Simple refutation of Joy Christian's simple refutation of Bell's simple theorem

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Abstract

I point out a simple algebraic error in Joy Christian's refutation of Bell's theorem. In substituting the result of multiplying some derived bivectors with one another by consultation of their multiplication table, he confuses the generic vectors which he used to define the table, with other specific vectors having a special role in the paper, which had been introduced earlier. The result should be expressed in terms of the derived bivectors which indeed do follow this multiplication table. When correcting this calculation, the result is not the singlet correlation any more. Moreover, curiously, his normalized correlations are independent of the number of measurements and certainly do not require letting n converge to infinity. At the same time, his unnormalized or raw correlations are identically equal to -1, independently of the number of measurements! Correctly computed, his standardized correlations are the bivectors $-\boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \wedge \boldsymbol{b}$, and they find their origin entirely in his normalization or standardization factors. I conclude that his research program has been set up around an elaborately hidden but trivial mistake.

In at least 11 papers on quant-ph author Joy Christian proposes a local hidden variables model for quantum correlations which disproves Bell's theorem "by counterexample" in a number of different settings, including the famous CHSH and GHZ versions. Fortunately one of these papers is just one page long and concentrates on the mathematical heart of his work. Unfortunately for his grand project, this version enables us to clearly see a rather shorter derivation of the desired correlations, which exposes an error in his own derivation. The error is connected with an unfortunate notational ambiguity at the very start of the paper. In a nutshell: the same symbols are used to denote both a certain fixed basis $(\boldsymbol{\beta}_j, j = 1, 2, 3)$ in terms of which two other bases are defined, $(\boldsymbol{\beta}_j(+1), j = 1, 2, 3)$ and $(\boldsymbol{\beta}_j(-1), j = 1, 2, 3)$, as well as to express the generic algebraic multiplication rules which these latter two bases satisfy. This tiny ambiguity, though harmless locally, is probably the reason why later on, when apparently using the multiplication tables for the new bases, he silently shifts from the derived bases to the original special basis.

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The reader will need a copy of the one page paper Christian (2011), arXiv:1103.1879 (the body of this paper is reproduced in the appendix). The context in which he works is so-called Geometric Algebra, which in this case means that we are working within the so-called even sub-algebra of standard Clifford algebra, or if you prefer, with quaternions.

At the start of the paper the author fixes a *bivector basis* $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3)$ satisfying (usual Kronecker and Levi-Civita symbols δ, ϵ) the multiplication rules

$$\boldsymbol{\beta}_j \boldsymbol{\beta}_k \;=\; -\delta_{jk} - \sum_l \epsilon_{jkl} \boldsymbol{\beta}_l. \tag{1}$$

He writes that $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, $\boldsymbol{\beta}_3$ are *defined* by this multiplication table, but that is not exactly true. One can say that the *algebra* of bivectors is defined by the multiplication table, but the bivectors themselves are clearly not, since different bases can have the same multiplication table. The *bivector algebra* is the algebra of formal real linear combinations of real numbers and $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, $\boldsymbol{\beta}_3$. It is therefore a four dimension real vector space, with on top of the vector space structure a multiplication operation or *bivector product*. This is defined by combining ordinary real multiplication with the 3×3 multiplication table of the bivector basis into the obvious 4×4 multiplication table for the four vector-space basis elements $+1$, $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, and $\boldsymbol{\beta}_3$.

The bivector product is associative but not commutative. Non-zero elements have multiplicative inverses, left and right inverses coincide. Vector-space scalar multiplication of elements of the algebra by real numbers is identical with algebraic multiplication of elements of the algebra, either from the left or the right, by elements in the one dimensional subspace generated by $+1$.

All this is no more than a standard definition of the quaternionic number system, which contains a unique copy of the real number system as well as many overlapping copies of the complex number system. Every quaternion has a real part and a quaternionic part. If the latter part is zero we call the quaternion real; if the former part is zero we call it purely quaternionic. If we prefer to talk about elements of the bivector algebra one can correspondingly identify within the algebra two special kinds of elements which we call *real* and *purely bivectorial* respectively. I will use the word "bivector" as synonym for "element of the bivector algebra". According to my terminology, each *bivector* can be uniquely decomposed into the sum of a *real number* and a *pure bivector*.

