Angelo Vistoli on Gerbes

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Abstract

There are notes from lectures by Angelo Vistoli at the summer school on Stacks in Mainz, 2015. They are being tex'd during the lectures by David, and subsequently improved by Raymond. As you will see, the notes are very rough, and there are surely still many things we missed; our excuse is that the aim is to give participants something useful to use immediately after the lecture, rather than produce a polished set of notes.

Comments and corrections are very welcome. If you want to edit the tex file yourself just let me know.

1 Lecture 1: torsors

Let $\mathcal{C}$ a site, for example:

1. $\mathcal{C}$ is the category of open subsets of a topological space, covers are jointly surjective collections of open subsets.
2. **Top**, the category of topological spaces (or topological spaces over a fixed topological space S);
3. S a scheme, then $\mathbf{Aff}/S$ has as objects affine schemes over S [1]. Then put a topology such as Zariski, étale, fppf cf. [Sta13, Tag 021L], fpqc cf. [Sta13, Tag 022A] (he gave simplified versions which work in affine setting).

Let $X \in \operatorname{ob}\mathcal{C}$, let $h_X : \mathcal{C}^{op} \to \mathbf{Set}$ by $h_X(S) = \operatorname{Hom}(S, X)$. If $\mathcal{C}$ is any of the above then h_X is a sheaf. If $\mathcal{C}$ is one of the above affine sites and X is any scheme, then can define h_X by the same formula, and it is still a sheaf. Moreover, $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(h_X, h_Y)$ is a bijection (in any of the above settings) - if X and Y are objects of $\mathcal{C}$ then this is Yoneda, otherwise needs a moment's thought.

[1] affine means in an absolute sense, not relative to S. So in general the identity is not a cover.

Let G be a sheaf of groups on $\mathcal{C}$, $P\colon \mathcal{C}^{op} \to \mathbf{Set}$, and $P \times G \to P$ a right action of G on P. Suppose $f\colon P \to X$ is an invariant map; this means that the following diagram commutes:

$$\begin{array}{ccc} P \times G & \xrightarrow{acts} & P \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\ P & \xrightarrow{f} & X. \end{array} \tag{1}$$

Then define $P \times G \to P \times_X P$ by $(p, g) \mapsto (p, pg)$ (the invariance is key, otherwise not a map over X).

Lemma 1.1 (Exercise)**.** *The following are equivalent:*

1. *$P \to X$ is an epi (of sheaves), and $P \times G \to P \times_X P$ is an isomorphism of functors;*

2. *there exists an epi $Y \to X$ of sheaves and a G-equivariant Y-isomorphism $Y \times_X P \to Y \times G$.*

Proof. 1 $\implies$ 2 is basically obvious, take $Y = P$. The other needs a little thought. □

Definition 1.2. $P \to X$ is a G-torsor if either of the two above equivalent conditions are satisfied. Note that P is just a sheaf, not a scheme/space in general.

Say $\mathcal{C} = \mathrm{Aff}/k$ and G/k is a group scheme of finite type (so h_G comes with a factorisation via $\mathbf{Grp}$). Now being of finite type is local in the fpqc topology.

Let $h_X = X$ be a k-scheme. If $P \to X$ is a G-torsor in fpqc then P is automatically an algebraic space, and $P \to X$ is flat and locally of finite presentation. So $P \to X$ is an fppf cover, so $P \to X$ is a torsor in fppf (not a-priori obvious). If G/k is smooth then it is even an etale torsor; we know $P \to X$ is smooth, and any smooth map has local sections in the étale topology.

Definition 1.3. Fix G, X a above. The category $\mathbf{Tors}_X$ has torsors as objects, and maps are G-equivariant arrows.

Exactly in the classical case, have

Lemma 1.4 (Exercise)**.**

1. *The category $\mathbf{Tors}_X$ is a groupoid, i.e. all arrows are isomorphisms.*

2. *A torsor P is trivial (i.e. $p \cong X \times G$) iff $P \to X$ has a section.*

Lemma 1.5 (Exercise)**.**

1. *If $P \to X$ is a G-torsor, then the action of G on P is free (pointwise, i.e. on each object X the action $G(X)$ on $P(X)$ is set-theoretically free).*

2. $X = P/G$ is a sheaf quotient. So $P \to P/G$ is universal invariant map, and P/G is the sheaf associated to $S \mapsto P(S)/G(S)$.

