Steve Gull's challenge

An impossible Monte Carlo simulation project in distributed computing

Richard Gill. Statistics group seminar. 11 January 2021

CLEARING UP MYSTERIES – THE ORIGINAL GOAL

E. T. Jaynes

Abstract : We show how the character of a scientific theory depends on one's attitude toward probability. Many circumstances seem mysterious or paradoxical to one who thinks that probabilities are real physical properties existing in Nature. But when we adopt the "Bayesian Inference" viewpoint of Harold Jeffreys, paradoxes often become simple platitudes and we have a more powerful tool for useful calculations. This is illustrated by three examples from widely different fields: diffusion in kinetic theory, the Einstein–Podolsky–Rosen (EPR) paradox in quantum theory, and the second law of thermodynamics in biology.

Opening lecture at MAXENT 8 (1998)

Cambridge, ca. 1984
Gull's overhead sheets
for a master level
math physics course

Better to talk of *spin* than *polarization*

The EPR problem: 2 spin-half particles in a singlet state: $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

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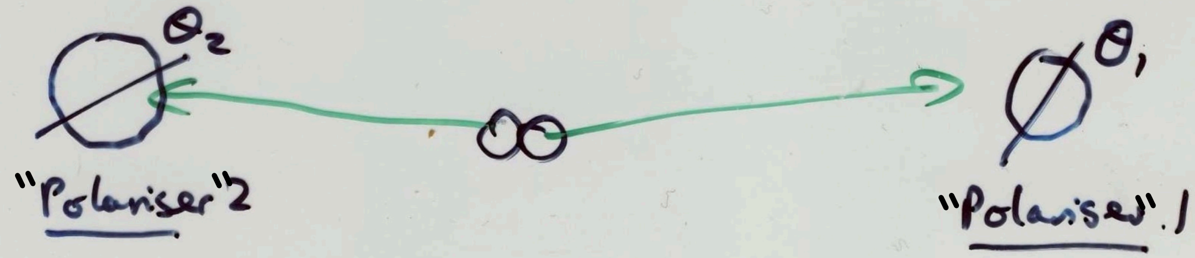
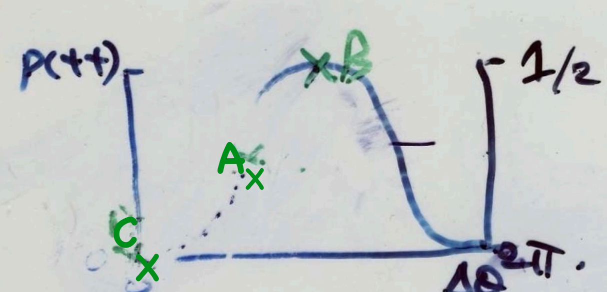


Presented (1998) at MaxEnt 8 in discussion by Gull of talk by E T Jaynes on Bell's theorem

QM says: $pr(++|\theta_1\theta_2) = \frac{1}{4}(1-\cos\Delta\theta)$

[you might like to prove this] $\Delta\theta = \theta_1 - \theta_2$



points A, B, C are easy to understand. [conservation laws etc.].

Jaynes had explained that Bell was mixed up

Jaynes was flummoxed by Gull's proof

The rest is awful. The problem (in words) is that the particles each have to "decide" which of + or − to answer for every (θ_1, θ_2) pair. (Loose change in the uncertainty principle $\sim\hbar$) But the particles "overhear" the questions $(\theta_{1,2})$ asked on the other side.

Therefore they didn't just travel out there alone

(You'll get more than the Mott prize for this!)

Write a computer program which is to run on two independent Personal Computers which mimics the QM predictions for the EPR setup. There must be no communication between the computers after the time of program load.

The program can be restricted to "polarisers" in (x-y) plane!

Each machine sets up a dialogue:

C: Particle # n : Please input Θ $(0-2\pi)$:

You give Θ_1 or Θ_2

C: Particle went into + channel (or −)

It then asks you for Θ for $n+1$

You task is to ensure: $pr(++|\Theta_1\Theta_2) = \frac{1}{4}(1-\cos\Delta\Theta)$

over long term performance. (the results are recorded and the correlations for the particles are worked out afterwards.