Christian next defines new sets of bivectors $\boldsymbol{\beta}_j(\lambda) = \lambda \boldsymbol{\beta}_j$, where $\lambda = +1$ or -1. These can also each be considered to form a bivector basis, but now have multiplication tables which depend on λ. To be precise:

$$\boldsymbol{\beta}_j(\lambda) \boldsymbol{\beta}_k(\lambda) \;=\; -\delta_{jk} - \sum_l \lambda \epsilon_{jkl} \boldsymbol{\beta}_l(\lambda). \tag{2}$$

He somewhat dangerously writes that these new bivectors are "defined" by the new algebraic rules

$$\boldsymbol{\beta}_j \boldsymbol{\beta}_k \;=\; -\delta_{jk} - \sum_l \lambda \epsilon_{jkl} \boldsymbol{\beta}_l. \tag{3}$$

Indeed there is a sense in which this is true, but within the context of the paper, it is clear that the two new "λ bivector bases" are defined in relation to the initially fixed (even

if arbitrary) basis, and then just "happen" to satisfy the new multiplication tables. Of course, the $+1$ bivector basis is the same as the original bivector basis, the -1 bivector basis is just -1 times the first.

At the start of the paper Christian defines "measurement functions" $A(\boldsymbol{a}, \lambda)$ and $B(\boldsymbol{b}, \lambda)$, where λ is a binary hidden variable (a fair coin toss, outcomes coded $+1$ and -1), and $\boldsymbol{a}$ and $\boldsymbol{b}$ are two unit vectors in ordinary real three dimensional space. The measurement functions are defined as two bivectors and the definition appears complicated but Christian claims, and that claim can easily be checked, that $A(\boldsymbol{a}, \lambda) = \lambda$ and $B(\boldsymbol{b}, \lambda) = -\lambda$, which of course are reals. I will give the definition and verify this claim in a moment, using a notation which will make life more easy. But first I want to point out an important consequence. Since $\lambda = \pm 1$, it follows that

$$A(\boldsymbol{a}, \lambda) B(\boldsymbol{b}, \lambda) = -\lambda^2 = -1, \tag{4}$$

independently of $\boldsymbol{a}$, $\boldsymbol{b}$ and λ (each within their respective domain). This striking relation is however not observed by Christian.

For completeness, here are Christian's definitions of A and B, expressed in a convenient notation which I will return to again later. Use the symbols $\boldsymbol{a}$ and $\boldsymbol{b}$ also to denote the pure bivectors $\sum_j a_j \boldsymbol{\beta}_j$ and $\sum_k b_k \boldsymbol{\beta}_k$. The reader may check that they are both square roots of -1. Define in the same way $\boldsymbol{a}(\lambda)$ and $\boldsymbol{b}(\lambda)$. Clearly,

$$\boldsymbol{a}(\lambda) = \lambda\, \boldsymbol{a}, \quad \boldsymbol{b}(\lambda) = \lambda\, \boldsymbol{b}.$$

Then we have

$$A(\boldsymbol{a}, \lambda) = -\boldsymbol{a}\, \boldsymbol{a}(\lambda), \quad B(\boldsymbol{b}, \lambda) = \boldsymbol{b}(\lambda)\, \boldsymbol{b}.$$

Because $\boldsymbol{a}^2 = \boldsymbol{b}^2 = -1$ and $\lambda^2 = 1$, it follows that

$$A(\boldsymbol{a}, \lambda) = \lambda, \quad B(\boldsymbol{b}, \lambda) = -\lambda, \quad A(\boldsymbol{a}, \lambda)\, B(\boldsymbol{b}, \lambda) = -1.$$

In view of this algebraic fact, it is curious that computation of the correlation between many independent A and B measurements should proceed so laboriously, and indeed, according to a curious definition, but let us accept the definition (his equation (5), and coincidentally mine too) which Christian elsewhere gives reasons for:

$$\mathcal{E}(\boldsymbol{a}, \boldsymbol{b}) = \lim_{n \to \infty} \frac{\frac{1}{n} \sum_{i=1}^{n} A(\boldsymbol{a}, \lambda^i) B(\boldsymbol{b}, \lambda^i)}{\left(-\sum_j a_j \boldsymbol{\beta}_j\right)\left(\sum_k b_k \boldsymbol{\beta}_k\right)}. \tag{5}$$