Lemma 1.6 (Exercise)**.** *If $P \times X \to P$ is a free action, then $P \to P/G$ is a G-torsor*

Let $P \to X$ be a G-torsor. Let F be a sheaf, and $G \times F \to F$ a left action. Then G acts on $P \times F$ by $(p, f)g = (pg, g^{-1}f)$, and the action is free. Write $P \times^G F = (P \times F)/G$. Then get a Cartesian diagram

$$\begin{array}{ccc} P \times F & \longrightarrow & P \\ \downarrow & & \downarrow \\ P \times^G F & \longrightarrow & X = P/G. \end{array}$$

A very important application of this is that if $\varphi\colon G \to H$ is a homomorphism of sheaves of groups and P is a G-torsor, you can apply the above. Find that $P \times^G H \to X$ is an H-torsor via

$$(p, h)h' = (p, hh')$$

and action of G commutes with this:

$$(p, h)g = (pg, \varphi(g)h)$$

and there is an induced X-map $P \to P \times^G H$, which is φ-equivariant ($p \mapsto (p, 1)$). So have a φ-equivariant map from a G-torsor to an H-torsor, and moreover this map is universal wrt that.

If $Q \to X$ is an H-torsor and $P \to Q$ is φ-equivariant, then this extends uniquely[2] to an H-equivariant map $P \times H \to Q$ (the one sending $(p, h) \mapsto f(p)h$. Thus we get an X-map of torsors $P \times^G H \to Q$, and since it is a map of torsors it is an isomorphism.

Say X is a terminal object in $\mathcal{C}$, and $P \to X$ is a G-torsor. The rule $F \mapsto P \times^G F$ defines a functor from sheaves with an action of G to sheaves.

Lemma 1.7 (Exercise)**.** *This sends products to products (even preserves all limits).*

If F is a sheaf of groups then $P \times^G F$ is a sheaf of groups.

If we take $F = G$ and G acts on itself by conjugation $gh = hgh^{-1}$, and $P \to X$ is a torsor, then $P \times^G G =: \tilde{G}$ is another sheaf of groups, a'twisted form of G'. If G is abelian then $\tilde{G} = G$. If not then things can happen!

[2]It is unique map respecting the maps from P.

Lemma 1.8 (Exercise)**.** $\tilde{G} = \mathbf{Aut}^G(P)$*, i.e. G-equivariant automorphisms of P as a bundle. The formula is* $T \mapsto \mathrm{Aut}^G(T \times_X P/T)$*.*

Then P is a $(\tilde{G}, G)$-bitorsor; $\tilde{G}$ acts on the left, G acts on the right, the actions commute, and P is a torsor for both [this is the definition of a bi-torsor].

If I is a (H, G)-bitorsor, then (exercise) there is automatically a map $H \to \mathrm{Aut}^G(I)$ (which is automatically an isomorphism), so $H = \tilde{G}$.

The group $\tilde{G}$ is called an 'inner form' of G [beware that terminology varies].

Remark 1.9. If P is trivial (i.e. $P = G \times G$) then $\tilde{G} = G$, nothing happens.

Example 1.10. Suppose that (V, q) and (V', q') are non-degenerate quadratic forms (same dimension n) on a field k with $2 \in k^\times$. Then we get two group schemes[3] $O(q)$ and $O(q')$, in general not isomorphic. Define a functor $I\colon \mathbf{Aff}_k \to \mathbf{Set}$, sending $\operatorname{Spec} A \mapsto$ (the set of isometries $V \otimes_k A \cong V' \otimes_k A$) (this set is non-empty for some finite separable field extension of k). Then $O(q)$ acts on the right on I and $O(q')$ acts on the left, so I is a $(O(q), O(q'))$-bitorsor. So any two orthogonal groups of the same dimension are inner forms of each other.

2 Lecture 2: Gerbes

Warning: all Auts and Isoms should be underlined, but are not. Rooms for afternoon sessions:
04-224: Algelo
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Some examples of torsors.

