\otimes n} = \bigoplus_{j=0,1/2}^{n/2} \mathbb{C}^{2j+1} \otimes \mathbb{C}^{d_j}$$

$$\rho_{\mathbf{u}/\sqrt{n}}^{\otimes n} = \bigoplus_{j=0,1/2}^{n/2} p_{\mathbf{u},n}(j)\, \rho_{\mathbf{u},n}(j) \otimes \mathrm{tr}_j$$



- Classical part: $p_{\mathbf{u},n}(j) = \mathbb{P}[L = j]$ with L the total spin

$$L \approx L_z \sim \mathrm{Bin}(\mu + u_z/\sqrt{n}, n) \xrightarrow{s.} N_{\mathbf{u}}$$

- Quantum part: embed conditional state $\rho_{\mathbf{u},j}$ isometrically into $L^2(\mathbb{R})$

$$V_j : \mathcal{H}_j \to L^2(\mathbb{R})$$

$$T_j : \rho_{\mathbf{u},j} \longmapsto V_j \rho_{\mathbf{u},j} V_j^*$$

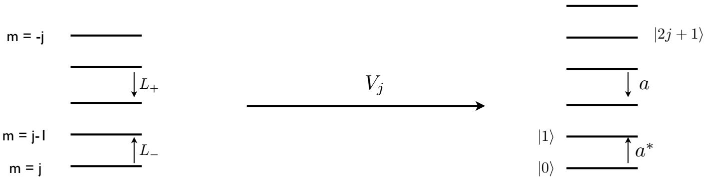
Isometric embedding

- Orthonormal bases

$$\begin{aligned} L_z|m,j\rangle &= m|m,j\rangle & (\ \mathbb{C}^{2j+1}\ ) \\ |k\rangle &= H_k(x)e^{-x^2/2} & (\ L^2(\mathbb{R})\ ) \end{aligned}$$

- Ladder operators

$$\begin{cases} L_+ & := L_x + iL_y \\ L_- & := L_x - iL_y \end{cases} \quad \text{and} \quad \begin{cases} a & := (Q+iP)/\sqrt{2} \\ a^* & := (Q-iP)/\sqrt{2} \end{cases}$$



Local asymptotic normality in d-dimensions

- Local model around $\rho_0 = \mathrm{Diag}(\mu_1, \ldots, \mu_d)$ with $\mu_1 > \mu_2 > \cdots > \mu_d > 0$

$$\rho_{\mathbf{u}/\sqrt{n}} = \begin{bmatrix} \mu_1 + h_1/\sqrt{n} & \ldots & z^*_{1,d}/\sqrt{n} \\ \vdots & \ddots & \vdots \\ z_{1,d}/\sqrt{n} & \ldots & \mu_d - \sum_{i=1}^{d-1} h_i/\sqrt{n} \end{bmatrix} \quad \mathbf{u} = (\mathbf{h}, \mathbf{z}) \in \mathbb{R}^{d-1} \times \mathbb{C}^{d(d-1)/2}$$

- Gaussian shift model: $N_{\mathbf{u}} \otimes \Phi_{\mathbf{u}}$
  - Classical part: $N_{\mathbf{u}} := N(\mathbf{z}, I_\mu^{-1})$
  - Quantum part: $\Phi_{\mathbf{u}} := \bigotimes_{1 \le j < k \le d} \Phi\left(\frac{z_{j,k}}{2\sqrt{\mu_j - \mu_k}} ; \frac{\mu_j + \mu_k}{\mu_j - \mu_k} \right)$

Local asymptotic normality in d-dimensions

Theorem

Let $\rho_{\mathbf{u},n} := \left(\rho_{\mathbf{u}/\sqrt{n}}\right)^{\otimes n}$ be the state of n i.i.d systems with $\mu_1 > \cdots > \mu_d > 0$.