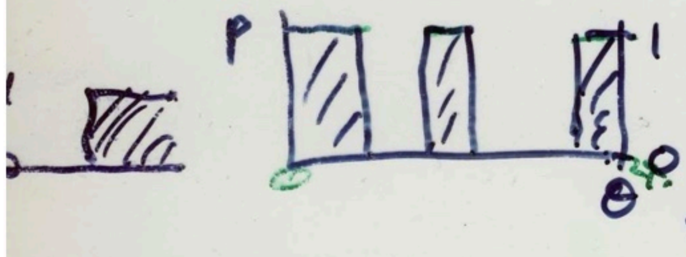
THIS IS A MATHEMATICAL PROJECT, THERE ARE NO PHYSICAL ASSUMPTIONS.

-3-

(1) Set $\Theta_1 = \Theta_2$ and repeatedly test alleged program. Any ++ or -- result fails it.

⟹ Programs are deterministic.
(any random component will catch it out.)

(2) There must be a function $p(\Theta, n)$ implied that represents what the answer would be if 'Θ' was asked at the n trial.



p = 0 or 1 for all Θ
can be different between trials

(3) "Correlation" of 2 of these is the $FT(|FT(p)|^2)$
(MP2)

(MP2 = previous year's math phys)

(4) But $\frac{1}{4}(1-\cos\Delta\Theta)$ only has 3 non-zero Fourier components.

(5) Therefore $FT(p)$ only has 3 non-zero components.

(6) Therefore can't be 0 or 1 everywhere.

(7) Oh dear ....

Failure to mimic EPR setup.

⇒ particles don't work like computers.

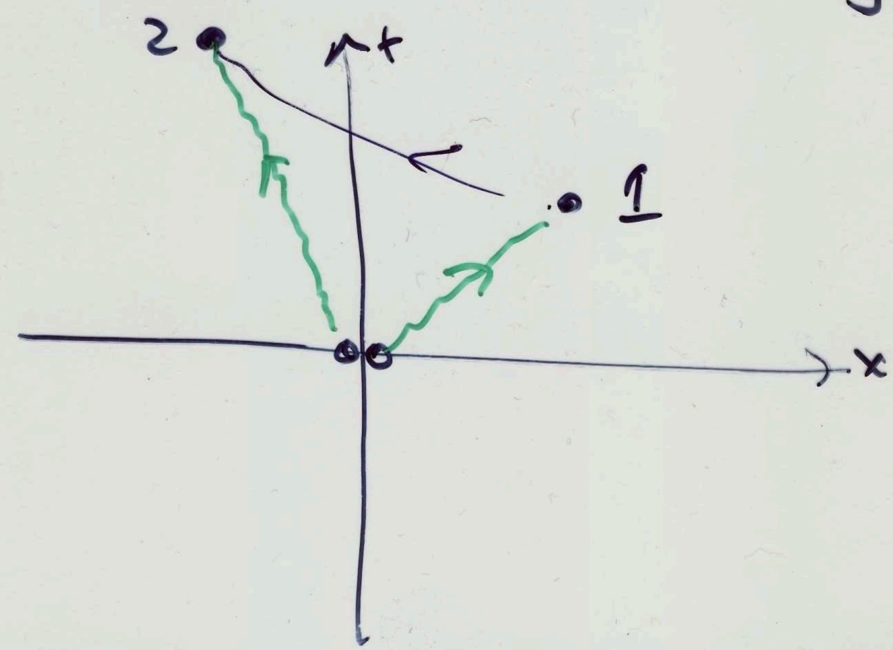
⇒ no local hidden variables.

⇒ NOT an INFERENCE problem.

Could easily write such a program to run in 1 computer with 2 screens.

Need communication ~~How~~? INFLUENCE problem

How can I make it relativistically invariant?