There is slight ambiguity in this expression concerning the division by two terms: is this division by a product, or is this two successive divisions? Division stands for multiplication by the inverse, but is this supposed to take place on the left or on the right? The answer is given by inspecting the calculations in Christian's equations (5) to (7): the two terms in the denominator of this fraction are supposed to divide left and right hand side of the numerator respectively, corresponding to their order in the denominator. Those are the crucial calculations which, end of his (7), lead to his desired result

$$\mathcal{E}(\boldsymbol{a}, \boldsymbol{b}) = -\boldsymbol{a} \cdot \boldsymbol{b}. \tag{6}$$

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The fact we have previously observed concerning the product of the A and B measurements enables us to make a grand short cut through Christian's derivation of the right hand side of his (7) from the left hand side of his (5). With again the convention that $\boldsymbol{a}$ and $\boldsymbol{b}$ also denote the pure bivectors $\sum_j a_j \boldsymbol{\beta}_j$ and $\sum_k b_k \boldsymbol{\beta}_k$, on substituting the value -1 of all the n products, we obtain $\mathcal{E}(\boldsymbol{a}, \boldsymbol{b}) = -(-\boldsymbol{a})^{-1} \boldsymbol{b}^{-1}$. Both of $\boldsymbol{a}$ and $\boldsymbol{b}$ are square roots of -1. From $\boldsymbol{a}\boldsymbol{a} = -1$ we find $\boldsymbol{a}(-\boldsymbol{a}) = 1 = (-\boldsymbol{a})\boldsymbol{a}$, hence $(-\boldsymbol{a})^{-1} = \boldsymbol{a}$ and $\boldsymbol{b}^{-1} = -\boldsymbol{b}$. Thus $\mathcal{E}(\boldsymbol{a}, \boldsymbol{b}) = +\boldsymbol{a}\boldsymbol{b}$. This bivector product must be evaluated as $-\sum_j a_j b_j - \sum_{jkl} \epsilon_{jkl} a_j b_k \boldsymbol{\beta}_l$. The second term does not vanish, unless $\boldsymbol{a} = \pm\boldsymbol{b}$. It is conventionally denoted $\boldsymbol{a} \wedge \boldsymbol{b}$ and it is a pure bivector whose coefficients are the three coefficients of the usual Euclidean vector cross product $\boldsymbol{a} \times \boldsymbol{b}$. Using the notational identification between pure bivectors and real vectors, we can even define the pure bivector wedge product by writing $\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$. We also define the pure bivector dot product $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}$ (left hand side a bivector, right hand side a real number). Conclusion: Christian's correlation is not $\mathcal{E}(\boldsymbol{a}, \boldsymbol{b}) = -\boldsymbol{a} \cdot \boldsymbol{b}$ but

$$\mathcal{E}(\boldsymbol{a}, \boldsymbol{b}) \;=\; -\,\boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \wedge \boldsymbol{b}. \tag{7}$$

However Christian prefers a more complicated derivation. Naturally, if he too follows accurately his own algebraic rules, he cannot obtain a different answer. However his derivation appears to make use of the law of large numbers. In the last complicated term before the end of his (7), there appears a sum over n terms each involving its own λ^i, and the argument is clearly that this cancels in the limit as $n \to \infty$, by the law of large numbers. Indeed, if this expression is correct, and unless $\boldsymbol{a} = \pm\boldsymbol{b}$, it is *only* in the limit that it vanishes, when we obtain $-\boldsymbol{a} \cdot \boldsymbol{b} + \frac{1}{2}\boldsymbol{a} \wedge \boldsymbol{b} - \frac{1}{2}\boldsymbol{a} \wedge \boldsymbol{b}$. Something has gone wrong here. The mistake is in the transition from his formulas (6) to (7) where Christian is using the λ-multiplication tables to simplify linear combinations of products of the $\boldsymbol{\beta}_j(\lambda^i)$. We have already written down the correct multiplication table, which expresses these products in terms of the same basis vectors $\boldsymbol{\beta}_l(\lambda^i)$. The last $\boldsymbol{\beta}_l$ appearing in (7) should actually be $\boldsymbol{\beta}_l(\lambda^i)$! The other λ^i in the same expression is the λ^i appearing on the right hand side of the λ^i-multiplication table: altogether, $\lambda^i \epsilon_{jkl} \boldsymbol{\beta}_l(\lambda^i)$. By definition $\boldsymbol{\beta}_l(\lambda^i) = \lambda^i \boldsymbol{\beta}_l$. Making this correct substitution gives us a factor $(\lambda^i)^2 = 1$. Finally we obtain the same result as I earlier got from a shorter route. Sanity has been restored.