Then there exist quantum channels T_n, S_n such that

$$\begin{aligned}
\lim_{n\to\infty} \sup_{\mathbf{u}\in\Theta_{n,\beta,\gamma}} \left\| T_n(\rho_{\mathbf{u},n}) - N_{\mathbf{u}} \otimes \Phi_{\mathbf{u}} \right\|_1 &= 0 \\
\lim_{n\to\infty} \sup_{\mathbf{u}\in\Theta_{n,\beta,\gamma}} \left\| S_n(N_{\mathbf{u}} \otimes \Phi_{\mathbf{u}}) - \rho_{\mathbf{u},n} \right\|_1 &= 0
\end{aligned}$$

where

$$\Theta_{n,\beta,\gamma} = \left\{ \mathbf{u} := (\mathbf{z}, \mathbf{d}) : \|\mathbf{z}\| \leq n^{\beta}, \|\mathbf{d}\| \leq n^{\gamma} \right\}, \text{ with } \beta < 1/9, \gamma < 1/4.$$

[Guta and Kahn, C.M.P. 2008]

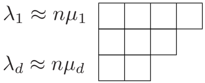
Blocks indexed by Young diagrams

- Block diagonal form

$$\left(\mathbb{C}^d\right)^{\otimes n} = \bigoplus_\lambda \mathcal{H}_\lambda \otimes \mathcal{K}_\lambda$$

$$\rho_{\mathbf{u}/\sqrt{n}}^{\otimes n} = \bigoplus_\lambda p_{\mathbf{u},n}(\lambda)\, \rho_{\mathbf{u},n}(\lambda) \otimes \mathrm{tr}_\lambda$$

- Young diagrams λ with d lines and n boxes



- Classical part: $p_{\mathbf{u},n} \approx \mathrm{Mult}\left(\mu_1 + \frac{h_1}{\sqrt{n}}, \ldots, \mu_d - \sum_i \frac{h_i}{\sqrt{n}}; n\right) \Longrightarrow N_{\mathbf{u}}$

Bases and ladder operators in $\mathcal{H}_\lambda$

- Non-orthogonal basis $|t, \lambda\rangle = |\mathbf{m}, \lambda\rangle$

  $\mathbf{m} = (m_{i,j} = \#\text{j's in row i}\} : i < j)$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

semi-standard Young tableau t

- Typical vectors are $\approx$ orthogonal

  If $|\mathbf{m}|, |\mathbf{l}| = O(n^\eta)$ with $\eta < 2/9$ then

$$|\langle \mathbf{m}, \lambda \,|\, \mathbf{l}, \lambda \rangle| = O(n^{-c(\eta)})$$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&2&2&3 \\ \hline 2&2&2&2&3&3 \\ \cline{1-6} 3&3&3 \\ \cline{1-3}\end{array}$$

typical Young tableau t

- Approximate ladder operators

$$L^*_{2,3} : \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&2&2&3 \\ \hline 2&2&2&2&3&3 \\ \cline{1-6} 3&3&3 \\ \cline{1-3}\end{array} \longrightarrow O(n^\eta) \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&2&3&3 \\ \hline 2&2&2&2&3&3 \\ \cline{1-6} 3&3&3 \\ \cline{1-3}\end{array} + O(n) \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&2&2&3 \\ \hline 2&2&2&3&3&3 \\ \cline{1-6} 3&3&3 \\ \cline{1-3}\end{array}$$

- Approximate isometry

$$V_\lambda : |\mathbf{m}\rangle \longmapsto \bigotimes_{1 \le j < k \le d} |m_{j,k}\rangle$$

Outlook

- Statistical inference is used more and more in quantum engineering
  - high dimensional estimation problems
  - system identification
  - low dimensional approximation of dynamics
- Remarkable coherence between classical and quantum statistics
  - Cramér-Rao bound(s)
  - Stein Lemma and Chernoff bound
  - Local asymptotic normality for i.i.d. states and quantum Markov chains
  - Quantum Sufficiency
- Open problems
  - Local asymptotic normality in infinite dimensions
  - General quantum decision theory
  - Statistics in dynamical framework

Madalin Guta's Quantum Statistics course

http://maths.dept.shef.ac.uk/magic/course.php?id=64

Madalin Guta's Lunteren lectures

http://www.maths.nottingham.ac.uk/personal/pmzmig/Lunteren.pdf

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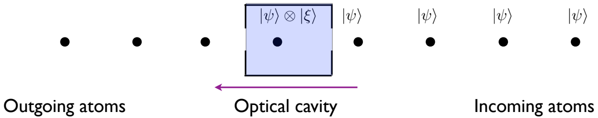
L.A.N. for Quantum Markov chains

- Dynamical system
- Ergodicity
- Local asymptotic normality
- Forgetfulness and Central Limit Theorem