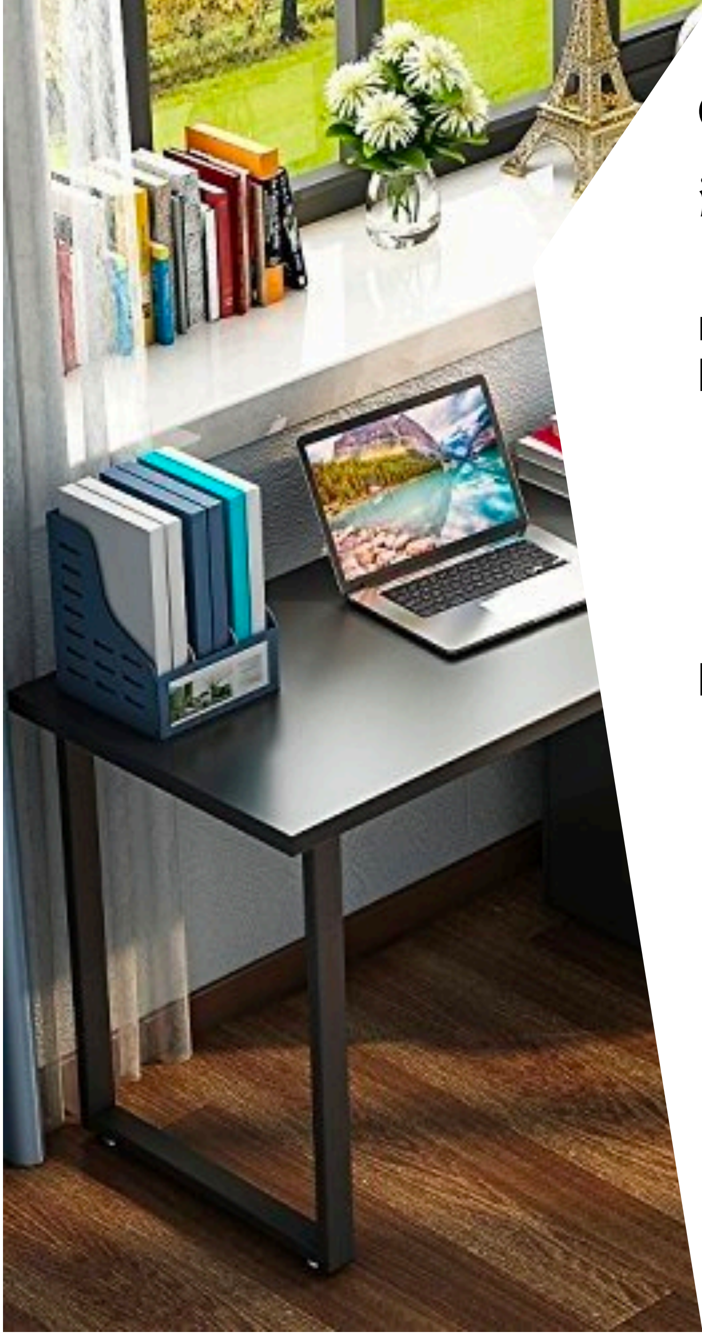
There is only one invariant "path" to get the influence from 1 to 2.

Backwards in time down the green lines.

⇒ Feynmann / Wheeler type theories.

Impossible?

- Every year there are several new papers disproving Bell's theorem
- … published in top journals, and authored by well known and well qualified scientists
- Some even supply computer code
- Jaynes' dismissive discussion of EPR, Bell, and all that continues to be quoted and to inspire professionals and amateurs alike



No communication at all, during the run!

Computer 1 runs dialogue;

inputs are angles θ_{1n}, $n = 1,2,\ldots$

```
n <- 1
Begin Loop
      Print: “input angle no.”, n
      Wait for input
      Compute output
      Output: +1 or -1
      n <- n +1
End Loop
```

Computer 2 runs dialogue;

inputs are angles θ_{2n}, $n = 1,2,\ldots$

```
n <- 1
Begin Loop
      Print: “input angle no.”, n
      Wait for input
      Compute output
      Output: +1 or -1
      n <- n + 1
End Loop
```

No communication at all, during the run!

Task: mean value of product of outputs,
given both inputs, converges to $-\cos(\theta_1 - \theta_2)$;
mean values of outputs, given inputs, converge to zero.

But how could there be any correlation at all?