Let $\mathcal{C} = \mathbf{Aff}_k$ with fppf topology, k a field, $G \to \operatorname{Spec} k$ affine group scheme of finite type. Then by descent theory, if S is a scheme, a G-torsor P/S is an affine map of schemes.

Examples:

1. $G = \mathrm{GL}_n = \operatorname{Spec} k[x_{i,j} : 1 \leq i, j \leq n]_{det}$. Then the functor sends an algebra A to $\mathrm{GL}_n(A)$. Basic result: GL_n torsors correspond to vector bundles of rank n (equivalence of categories, allowing only isomorphisms of vector bundles).

   Idea of proof of equivalence: say that $P \to S$ is a GL_n-torsor, then $P \times^{\mathrm{GL}_n} \mathcal{O}^n|_{X_{zar}}$ is a vector bundle. If E is a vector bundle on S then define

$$P\colon \mathbf{Aff}_k^{op} \to \mathbf{Set}$$
$$U \mapsto \{(f, \varphi) : f\colon U \to S, \varphi\colon \mathcal{O} \cong f^*E\}$$

[3]Here $O(q)(T)$ is the set of isomorphisms from V_T to itself which preserve the form q.

2. V an n-dimensional vector space over k (assume $2 \in k^\times$), q an orthonormal quadratic form on V, $G = O(q)$ (so $G(A)$ is the set of A-isometries[4] from $A \otimes_k V$ to itself.

   Given $P \to S$ an $O(q)$-torsor, then $P \times^{O(q)} V$ is a vector bundle of rank n on S together with a non-degenerate quadratic form. Conversely, if E is a vector bundle of rank n on S with a non-degenerate quadratic form then set $P := \mathrm{Isometries}_S(\mathcal{O}_S \otimes_k V, E)$. Then $O(q)$ acts on this P by composition, and P becomes a torsor under this group. In this way you get an equivalence between $O(q)$-torsors on S and vector bundles of rank n with a non-degenerate quadratic form.

3. $G = \mathrm{PGL}_n = \mathrm{GL}_n / \mathbb{G}_m$ (sheaf quotient!). Now PGL_n is the automorphism group of[5] $M_{n,k}$ and also of $\mathbb{P}_k^{n-1}$.

   If $P \to \operatorname{Spec} k$ is a PGL_n-torsor then $P \times^{\mathrm{PGL}_n} M_{n,k}$ is a k-algebra. The k-algebras that become matrix algebras after an extension are the central simple algebras (could use this as a definition).

   The product $P \times^{\mathrm{PGL}_n} \mathbb{P}_k^{n-1}$ is a variety that becomes $\mathbb{P}^{n-1}$ after a field extension (a Brauer-Severi variety). If C is a CSA of dimension n, then

$$\begin{gathered}\mathbf{Isom}_k(M_{n,k}, C)\\ A \mapsto \mathrm{Isom}_A(A \otimes_k M_{n,k}, A \otimes_k C)\end{gathered}$$

   is a PGL_n-torsor.

   So you get correspondences between (PGL_n-torsors), (CSA of dimension n) and (Brauer-Severi varieties of dimension $n-1$). To go from a CSA to a BS, send the CSA C to the set of left ideals of $\mathrm{codimension}_k = n$.

2.1 Gerbes

Definition 2.1. Let $\mathcal{C}$ be a site (cat with Grothendieck topology) with a terminal object pt. A *gerbe* $\Gamma \to \mathcal{C}$ is a fibred category such that

1. Γ is a stack;

2. there exists a covering $U_i \to pt$ such that $\forall i$, $\Gamma(U_i) \neq \emptyset$.

3. If $S \in \operatorname{ob} \mathcal{C}$ and ξ, $\eta \in \Gamma(S)$ then there exists a covering $\{S_i \to S\}$ such that $\xi_{S_i} \cong \eta_{S_i}$ in $\Gamma(S_i)$.