Quantum Markov chains



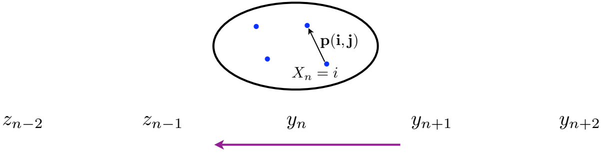
- Examples: quantum optical networks, atom maser, solid state cavity QED...
- Dynamics: unitary 'scattering' of atoms by cavity

$$U : M(\mathbb{C}^d \otimes \mathbb{C}^k) \to M(\mathbb{C}^d \otimes \mathbb{C}^k)$$

- System identification: estimate U by measuring outgoing atoms

[Kümmerer and Maassen, C.M.P. 1987]

Classical analogue



- Bernoulli shift Y_n
- Markov chain X_n driven by Y_n

$$X_{n+1} = F(X_n, Y_n)$$

- Observed (scattered) process Z_n

$$Z_n = S(X_n, Y_n)$$

Examples

- Jaynes-Cummings coupling

$$U : \mathbb{C}^2 \otimes \ell^2(\mathbb{N}) \to \mathbb{C}^2 \otimes \ell^2(\mathbb{N})$$

$$U = \exp\left[\alpha(\sigma_- \otimes a^* - \sigma_+ \otimes a) + i\beta\sigma_z + i\gamma a^* a\right]$$

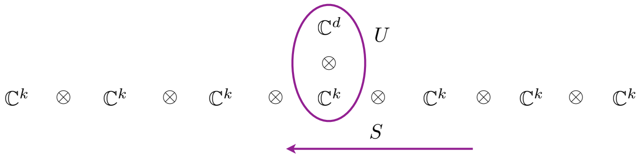
- Continuous-time quantum Markov process

$$U_t : \mathbb{C}^d \otimes \mathcal{F}(L^2(\mathbb{R}_+)) \to \mathbb{C}^d \otimes \mathcal{F}(L^2(\mathbb{R}_+))$$

$$dU_t = \left\{L \otimes dA_t^* - L^* \otimes dA_t - \frac{1}{2}L^* L\, dt - iH\, dt\right\} U_t \qquad \text{(QSDE)}$$

Hilbert space evolution

- 'system' $\mathbb{C}^d$, 'noise unit' $\mathbb{C}^k$, interaction unitary U



- One step joint evolution: $W = S \circ U$

Hilbert space evolution

- 'system' $\mathbb{C}^d$, 'noise unit' $\mathbb{C}^k$, interaction unitary U

$$\begin{array}{ccccccccccccc} & & & & & & |\xi\rangle & & & & & & \\ & & & & & & \otimes & & & & & & \\ |\psi\rangle & \otimes & |\psi\rangle & \otimes & |\psi\rangle & \otimes & |\psi\rangle & \otimes & |\psi\rangle & \otimes & |\psi\rangle & \otimes & |\psi\rangle \end{array}$$

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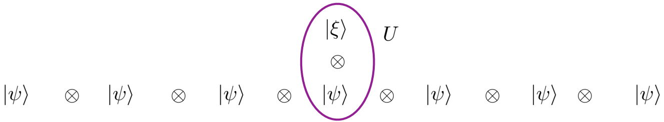

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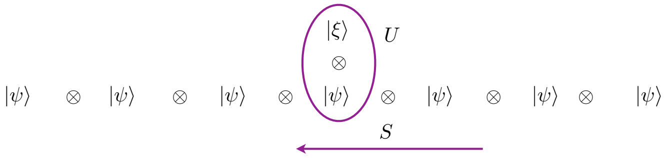

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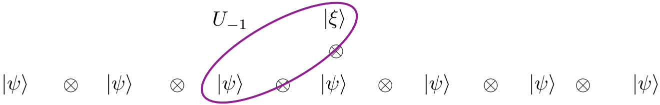

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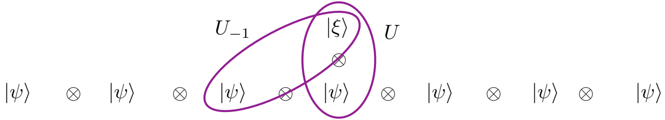