Discussion

I have tried in this paper to separate mathematics from physical interpretation. In this closing passage however, I will bring the two together again.

There is no limitation in Bell's theorem on the space in which the hidden variables live. The "measurement functions" could be imagined to "perform" or "implement" calculations in any suitable algebraic or other mathematical framework, and hidden variables can include elements of any exotic mathematical space. Moreover there is no objection whatever within Bell's theorem to allow the outcomes, though of necessity encoded as $+1$ and -1, to be thought of being members of a larger mathematical space than the set $\{+1, -1\}$. In the context of CHSH, all hidden variables can be reduced to, or subsumed in, the outcomes

of the other measurements to the two which were actually done. "Realism", however it is defined, comes down, effectively, to the mathematical existence of the outcomes of unperformed experiments, alongside of those which were actually performed. "Locality" refers to the attempt to "locate" those counterfactual outcomes in the "obvious" region of space and time. Alongside of the assumptions of realism and locality (the second only being meaningful given the first) we need an assumption of freedom: the freedom of the experimenter to perform either measurement. This does not need to involve metaphysical assumptions either of free will or of existence of true randomness. It just involves the assumption that the physical processes going on at one measurement location can't have access to the measurement choice made at the other location, till after the measurement outcome has been committed to.

And of course, Bell's theorem applies to ordinary correlations computed in the ordinary way on Christian's outcomes A and B, which as we have seen are actually always perfectly anti correlated, whatever the measurement settings. No violation of Bell's theorem (CHSH inequality version).

In real experiments, ordinary correlations are computed in the ordinary way on binary outcomes and *do* violate the CHSH inequality.

So even if Christian's algebra had been correct, what relevance does it have to the real world? As we have seen, Christian's work applies to correlations obtained by dividing the raw correlation between measurement outcomes by the pure bivectors $-\boldsymbol{a}$ and $\boldsymbol{b}$, and within his own model would lead to standardized correlations which are not even real numbers. This simply has got nothing at all to do with Bell's own programme, as far as it is usually interpreted.

Some of those writing critical evaluation of Christian's work have expressed the hope that it might at least provide a mathematical framework for the theoretical side of quantum mechanics, in which the usual structure of Hilbert spaces, projection operators, and so on, could be entirely replaced by a mathematical structure having a much closer connection with, for instance, the geometry of the real world. It seems to this author that that is indeed a legitimate quest. However in view of the failure of this particular attempt, those wanting to do this job are going to have to look elsewhere.

There remains a psychological question, why so strong a need is felt by so many researchers to "disprove Bell" in one way or another? At a rough guess, at least one new proposal comes up per year. Many pass by unnoticed, but from time to time one of them attracts some interest and even media attention.

Having studied a number of these proposals in depth, I see two main strategies of would-be Bell-deniers. But please notice, I do not mean to imply that these strategies are deliberate: I believe they are found "accidentally"; I have no doubt of the sincerity of the proposers.

The first strategy (the strategy, I would guess, in the case in question) is to build elaborate mathematical models of such complexity and exotic nature that the author him or herself is probably the only person who ever worked through all the details. Somewhere in the midst of the complexity a simple mistake is made, usually resulting from suppression of an important index or variable. There is a hidden and non-local hidden variable.

The second strategy is to simply build elaborate versions of detection loophole models.

Sometimes the same proposal can be interpreted in both ways at the same time. Interpreting the proposal as the result of a hidden mistake or as a detection loophole model are both interpretations of the reader, not of the writer.