[4]i.e. preserving q
[5]i.e. *algebra* of $n \times n$ matrices over k

Definition 2.2 (Stack). 1. If $S \in \operatorname{ob} \mathcal{C}$ and ξ, $\eta \in \Gamma(S)$ define

$$\begin{aligned} \mathbf{Isom}_S(\xi, \eta) \colon (\mathcal{C}/S)^{op} &\to \mathbf{Set} \\ T \to S &\mapsto \mathrm{Isom}_{\Gamma(T)}(\xi|_T, \eta|_T). \end{aligned}$$

Then this functor should be a sheaf.

2. 'You can glue objects, with morphisms satisfying a cocycle condition' (David: I think this is 'every descent datum is effective', cf [Sta13, Tag 026E]).

Remark 2.3. Let $\xi \in \Gamma(S)$. Then get (from conditions in above definition in order):

1. define

$$\begin{aligned} \mathbf{Aut}_S(\xi) \colon (\mathcal{C}/S)^{op} &\to \mathbf{Set} \\ T \to S &\mapsto \mathrm{Aut}_T(\xi|_T). \end{aligned}$$

Then this $\mathbf{Aut}_S(\xi)$ is a sheaf of groups on $\mathcal{C}/S$.

2. If $\xi, \eta \in \Gamma(S)$, then $\mathbf{Aut}_S(\xi)$ acts on $\mathbf{Isom}_S(\xi, \eta)$ on the right.

3. $\mathbf{Isom}_S(\xi, \eta)$ is an $\mathbf{Aut}_S(\xi)$-torsor.

Remark 2.4 (Examples of gerbes). 1. If G is a sheaf of groups on $\mathcal{C}$, then the classifying stack $BG \to \mathcal{C}$ has objects G-torsors $P \to S$ and morphisms are

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

where $Q \to P$ is G-equivariant.

What about the conditions of the definition?

(a) is an exercise.

(b) We take the trivial covering $pt \to pt$, and the torsor is $\Gamma(pt) \ni G \to pt$.

(c) clear.

Let $\Gamma \to \mathcal{C}$ be a Gerbe, and $\xi_0 \in \Gamma(pt)$. The functor $(\mathcal{C}/S)^{op} \to \mathbf{Set}$ given by $\xi \mapsto \mathbf{Isom}_S(\xi_0|_S, \xi)$ is an $\mathbf{Aut}(\xi_0)$-torsor. Write $G = \mathbf{Aut}(\xi_0)$. Define a functor $\Gamma \to BG$ over $\mathcal{C}$ by sending ξ to $\mathbf{Isom}_S(\xi_0|_S, \xi)$, which is an equivalence. Proof: see notes on descent by torsors. Then ξ_0 corresponds to the trivial torsor $G \to pt$.

Example: $\mathcal{C} = \mathbf{Aff}_k$ with étale topology. Then $Q_n \to \mathbf{Aff}_k$ is define by $Q_n(U) = \{(E, q)\}$ where E is a VB on U and q a non-degenerate quadratic form on E.

Then Q_n is a gerbe (cf. previous discussion). If $(V, q) \in Q_n(k)$ then get an equivalence between Q_n and $BO(q)$. This explains why all these groups $O(q)$ give the same classifying space - they are different sections of the same gerbe.

More generally: Sheaves of groups correspond to gerbes with a section. Different section corresponds to different groups.

2.1.1 Example of a gerbe without a section:

Suppose
$$1 \to G' \to G \to G'' \to 1$$
is a SES of sheaves of groups. Say $P'' \to pt$ is a G''-torsor. Then we can define a gerbe $\Gamma \to \mathcal{C}$ by setting $\Gamma(S)$ to be the set of $G \to G''$equivariant S-maps $P \to P''_S$ of G-torsors. IOW, Γ is the gerbe of liftings of P'' to G. Exercise: this is a gerbe.

For example, take
$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1.$$
Then GL_n-torsors over k are trivial (because always trivial Zariski locally). Hence if $P \to k$ is a non-trivial PGL_n-torsor then Γ has no sections! This corresponds to examples from Max's lectures.

3 Lecture 3

Recall: Let G, H sheaves of groups on $\mathcal{C}$. The following are equivalent:

1. H is an inner form of G
2. $\exists$ an (H, G)-bitorsor
3. BG and BH are equivalent

Say $\mathscr{C} = \mathbf{Aff}_k$ and G, H affine group schemes. Then can associate to G the category
$$\mathrm{Rep}(G) = \{\text{reps of } G\},$$
which has canonical structure of a k-linear tensor category.