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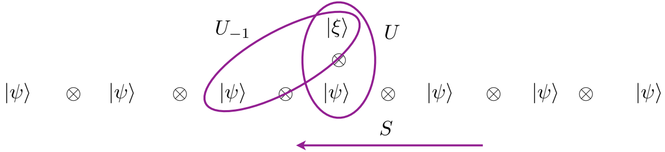

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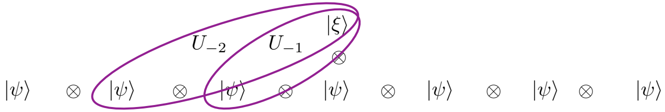
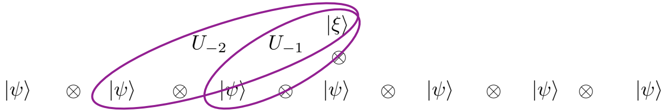


- One step joint evolution: $W = S \circ U$
- Output state after n steps

$$|\psi_n\rangle := U_{-1} \circ \cdots \circ U_{-n} \, |\xi\rangle \otimes |\psi\rangle^{\otimes n} \in \mathbb{C}^d \otimes \mathbb{C}^k$$

Back to quantum Markov chains

- 'system' $\mathbb{C}^d$, 'noise unit' $\mathbb{C}^k$, interaction unitary U



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Markov (transition) semigroup and ergodicity

- $T : M(\mathbb{C}^d) \to M(\mathbb{C}^d)$ describes the 'reduced' evolution of the system

$$X \mapsto T(X) := \left\langle \psi \right| U^{-1} \left(X \otimes \mathbf{1} \right) U \left| \psi \right\rangle$$

$$\begin{matrix} & & & X & & & \\ & & & \otimes & & & \\ \mathbf{1} \otimes & \mathbf{1} \otimes & \mathbf{1} \otimes & \mathbf{1} & \otimes \mathbf{1} & \otimes \mathbf{1} & \otimes \mathbf{1} \end{matrix}$$

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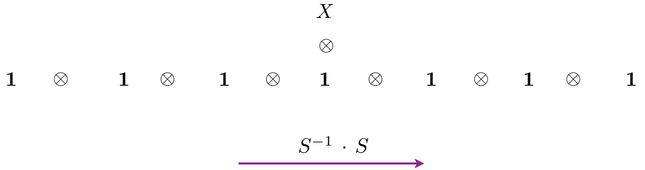

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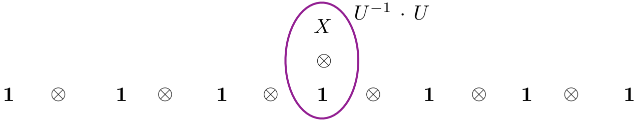

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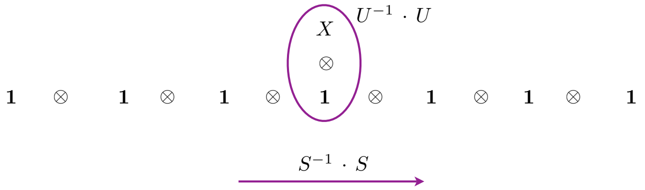

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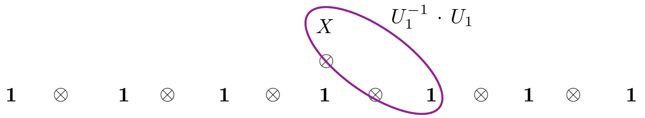

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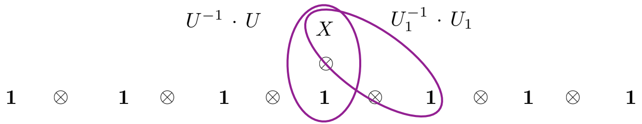

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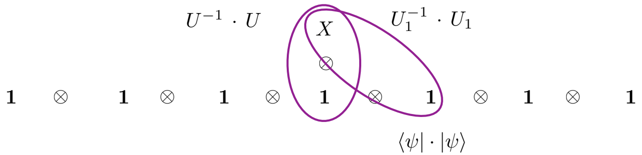