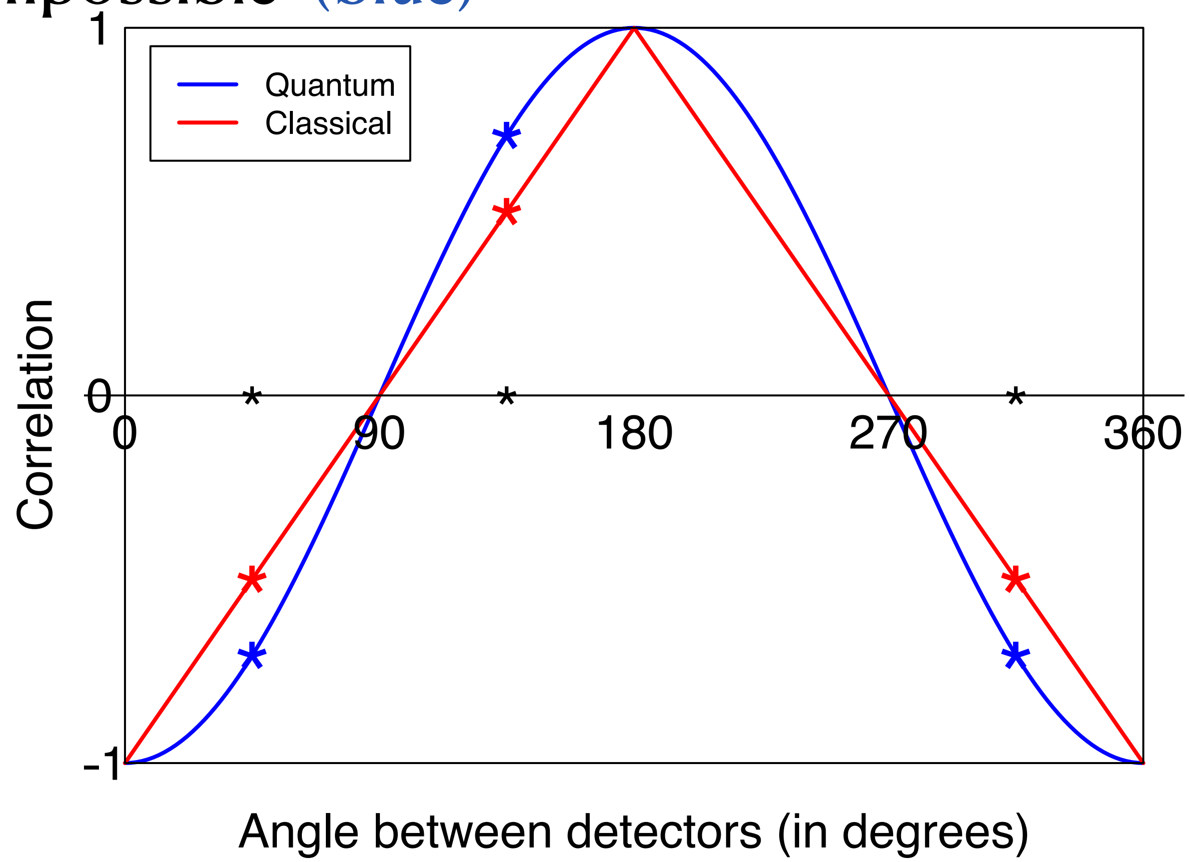
EASY! Both computers run the same RNG (same seed, same parameters) or have the same hard disk full of previously collected random numbers

For instance, for trial "*n*" both computers use same uniform random angle $\phi_n \in [0, 2\pi)$
Computer 1 outputs $\operatorname{sign}\cos(\phi_n - \theta_{1n})$
Computer 2 outputs $-\operatorname{sign}\cos(\phi_n - \theta_{2n})$
This generates the "triangle wave" $1 - 2\left|((\theta_1 - \theta_2)/2\pi \bmod 1) - \tfrac{1}{2}\right|$ (red)

It's also possible to generate the correlation $-\tfrac{2}{3}\cos(\theta_1 - \theta_2)$

cf. Peres–Horodecki criterion

But $-\cos(\theta_1 - \theta_2)$ is impossible (blue)