We define $\mathrm{Vect}(BG)$ to be the category of vector bundles (and suitable morphisms...) on BG [David: i.e. quasicoherent sheaves which are vector bundles on every scheme. Quasicoherent in sense of Max; recall:

Definition 3.1. An $\mathcal{O}_{\mathscr{X}}$-module F is *quasicoherent* if

1. for all $x \in \mathscr{X}$, the restriction $F|_x$ is a quasicoherent $\mathcal{O}_{\varphi(x)}$-module
2. for every $f\colon y \to x \in \mathscr{X}$, the natural arrow $F|_x \to f_* F|_y$ is adjoint in qcoh to an isomorphism. ('Cartesian')

Condition (2) is a pedantic version of saying that $f^*(F|_x) \to F_y$ is an isomorphism.]

We define a map $\mathrm{Rep}(G) \to \mathrm{Vect}(BG)$ by sending a representation $G \mapsto GL(V)$ to the vector bundle which on a G-torsor P has fibre $P \times^G V$ (doing this functorially gets you a vector bundle - what we have done above is to associate to each point of BG a vector space, so there are of course some things to check).

Then descent theory implies that $\mathrm{Rep}(G) \to \mathrm{Vect}(BG)$ is an equivalence of tensor categories.

If $BG \cong BH$ then (eg. by the above) we get an isomorphism $\mathrm{Rep}(G) \tilde{\to} \mathrm{Rep}(H)$.

If q, q' are non-degenerate quadratic forms of the same dimension n then combining this with things form the previous lectures we get an iso $\mathrm{Rep}\, O(q) \tilde{\to} \mathrm{Rep}\, O(q')$.

Part of the package of Tannakian duality is that if we have an equivalence of k-linear tensor cats $\mathrm{Rep}(G) \to \mathrm{Rep}(H)$ then we get an equivalence of BG with BH. But to recover the group itself you need a fibre functor, and different fibre functors give different groups.

3.1 Bands

Let $\Gamma \to \mathcal{C}$ be a gerbe, and $S \in \mathrm{ob}\, \mathcal{C}$. Let $\xi, \eta \in \mathrm{ob}\, \Gamma(S)$. Given an isomorphism $\varphi\colon \xi \tilde{\to} \eta$ then get an induced isomorphism $\mathbf{Aut}_S(\xi) \to \mathbf{Aut}_S(\eta)$ by conjugating with $\varphi : \alpha \mapsto \varphi\alpha\varphi^{-1}$ (or the other way around...?).

If you change φ then the above equivalence will change (in general).

On the other hand, if the sheaf is abelian then it is independent of the choice of automorphism (cf. fundamental groups)!

We say Γ is *abelian* is for all $S \in \mathcal{C}$ and all $\xi \in \Gamma(S)$ we have that $\mathbf{Aut}_S(\xi)$ is abelian. Assume Γ abelian. Then for any $\xi, \eta \in \Gamma(S)$ we get a canonical isomorphism $\mathbf{Aut}_S(\xi) \tilde{\to} \mathbf{Aut}_S(\eta)$. Omitting details, you get a sheaf of abelian groups G on $\mathcal{C}$ such that for each object $S \in \mathcal{C}$ and each $\xi \in \Gamma(S)$, there is a canonical isomorphism $G_S \to \mathbf{Aut}_S(\xi)$.

This G is called the *band* of Γ, written $\mathrm{Band}_{\mathcal{C}}(\Gamma)$.

Definition 3.2. Let G be a sheaf of abelian groups on $\mathcal{C}$. A *gerbe banded by* G is an abelian gerbe $\Gamma \to \mathcal{C}$ together with an isomorphism $G \to \mathrm{Band}_{\mathcal{C}}(\Gamma)$.

[David: basic example: if G is an abelian group scheme then BG should be a gerbe banded by G. Indeed, by lemma 1.8 we have that, for a G-torsor P, the formula $\mathbf{Aut}^G(P) = \tilde{G}$ holds, and if G is abelian then $\tilde{G} = G$.]