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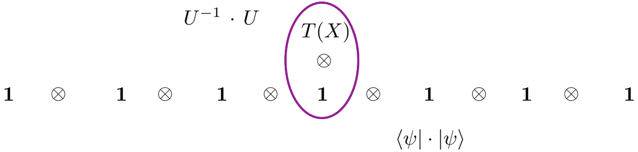

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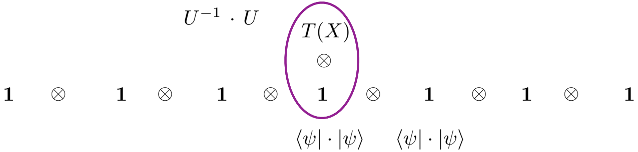

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$$\begin{array}{ccccccccccccc} & & & & & & T^2(X) & & & & & & \\ & & & & & & \otimes & & & & & & \\ \mathbf{1} & \otimes & \mathbf{1} & \otimes & \mathbf{1} & \otimes & \mathbf{1} & \otimes & \mathbf{1} & \otimes & \mathbf{1} & \otimes & \mathbf{1} \\ & & & & & & \langle\psi|\cdot|\psi\rangle & & \langle\psi|\cdot|\psi\rangle & & & & \end{array}$$

- after n steps

$$X \mapsto T_n(X) := \left\langle\, \psi^{\otimes n} \,\middle|\, \hat{U}^{-n}\, (X \otimes \mathbf{1})\, \hat{U}^{n} \,\middle|\, \psi^{\otimes n} \right\rangle = T^n(X)$$

Mixing quantum Markov chain

- The Markov chain (transition operator T) is called mixing if
  - $T(X) = X$ if and only if $X = \alpha\mathbf{1}$
  - All other eigenvalues λ satisfy $|\lambda| < 1$.

- Convergence to equilibrium

  If T is mixing then there exists a unique invariant state ρ_∞ on $M(\mathbb{C}^d)$ and

$$\lim_{n\to\infty} T_*^n(\sigma) = \rho_\infty, \qquad \text{for all initial states } \sigma$$

- Classical analogue

  Finite state irreducible aperiodic chain (Perron-Frobenius Therem)

L.A.N. for (one parameter) coupling constant

Theorem

- $U_\theta = \exp(i\theta H) \in \mathcal{U}(\mathbb{C}^d \otimes \mathbb{C}^k)$ with unknown coupling θ.
- Mixing transition operator $T_\theta(X) := \langle\, \psi \,|\; U_\theta^{-1}\,(X \otimes \mathbf{1})\, U_\theta \;|\, \psi \rangle$.

Then the output state (statistical model)

$$|\,\psi_{u,n}\rangle \;:=\; \left(S \circ U_{\theta_0 + u/\sqrt{n}}\right)^n \left|\xi \otimes \psi^{\otimes n}\right\rangle$$

is asymptotically normal, i.e

$$\lim_{n\to\infty} \langle \psi_{u,n} \,|\, \psi_{v,n} \rangle = \left\langle \phi_{\sqrt{2V}u} \,|\, \phi_{\sqrt{2V}v} \right\rangle = \exp\left(-V(u-v)^2/2\right),$$

where $\{\,|\,\phi_{\sqrt{2V}u}\rangle : u \in \mathbb{R}\}$ is the quantum Gaussian shift with Fisher info $4V$.

Fisher information = variance of generator

- The 'variance' V is given by

$$V = V(H,H) := \mathbb{E}\left(H^2\right) + 2\sum_{k=1}^{\infty} \mathbb{E}\left(H \circ \left(W_{\theta_0}^{-k} H W_{\theta_0}^{k}\right)\right)$$
$$= \mathbb{E}\left(H^2\right) + 2\mathbb{E}\left(U_{\theta_0}^{-1}\left(H \circ (\mathrm{Id} - T_{\theta_0})^{-1}(K)\right) U_{\theta_0}\right)$$

where

  - $\mathbb{E} := \rho_\infty \otimes |\psi\rangle\langle\psi|^{\otimes\infty}$ is the stationary state at θ_0
  - $K := \langle\psi|H|\psi\rangle$ is the conditional expectation of H onto the system,
  - $A \circ B := (AB + BA)/2$