According to the Anna Karenina principle of evolutionary biology, in order for things to succeed, everything has to go exactly right, while for failure, it suffices if any one of a myriad factors is wrong. Since errors are typically accidental and not recognized, an apparently logical deduction which leads to a manifestly incorrect conclusion does not need to allow a unique diagnosis. If every apparently logical step had been taken with explicit citation of the mathematical rule which was being used, and in a specified context, one could say where the first misstep was taken. But mathematics is almost never written like that, and for good reasons. The writer and the reader, coming from the same scientific community, share a host of "hidden assumptions" which can safely be taken for granted, as long as no self-contradiction occurs. Saying that the error actually occurred in such-and-such an equation at such-and-such a substitution depends on various assumptions. The author who still sincerely believes in his result will therefore claim that the diagnosis is wrong because the wrong context has been assumed.

We can be grateful for Christian that he has had the generosity to write his one page paper with a more or less complete derivation of his key result in a more or less completely explicit context, without distraction from the author's intended physical interpretation of the mathematics. The mathematics should stand on its own, the interpretation is "free". My finding is that in this case, the mathematics does not stand on its own.

Postscript

After posting this paper on arXiv.org I became aware that others, not surprisingly, had already published interesting critiques on Christian's work. In particular, Florin Moldoveanu has published a comprehensive review, Moldoveanu (2011), which also cites many other works. His "error 1" is the same as the error on which I have focussed. I had the advantage of the availability of Christian's "one page paper", and restrict myself strictly to that particular presentation. Moldoveanu has carried out a review of the whole corpus of works, which is complicated by the fact that different notations and different definitions are used in different papers.

I also found it useful to strictly separate mathematical consistency from physical and metaphysical interpretation. If the mathematics is fatally flawed at the outset, then we need not spend energy on tracking down further errors, or on debating the soundness of the ideas.

After my own preprint was posted on arXiv.org, Christian has published a refutation of it, see Christian (2012a).

Also, his magum opus, the book Christian (2012b), has now been published. The first twenty-five pages (which consists of 14 pages of front-matter and 11 numbered pages) are freely available as a pdf file. The reader will find the identical error as in the one-page

paper on page number 10 of the book, in the transition from formula (1.23) to (1.24), where Christian appeals to the multiplication table (1.8).

Christian argues that he is not making an error at this point, but introducing a new postulate. In that case it is curious that he does not draw attention to the fact that he is introducing a daring new ingredient into his model, especially in view of the fact that the new postulate contradicts the earlier made (and used) postulates.

Christian's work has stimulated a number of interesting discoveries and inventions. I would especially like to draw attention to Sascha Vongehr's "quantum Randi challenge" on science20.com, see Vongehr (2011). The idea is to insist that those who believe Bell got it all wrong, to deliver by providing computer programs which simulate their local realistic violation of Bell's inequality. A successful simulation will get the attention of science journalists and science communicators and educators, and thereby of the whole scientific community, without having to pass the barrier of hostile peer review.

References

[1] Christian, J., (2011), *Disproof of Bell's Theorem*, arXiv:1103.1879,
`http://arxiv.org/abs/1103.1879`

[2] Christian, J., (2012a), *Refutation of Richard Gill's Argument Against my Disproof of Bell's Theorem*, arXiv:1203.2529, `http://arxiv.org/abs/1203.2529`

[3] Christian, J., (2012b), *On the Origins of Quantum Correlations*, arXiv:1201.0775, `http://arxiv.org/abs/1201.0775`. This preprint reproduces the first chapter of the book "Disproof of Bell's Theorem. Illuminating the Illusion of Entanglement" by J.J. Christian, published by Brown Walker,
`http://www.brownwalker.com/book.php?method=ISBN&book=1599425645`

[4] Moldoveanu, F. (2011), *Disproof of Joy Christian's "Disproof of Bell's theorem"*, arXiv:1109.0535, `http://arxiv.org/abs/1109.0535`

[5] Vongehr, S. (2011), The official quantum Randi challenge,
`http://www.science20.com/alpha_meme/official_quantum_randi_challenge-80168`

