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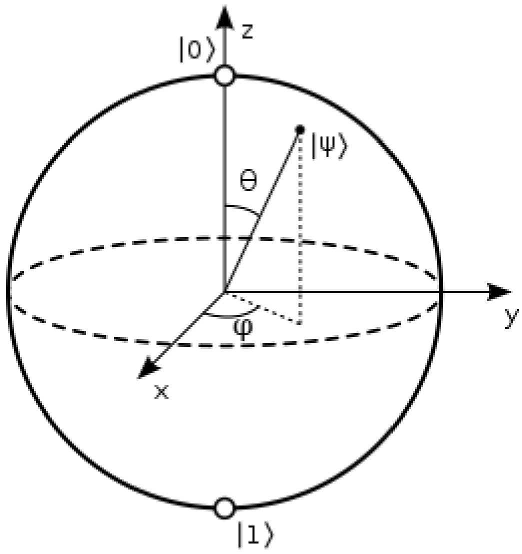
Bloch sphere

In quantum mechanics, the **Bloch sphere** is a geometrical representation of the pure state space of a two-level quantum mechanical system (qubit), named after the physicist Felix Bloch.[1]



Bloch sphere

Quantum mechanics is mathematically formulated in Hilbert space or projective Hilbert space. The space of pure states of a quantum system is given by the one-dimensional subspaces of the corresponding Hilbert space (or the "points" of the projective Hilbert space). For a two-dimensional Hilbert space, this is simply the complex projective line $\mathbb{CP}^1$. This is the Bloch sphere.

The Bloch sphere is a unit 2-sphere, with antipodal points corresponding to a pair of mutually orthogonal state vectors. The north and south poles of the Bloch sphere are typically chosen to correspond to the standard basis vectors $|0\rangle$ and $|1\rangle$, respectively, which in turn might correspond e.g. to the spin-up and spin-down states of an electron. This choice is arbitrary, however. The points on the surface of the sphere correspond to the pure states of the system, whereas the interior points correspond to the mixed states.[2][3] The Bloch sphere may be generalized to an *n*-level quantum system, but then the visualization is less useful.

For historical reasons, in optics the Bloch sphere is also known as the Poincaré sphere and specifically represents different types of polarizations. Six common polarization types exist and are called Jones vectors. Indeed Henri Poincaré was the first to suggest the use of this kind of geometrical representation at the end of 19th century,[4] as a three-dimensional representation of Stokes parameters.

The natural metric on the Bloch sphere is the Fubini–Study metric. The mapping from the unit 3-sphere in the two-dimensional state space $\mathbb{C}^2$ to the Bloch sphere is the Hopf fibration, with each ray of spinors mapping to one point on the Bloch sphere.

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Definition

Given an orthonormal basis, any pure state $|\psi\rangle$ of a two-level quantum system can be written as a superposition of the basis vectors $|0\rangle$ and $|1\rangle$, where the coefficient or amount of each basis vector is a complex number. Since only the relative phase between the coefficients of the two basis vectors has any physical meaning, we can take the coefficient of $|0\rangle$ to be real and nonnegative.

We also know from quantum mechanics that the total probability of the system has to be one: $\langle\psi|\psi\rangle = 1$, or equivalently $\| \, |\psi\rangle \, \|^2 = 1$. Given this constraint, we can write $|\psi\rangle$ using the following representation:

$$|\psi\rangle = \cos(\theta/2)|0\rangle \; + \; e^{i\phi} \sin(\theta/2)|1\rangle = \cos(\theta/2)|0\rangle \; + \; (\cos\phi + i\sin\phi) \; \sin(\theta/2)|1\rangle$$, where $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$.

Except in the case where $|\psi\rangle$ is one of the ket vectors (see Bra-ket notation) $|0\rangle$ or $|1\rangle$, the representation is unique. The parameters θ and ϕ, re-interpreted in spherical coordinates as respectively the colatitude with respect to the *z*-axis and the longitude with respect to the *x*-axis, specify a point

$$\vec{a} = (\sin\theta\cos\phi, \; \sin\theta\sin\phi, \; \cos\theta) = (u, v, w)$$

on the unit sphere in $\mathbb{R}^3$.

For mixed states, one considers the density operator. Any two-dimensional density operator ρ can be expanded using the identity I and the Hermitian, traceless Pauli matrices $\vec{\sigma}$:

$$\rho = \frac{1}{2}\left(I + \vec{a}\cdot\vec{\sigma}\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a_x}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{a_y}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{a_z}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\vec{a} \in \mathbb{R}^3$ is called the **Bloch vector**.