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$$= \mathbb{E}\left(H^2\right) + 2\mathbb{E}\left(U_{\theta_0}^{-1}\left(H \circ (\mathrm{Id} - T_{\theta_0})^{-1}(K)\right) U_{\theta_0}\right)$$

  where

  - $\mathbb{E} := \rho_\infty \otimes |\psi\rangle\langle\psi|^{\otimes\infty}$ is the stationary state at θ_0
  - $K := \langle\psi|H|\psi\rangle$ is the conditional expectation of H onto the system,
  - $A \circ B := (AB + BA)/2$

- Interpretation:
  - limit model is family of coherent states $\tilde{\phi}_{\sqrt{2V}u} = \exp(iu\,\mathbb{G}(H))$
  - for optimal estimation of u measure conjugate variable of $\mathbb{G}(H)$

More insight into the limit model

- Forgetful quantum Markov chains
- Central Limit Theorem

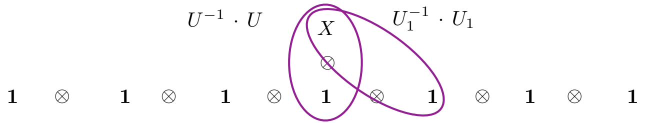
Forgetful Markov chains [Kretschmann and Werner 2005]

- A quantum Markov chain is called forgetful if there exist linear maps

$$R_n : M(\mathbb{C}^d) \to M(\mathbb{C}^k)^{\otimes n}$$

such that

$$\lim_{n\to\infty} \| W^{-n} \, (X \otimes \mathbf{1}) \, W^n - \mathbf{1} \otimes R_n(X) \| = 0, \qquad X \in M(\mathbb{C}^d)$$



Forgetful Markov chains [Kretschmann and Werner 2005]

- A quantum Markov chain is called forgetful if there exist linear maps

$$R_n : M(\mathbb{C}^d) \to M(\mathbb{C}^k)^{\otimes n}$$

such that

$$\lim_{n\to\infty} \|W^{-n}\,(X \otimes \mathbf{1})\,W^n - \mathbf{1} \otimes R_n(X)\| = 0, \qquad X \in M(\mathbb{C}^d)$$

- Example: the creation-annihilation interaction on $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$U_\alpha := \exp(-\alpha(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+))$$

is forgetful in a neighbourhood of $\alpha = \pi/2$, and

- Conjecture: U_α is forgetful for all $\alpha \in (0, \pi)$

Properties of forgetful Markov chains

- Forgetfulness is equivalent to asymptotic abelianess

$$\lim_{n\to\infty} \left\| \left[W^{-n} \left(X \otimes \mathbf{1} \right) W^{n} ,\ Y \otimes \mathbf{1} \right] \right\| = 0, \qquad X, Y \in M\mathbb{C}^{d})$$

- Forgetfulness implies mixing

- Controllability

  The system can be driven to any state asymptotically

- Observability

  Any measurement on the system can be performed indirectly

- Forgetfulness implies asymptotic completeness

  [Kümmerer and Maassen, Q.P.I.D.A. 2000]

CLT for forgetful Markov chains

- Forgetful Markov chain with unitary $U \in M(\mathbb{C}^d \otimes \mathbb{C}^k)$
- 'Local observable' $A \in M(\mathbb{C}^d \otimes \mathbb{C}^k)$ such that $\mathbb{E}(A) = 0$
- Fuctuation operator associated to A

$$\mathbb{F}_n(A) := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} A(k), \qquad A(k) := W^{-k} A W^k$$

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Then $\mathbb{F}_n(A)$ converges in distribution to $N(0, V(A, A))$ where

$$\begin{aligned} V(A, A) &:= \mathbb{E}\left(A^2\right) + 2 \sum_{k=1}^{\infty} \mathbb{E}\left(A \circ \left(W^{-k} A W^k\right)\right) \\ &= \mathbb{E}\left(A^2\right) + 2\mathbb{E}\left(A \circ \left(U^{-1} \left(\mathrm{Id} - T\right)^{-1} (B) U\right)\right), \qquad B := \langle \psi | A | \psi \rangle \end{aligned}$$

Outlook

- Statistical inference is used more and more in quantum engineering
  - high dimensional estimation problems
  - system identification
  - low dimensional approximation of dynamics
- Remarkable coherence between classical and quantum statistics
  - Cramér-Rao bound(s)
  - Stein Lemma and Chernoff bound
  - Local asymptotic normality for i.i.d. states and quantum Markov chains
  - Quantum Sufficiency
- Open problems
  - Local asymptotic normality in infinite dimensions
  - General quantum decision theory
  - Statistics in dynamical framework

