Tsirelson's bound

A **Tsirelson bound** is an upper limit to quantum mechanical correlations between distant events. Given that quantum mechanics is non-local, i.e., that quantum mechanical correlations violate Bell inequalities, a natural question to ask is "how non-local can quantum mechanics be?", or, more precisely, by how much can the Bell inequality be violated. The answer is precisely the Tsirelson bound for the particular Bell inequality in question. In general this bound is lower than what would be algebraically possible, and much research has been dedicated to the question of why this is the case.

The Tsirelson bounds are named after Boris S. Tsirelson (or Cirel'son, in a different transliteration), the author of the paper[1] in which the first one was derived.

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Tsirelson bound for the CHSH inequality

The first Tsirelson bound was derived as an upper bound on the correlations measured in the CHSH inequality. It states that if we have four (Hermitian) dichotomic observables A_0, A_1, B_0, B_1 (i.e., two observables for Alice and two for Bob) with outcomes $+1, -1$ such that $[A_i, B_j] = 0$ for all i, j, then

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2\sqrt{2}$$

For comparison, in the classical (or local realistic case) the upper bound is 2, whereas if any arbitrary assignment of $+1, -1$ is allowed it is 4. The Tsirelson bound is attained already if Alice and Bob each makes measurements on a qubit, the simplest non-trivial quantum system.

Lots of proofs have been developed for this bound, but perhaps the most enlightening one is based on the Khalfin-Tsirelson-Landau identity. If we define an observable

$$\mathcal{B} = A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1$$

and $A_i^2 = B_j^2 = \mathbb{I}$, i.e., if the outcomes of the observables are associated to projective measurements, then

$$\mathcal{B}^2 = 4\mathbb{I} - [A_0, A_1][B_0, B_1]$$

If $[A_0, A_1] = 0$ or $[B_0, B_1] = 0$, which can be regarded as the classical case, it already follows that $\langle \mathcal{B} \rangle \leq 2$. In the quantum case, we need only notice that $\|[A_0, A_1]\| \leq 2\|A_0\|\|A_1\| \leq 2$ and the Tsirelson bound $\langle \mathcal{B} \rangle \leq 2\sqrt{2}$ follows.

Tsirelson bounds for other Bell inequalities

Obtaining a Tsirelson bound for a given Bell inequality is in general a hard problem that has to be solved on a case-by-case basis, although there are numerical algorithms that can upperbound it [2]. The exact values are known for a few more Bell inequalities:

For the Braunstein-Caves inequalities we have that

$$\langle BC_n \rangle \leq n\cos(\pi/n)$$

For the WWŻB inequalities the Tsirelson bound is

$$\langle WWZB_n \rangle \leq 2^{\frac{n-1}{2}}$$

Finding the Tsirelson bound for the I_{3322} inequality is a notorious open problem in quantum information theory.

Tsirelson bounds from physical principles

Lots of research have been dedicated to find a physical principle that explains why quantum correlations go only up to the Tsirelson bound and nothing more. Three such principles have been found: no-advantage for non-local computation [3], Information causality[4] and macroscopic locality[5]. That is to say, if one could achieve a CHSH correlation exceeding Tsirelson's bound, all such principles would be violated. Tsirelson's bound also follows if the Bell experiment admits a strongly positive quansal measure[6].

See also

- Bell's theorem
- EPR paradox
- CHSH inequality
- Quantum pseudo-telepathy

References

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