Standard CHSH experiment

Alice chooses from $\{0, \tfrac{1}{2}\pi\}$
Bob from $\{\tfrac{1}{4}\pi, \tfrac{3}{4}\pi\}$
Differences (mod 2π):
$\tfrac{1}{4}\pi, 2\pi - \tfrac{1}{4}\pi, \tfrac{3}{4}\pi, \tfrac{1}{4}\pi$

Bell's theorem

Bell (1964)

- There exists no probability space $(\Omega, \mathbb{P})$ with r.v.'s A_θ, B_ϕ taking values in $\{-1, +1\}$, $\theta, \phi \in [0,2\pi)$, such that

$$\forall \theta, \phi : \mathbb{E}(A_\theta) = 0 = \mathbb{E}(B_\phi), \quad \mathbb{E}(A_\phi B_\theta) = -\cos(\theta - \phi)$$

- Thus: there do not exist functions A, B (taking values $+/-1$) and a probability measure $\mathbb{P}$ such that

$$\int A(\theta_1, \omega) B(\theta_2, \omega) \mathbb{P}(\mathrm{d}\omega) = -\cos(\theta_1 - \theta_2)$$

- Bell's proof used only two particular choices for each of θ, ϕ, and showed that approximate equality is not possible either

Gull's proof

(I make extra <u>assumptions</u>, but still need more <u>rigour</u>)

- I will suppose that the two computers contain an i.i.d. sequence of random elements $\omega_1, \omega_2, \ldots$ drawn from $(\Omega, \mathbb{P})$
- I will suppose that Computer 1and Computer 2 contain modules which implement the functions $A(\cdot\,,\cdot)$ and $B(\cdot\,,\cdot)$ defined on $[0,2\pi) \times \Omega$
- **Step 1**: imagine both parties always submit the same angles. (One can rerun the programs from the same starting state as often as we like, with runs as long as we like). The correlation must always be -1. Therefore $B \equiv -A$

Gull's proof (continued)

(I made extra assumptions, but still need more rigour)

- **Step 2**. Imagine party 1 submits a very long sequence of uniformly distributed random angles θ_n, and party 2 submits the same sequence shifted (mod 2π) by the amount δ
- Thanks to the extra assumptions, the pairs of outcomes can be denoted by $A(\theta_n, \omega_n), -A(\theta_n + \delta, \omega_n)$ where (θ_n, ω_n) are i.i.d pairs from the probability measure Uniform $\times\, \mathbb{P}$ on $[0,2\pi] \times \Omega$
- We can expand the bounded random function $A(\phi)$ on the circle in its random Fourier series $\sum_n c_n \exp(in\phi)$, where the summation is over $n \in \mathbb{Z}$ and the complex numbers c_n are random (i.e., depend on ω)
- Because A is real, for $n = -n'$ we have $c_n = \overline{c}_{n'}$

Gull's proof (continued)

(I made extra assumptions, but still need more rigour)

- **Step 3**, take expectation value of $-A(\theta, \omega)A(\theta + \delta, \omega)$ (average over θ, ω), substitute for A (twice) by Fourier series: $\sum_n c_n \exp(in\theta)$ and $\sum_{n'} c_{n'} \exp(in'(\theta + \delta))$
- This gives a double summation over n, n', and integrals over θ, ω.
  The integration over θ of $\exp(i(n + n'))\theta)$ is zero unless $n = n'$. We finish with a single summation $\sum_{n \in \mathbb{Z}} \mathbb{E}|c_n|^2 \exp(in\delta)$
- But by assumption, the quantity whose expectation value we took (the empirical correlation between the outcomes as a function of the difference between the setting pairs) must converge to $-\cos(\delta) = -\frac{1}{2}(\exp(i\delta) + \exp(-i\delta))$. Therefore all c_n are zero except when $n = \pm 1$
- This is a contradiction since A only takes the values ± 1

Conclusion

Gull's proof works, at least, as theoretical physics, though not perhaps yet as mathematical physics