Appendix

The body of the one page paper arXiv:1103.1879v1

Central to Bell's theorem [1] is the claim that no local and realistic model can reproduce the correlations observed in the EPR-Bohm experiments. Here we construct such a model. Let Alice and Bob be equipped with the variables

$$A(\mathbf{a}, \lambda) = \{ -a_j\,\beta_j \}\,\{ a_k\,\beta_k(\lambda) \} = \begin{cases} +1 & \text{if} \quad \lambda = +1 \\ -1 & \text{if} \quad \lambda = -1 \end{cases} \tag{1}$$

$$\text{and } B(\mathbf{b}, \lambda) = \{ b_j\,\beta_j(\lambda) \}\,\{ b_k\,\beta_k \} = \begin{cases} -1 & \text{if} \quad \lambda = +1 \\ +1 & \text{if} \quad \lambda = -1, \end{cases} \tag{2}$$

where the repeated indices are summed over x, y, and z; the fixed bivector basis $\{ \beta_x,\, \beta_y,\, \beta_z \}$ is defined by the algebra

$$\beta_j\,\beta_k = -\delta_{jk} - \epsilon_{jkl}\,\beta_l\,; \tag{3}$$

and—together with $\beta_j(\lambda) = \lambda\,\beta_j$—the λ-dependent bivector basis $\{ \beta_x(\lambda),\, \beta_y(\lambda),\, \beta_z(\lambda) \}$ is defined by the algebra

$$\beta_j\,\beta_k = -\delta_{jk} - \lambda\,\epsilon_{jkl}\,\beta_l, \quad \text{where } \lambda = \pm 1 \text{ is a fair coin [2]}, \tag{4}$$

δ_{jk} is the Kronecker delta, ϵ_{jkl} is the Levi-Civita symbol, $\mathbf{a} = a_x\,\mathbf{e}_x + a_y\,\mathbf{e}_y + a_z\,\mathbf{e}_z$ and $\mathbf{b} = b_x\,\mathbf{e}_x + b_y\,\mathbf{e}_y + b_z\,\mathbf{e}_z$ are unit vectors, and the indices j, k, $l = x$, y, or z. The correlation between $A(\mathbf{a}, \lambda)$ and $B(\mathbf{b}, \lambda)$ then works out to be

$$\mathcal{E}(\mathbf{a}, \mathbf{b}) = \frac{\lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{i=1}^{n} A(\mathbf{a}, \lambda^i)\,B(\mathbf{b}, \lambda^i) \right\}}{\{ -a_j\,\beta_j \}\,\{ b_k\,\beta_k \}} = \lim_{n \to \infty} \left[\frac{1}{n} \sum_{i=1}^{n} \frac{A(\mathbf{a}, \lambda^i)\,B(\mathbf{b}, \lambda^i)}{\{ -a_j\,\beta_j \}\,\{ b_k\,\beta_k \}} \right] \tag{5}$$

$$= \lim_{n \to \infty} \left[\frac{1}{n} \sum_{i=1}^{n} \{ a_j\,\beta_j \}\,\{ A(\mathbf{a}, \lambda^i)\,B(\mathbf{b}, \lambda^i) \}\,\{ -b_k\,\beta_k \} \right] = \lim_{n \to \infty} \left[\frac{1}{n} \sum_{i=1}^{n} \{ a_j\,\beta_j(\lambda^i) \}\,\{ b_k\,\beta_k(\lambda^i) \} \right] \tag{6}$$

$$= -a_j\,b_j - \lim_{n \to \infty} \left[\frac{1}{n} \sum_{i=1}^{n} \{ \lambda^i\,\epsilon_{jkl}\,a_j\,b_k\,\beta_l \} \right] = -a_j\,b_j + 0 = -\mathbf{a} \cdot \mathbf{b}, \tag{7}$$

where the denominators in (5) are standard deviations. The corresponding CHSH string of expectation values gives

$$|\mathcal{E}(\mathbf{a}, \mathbf{b}) + \mathcal{E}(\mathbf{a}, \mathbf{b}') + \mathcal{E}(\mathbf{a}', \mathbf{b}) - \mathcal{E}(\mathbf{a}', \mathbf{b}')| \leq 2\sqrt{1 - (\mathbf{a} \times \mathbf{a}') \cdot (\mathbf{b}' \times \mathbf{b})} \leq 2\sqrt{2}. \tag{8}$$

Evidently, the variables $A(\mathbf{a}, \lambda)$ and $B(\mathbf{b}, \lambda)$ defined above respect both the remote parameter independence and the remote outcome independence (which has been checked rigorously [2][3][4][5][6][7]). This contradicts Bell's theorem.