It is this vector that indicates the point within the sphere that corresponds to a given mixed state. Specifically, as a basic feature of the Pauli vector, the eigenvalues of ρ are $\frac{1}{2}(1 \pm |\vec{a}|)$. Density operators must be positive-semidefinite, so we conclude that $|\vec{a}| \le 1$. For pure states, we must then have

$$\operatorname{tr}(\rho^2) = \frac{1}{2}\left(1 + |\vec{a}|^2\right) = 1 \quad \Leftrightarrow \quad |\vec{a}| = 1 \,,$$

in accordance with the above.

As a consequence, the surface of the Bloch sphere represents all the pure states of a two-dimensional quantum system, whereas the interior corresponds to all the mixed states.

u,v,w Representation

The Bloch vector $\vec{a} = (u, v, w)$ can be represented in the following basis, with reference to the density operator ρ:[5]

$$u = \rho_{10} + \rho_{01} = 2\mathrm{Re}(\rho_{01})$$
$$v = i(\rho_{01} - \rho_{10}) = 2\mathrm{Im}(\rho_{10})$$
$$w = \rho_{00} - \rho_{11}$$

where

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+w & u-iv \\ u+iv & 1-w \end{pmatrix}.$$

This basis is often used in laser theory, where w is known as the population inversion.[6]

Pure states

Consider an n-level quantum mechanical system. This system is described by an n-dimensional Hilbert space H_n. The pure state space is by definition the set of 1-dimensional rays of H_n.

Theorem. Let $\mathrm{U}(n)$ be the Lie group of unitary matrices of size n. Then the pure state space of H_n can be identified with the compact coset space

$$\mathrm{U}(n)/(\mathrm{U}(n-1) \times \mathrm{U}(1)).$$

To prove this fact, note that there is a natural group action of $\mathrm{U}(n)$ on the set of states of H_n. This action is continuous and transitive on the pure states. For any state $|\psi\rangle$, the isotropy group of $|\psi\rangle$, (defined as the set of elements g of $\mathrm{U}(n)$ such that $g|\psi\rangle = |\psi\rangle$) is isomorphic to the product group

$$\mathrm{U}(n-1) \times \mathrm{U}(1).$$

In linear algebra terms, this can be justified as follows. Any g of $\mathrm{U}(n)$ that leaves $|\psi\rangle$ invariant must have $|\psi\rangle$ as an eigenvector. Since the corresponding eigenvalue must be a complex number of modulus 1, this gives the U(1) factor of the isotropy group. The other part of the isotropy group is parametrized by the unitary matrices on the orthogonal complement of $|\psi\rangle$, which is isomorphic to $\mathrm{U}(n - 1)$. From this the assertion of the theorem follows from basic facts about transitive group actions of compact groups.

The important fact to note above is that the *unitary group acts transitively* on pure states.

Now the (real) dimension of $\mathrm{U}(n)$ is n^2. This is easy to see since the exponential map

$$A \mapsto e^{iA}$$

is a local homeomorphism from the space of self-adjoint complex matrices to $\mathrm{U}(n)$. The space of self-adjoint complex matrices has real dimension n^2.

Corollary. The real dimension of the pure state space of H_n is $2n - 2$.

In fact,

$$n^2 - ((n-1)^2 + 1) = 2n - 2.$$

Let us apply this to consider the real dimension of an m qubit quantum register. The corresponding Hilbert space has dimension 2^m.

Corollary. The real dimension of the pure state space of an m-qubit quantum register is $2^{m+1} - 2$.

Density operators

Formulations of quantum mechanics in terms of pure states are adequate for isolated systems; in general quantum mechanical systems need to be described in terms of density operators. The Bloch sphere parametrizes not only pure states but mixed states for 2-level systems. The density operator describing the mixed-state of a 2-level quantum system (qubit) corresponds to a point *inside* the Bloch sphere with the following coordinates:

$$(\Sigma p_i x_i, \Sigma p_i y_i, \Sigma p_i z_i),$$

where p_i is the probability of the individual states within the ensemble and x_i, y_i, z_i are the coordinates of the individual states (on the *surface* of Bloch sphere). The set of all points on and inside the Bloch sphere is known as the *Bloch ball.*

For states of higher dimensions there is difficulty in extending this to mixed states. The topological description is complicated by the fact that the unitary group does not act transitively on density operators. The orbits moreover are extremely diverse as follows from the following observation:

Theorem. Suppose A is a density operator on an n level quantum mechanical system whose distinct eigenvalues are μ_1, ..., μ_k with multiplicities n_1, ...,n_k. Then the group of unitary operators V such that $V A V^* = A$ is isomorphic (as a Lie group) to

$$\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k).$$

In particular the orbit of A is isomorphic to

$$\mathrm{U}(n)/(\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k)).$$

It is possible to generalize the construction of the Bloch ball to dimensions larger than 2, but the geometry of such a "Bloch body" is more complicated than that of a ball.[7]

See also

- Specific implementations of the Bloch sphere are enumerated under the qubit article.
- Atomic electron transition
- Gyrovector space
- Versors

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