- In fact we don't *need* it: there are proofs of stronger results with weaker assumptions, using 1969 CHSH inequality, strengthened by use of martingale theory to take care of time & memory. [RDG 2003 – Delft quantum physicists David Elkouss & Stephanie Wehner 2016]
- Open problem: can we prove Gull's theorem without making the i.i.d. assumption, and the memorylessness assumptions, which Gull seems to need? [I seemed to need them to make his outline proof work; he doesn't make those assumptions explicitly]
- Could we even let Alice and Bob each submit a large number *N* of angles in one batch, and allow those two computers to process all the angles arbitrarily?
- This question should be considered also for a more traditional approach via CHSH in which Alice submits a fair Bernoulli sequence of angles taken from the pair $\{0,\frac{1}{2}\pi\}$ and Bob (independently) from $\{\frac{3}{4}\pi\ \frac{1}{4}\pi\}$
- Idea: avoid measurability issues by just considering settings which are whole numbers of degrees and use discrete Fourier transform
- How to show that it is also not possible to even *approximately* reproduce the negative cosine?

Reference

(And a note on quantum computers and quantum internet)

- *Gull's theorem revisited,* Richard D. Gill & Dilara Karakozak (2020) https://arxiv.org/abs/2012.00719 (presently at version 5)

Note: in principle one could produce the negative cosine, or close to produce it, by using quantum internet to set up *N* entangled qubit pairs in the quantum memories of two separated quantum computers. Unlike the experiment with separated classical computers, one could not test by giving a clone of the same computer different sets of inputs. So once you have checked (close to) perfect anti-correlation in "**step 1**", you cannot try anything else. You have to start all over again.

Appendix

Remarks

- Can we get uniform convergence in sup norm, a.s., of all sample correlations?

$$\widehat{\rho}_N(\theta, \phi) = N^{-1} \sum_{n=1}^{N} A_\theta(\omega_n) B_\phi(\omega_n)$$

- New experiments to minimise statistical errors?
- Tests of circular symmetry?
- The grasshopper problem

D. Chistikov, O. Goulko, A. Kent, M. Paterson(2020)
Globe-hopping. *Proc. R. Soc. A* **476**: 20200038.
http://dx.doi.org/10.1098/rspa.2020.0038

Appendix

More remarks

- The usual CHSH experiment has: two settings each for Alice and Bob; repeatedly chosen anew by fair coin tosses
- Say "trial *n* results in success" if the outcomes are equal and the two settings are not both "setting no. 2", otherwise "fail"
- Let *S* be the total number of successes in *N* trials
- On distributed classical computers (allowed to communicate between each two trials) *S* is stochastically less than or equal to Binom(*N*, 0.75) distributed
- Quantum computers connected by quantum internet could achieve Binom(*N*, 0.85)