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Weak and strong convergence for pure states models

Let $\mathcal{Q}_n := \{|\psi_{\theta,n}\rangle \ : \ \theta \in \Theta\}$ and $\mathcal{Q} := \{|\psi_{\theta}\rangle \ : \ \theta \in \Theta\}$

- $\mathcal{Q}_n$ converges weakly to $\mathcal{Q}$ if

$$\lim_{n\to\infty} \langle \psi_{\theta_1,n} \,|\, \psi_{\theta_2,n} \rangle = \langle \psi_{\theta_1} \,|\, \psi_{\theta_2} \rangle, \qquad \text{(for some choice of phases!)}$$

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$$\lim_{n\to\infty} \sup_{\theta\in\Theta} \| T_n \left(|\psi_{\theta,n}\rangle\langle\psi_{\theta,n}|\right) - |\psi_\theta\rangle\langle\psi_\theta| \|_1 = 0$$

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Theorem

- Strong convergence implies weak convergence
- If Θ is finite weak convergence is equivalent to strong convergence

Weak LAN for pure spin states

- $|\psi_\theta\rangle := \exp(i\theta\sigma_x)|\uparrow\rangle = \cos\theta\,|\uparrow\rangle + \sin\theta\,|\downarrow\rangle$

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Local asymptotic normality

$\{|\psi_{u,n}\rangle : u \in \mathbb{R}\}$ converges weakly to the Gaussian model $\{|\phi(\sqrt{2}u)\rangle : u \in \mathbb{R}\}$

$$\langle\psi_{u,n}|\psi_{v,n}\rangle = \cos((u-v)/\sqrt{n})^n \longrightarrow e^{-\frac{1}{2}(u-v)^2} = \langle\phi(\sqrt{2}u)|\phi(\sqrt{2}v)\rangle$$

Sufficient subalgebra

- Equivalent models

  $\mathcal{Q} := \{\rho_\theta \ : \ \theta \in \Theta\} \cong \mathcal{R} := \{\sigma_\theta \ : \ \theta \in \Theta\}$ if there exist channels T, S

$$T(\rho_\theta) = \sigma_\theta \qquad \text{and} \qquad S(\sigma_\theta) = \rho_\theta, \qquad \theta \in \Theta$$

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  $\mathcal{Q} := \{\rho_\theta \ : \ \theta \in \Theta\}$ with $\rho_\theta \in \mathcal{T}_1(\mathcal{H})$. $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is sufficient for $\mathcal{Q}$ if

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- Connes cocycles: $[D\rho_\theta, D\rho_{\theta_0}]_t := \rho_\theta^{it}\rho_{\theta_0}^{-it}$

Theorem [Petz and Jencova, C.M.P. 2005]

$\mathcal{A} := \mathrm{Alg}([D\rho_\theta, D\rho_{\theta_0}]_t : \theta \in \Theta, t \in \mathbb{R})$ is the minimal sufficient algebra for $\mathcal{Q}$.

Weak convergence of quantum statistical models

- Weak convergence [Guta and Jencova, C.M.P. 2007]

  $\mathcal{Q}_n := \{\rho_{\theta,n} : \theta \in \Theta\})$ converges weakly to $\mathcal{Q} := \{\rho_\theta : \theta \in \Theta\}$ if

$$\lim_{n\to\infty} \mathrm{Tr}\left(\rho_{\theta_0,n} \prod_{i=1}^{k} [D\rho_{\theta_i,n}\,,\, D\rho_{\theta_0,n}]_{t_i}\right) = \mathrm{Tr}\left(\rho_{\theta_0} \prod_{i=1}^{k} [D\rho_{\theta_i}\,,\, D\rho_{\theta_0}]_{t_i}\right)$$

- Asymptotic normality ($\rho_\theta \in M(\mathbb{C}^d)$)

$$\left\{\rho_{u,n} := \rho^{\otimes n}_{\theta_0 + u/\sqrt{n}} : u \in \mathbb{R}^{d^2-1}\right\} \xrightarrow{\mathrm{w}} \{\Phi_u : u \in \mathbb{R}^{d^2-1}\}$$

- Complete convergence theory to be developed