Appendix

Yet more remarks

entropy

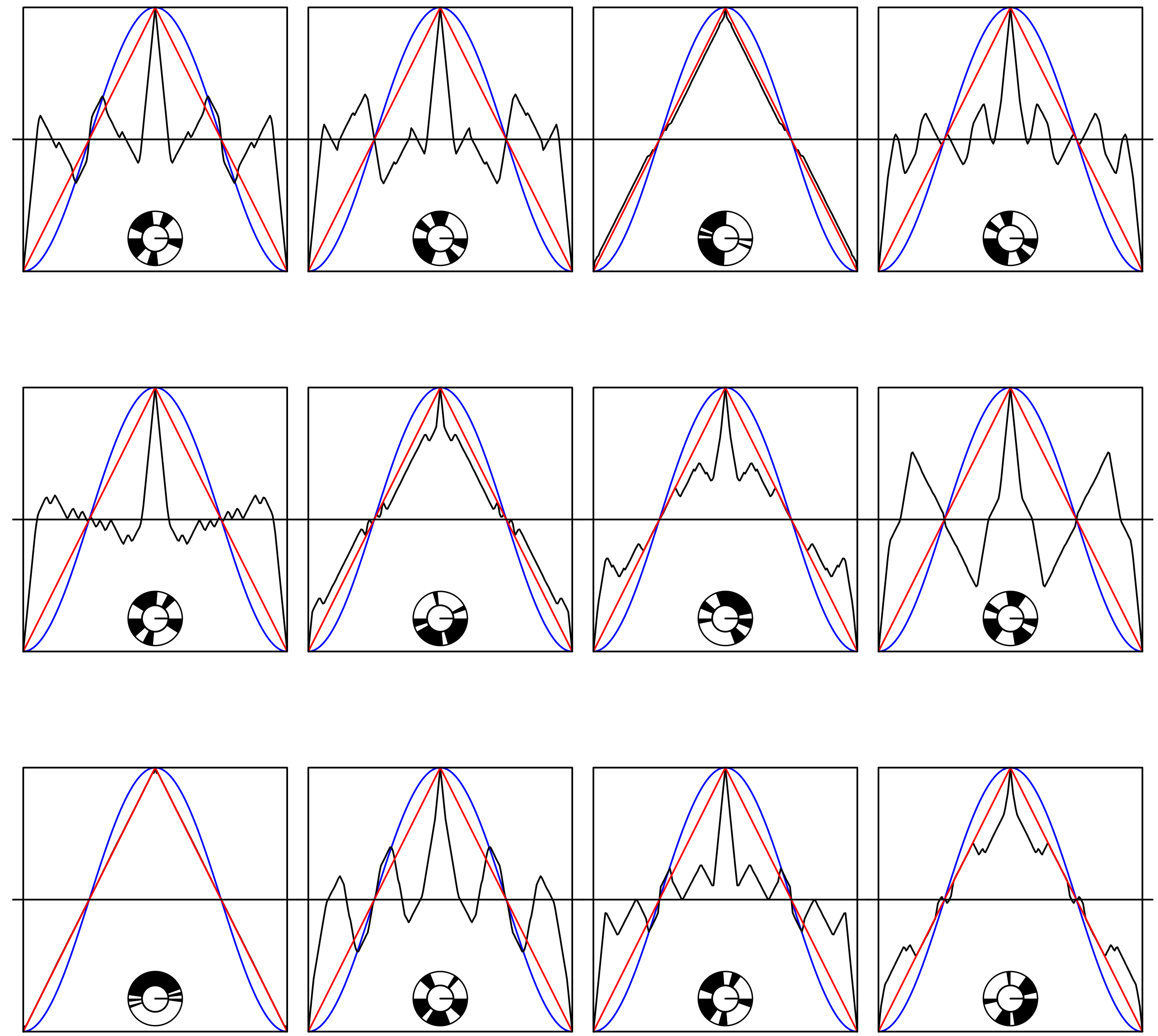
MDPI

Article

The Triangle Wave Versus the Cosine: How Classical Systems Can Optimally Approximate EPR-B Correlations

Richard David Gill

Entropy **2020**, *22*, 287; doi:10.3390/e22030287



A sample of classical correlation functions obtained from the coloured spinning disk model

- Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  - Notice: self-adjoint, square to Id (2×2 identity), anti-commute; eigenvalues ± 1

- Let $\overrightarrow{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$; for $\overrightarrow{a} \in \mathbb{R}$, $\|\overrightarrow{a}\|^2 = 1$ define $\sigma_{\overrightarrow{a}} = \overrightarrow{a} \cdot \overrightarrow{\sigma}$

  - self-adjoint, squares to (identity), eigenvalues ± 1

- Let $|z+\rangle, |z-\rangle \in \mathbb{C}^2$ as the corresponding normalised eigenvectors of σ_z, etc.

- State-vector of the singlet state

$$\Psi = \frac{1}{\sqrt{2}} \left(\, |z+\rangle \otimes |z-\rangle - |z-\rangle \otimes |z+\rangle \, \right) \in \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$$

- Mean values of measurement outcomes of spin of either particle in any direction

$$\langle \Psi | \sigma_{\overrightarrow{a}} \otimes \mathrm{Id} | \Psi \rangle = 0 = \langle \Psi | \mathrm{Id} \otimes \sigma_{\overrightarrow{b}} | \Psi \rangle$$

- Correlations (expectation of product) are negative cosine

$$\langle \Psi | \sigma_{\overrightarrow{a}} \otimes \sigma_{\overrightarrow{b}} | \Psi \rangle = -\cos(\overrightarrow{a} \cdot \overrightarrow{b})$$