Let Γ, Δ be abelian gerbes on $\mathcal{C}$. Let $\Phi\colon \Gamma \to \Delta$ be a functor over $\mathcal{C}$. Then we get a homomorphism of sheaves $\mathrm{Band}\, \Gamma \to \mathrm{Band}\, \Delta$, by sending an automorphism of an object of Γ to the corresponding automorphism of the image object in Δ (the map Φ is a functor!). Need to check that everything descends - details omitted.

If Γ is banded by G and Δ is banded by H and $\varphi\colon G \to H$ a homomorphism of sheaves of groups, then we say Φ is φ-equivariant if TFDC:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ \downarrow{=} & & \downarrow{=} \\ \mathrm{Band}(\Gamma) & \xrightarrow{\mathrm{Band}\,\Phi} & \mathrm{Band}(\Delta) \end{array}$$

If $G = H$ and $\varphi = id$ then we say Φ is G-equivariant if it is φ-equivariant.

Exercise: If Γ, Δ are gerbes banded by G and $\Phi\colon \Gamma \to \Delta$ is a functor over $\mathcal{C}$ and Φ is G-equivariant then Φ is an equivalence. [This should be very similar to the classical statement that a map of torsors is an equivalence (here 'map of torsors' includes equivariance)].

Lemma 3.3. *Let $\varphi\colon G \to H$ be a homomorphism of sheaves of abelian groups. Let Γ be a gerbe banded by G. Then there exists a gerbe Δ banded by H and a (unique up to an equivalence unique up to 2-equivalence [6]) φ-equivariant functor $\Phi\colon \Gamma \to \Delta$.*

We want to construct something like $\Delta = \Gamma \times^G H$, but this doesn't make sense as written because there is no action of G on Γ. So we need to work a little more.

Idea of construction: we start with Γ and then 'modify' it until we get someting that works as Δ. First modify the arrows. Define $\tilde{\Delta}$ to have the same objects as Γ, but given $S \in \mathcal{C}$, we define

$$\mathrm{Isom}_{\Delta/S}(\eta, \xi) = \left(\mathrm{Isom}_{\Gamma/S}(\eta, \xi) \times^G H\right)(S)$$

Here $\mathrm{Isom}_{\Gamma/S}(\eta, \xi)$ is a G-torsor by definition, so $\mathrm{Isom}_{\Gamma/S}(\eta, \xi) \times^G H$ is an H-torsor living over S, and we have taken the set of sections. However, this $\tilde{\Delta}$ is not a gerbe. The hom functors are certainly sheaves, but there are problems with glueing objects. So we 'stackify' (analogous to sheafification, but a bit more complicated). Details are a little unpleasant, and there are set theoretical issues. We end up with a map $\tilde{\Delta} \to \Delta$.

Theorem 3.4. *Suppose that G is a a sheaf of abelian groups on $\mathcal{C}$ Then*

1. *Isomorphism classes of G-torsors on $\mathcal{C}$ are parametrised by $H^1(\mathcal{C}, G)$*
2. *equivalence classes of (gerbes banded by G) are parametrised by $H^2(\mathcal{C}, G)$.*

[6] 2-cat fun

Sketch proof. Define $\tilde{H}^1(G)$ to be the set of isomorphism classes of G-torsors on $\mathcal{C}$, and define $\tilde{H}^2(G)$ to be equivalence classes of gerbes banded by G. Firstly, these are abelian groups: the group law is given by the following procedure:

say $P \to pt$ and $Q \to pt$ are G-torsors, then $P \times Q$ is a $G \times G$ torsor. Then define $(P \times Q) \times^{G\times G} G$ via the multiplication map $G \times G \to G$ (a hom because G is abelian). Then define $[P] + [Q] = [(P \times Q) \times^{G\times G} G]$. The identity is given by the trivial torsor, and the inverse is given by putting a $\star^{-1}$ in the action.

We also need a group law on the $\tilde{H}^2$ (gerbes), which is constructed in more or less the same way.

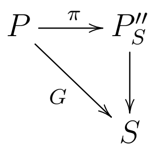
Then $\tilde{H}^1(G)$ and $\tilde{H}^2(G)$ are functorial in G, so we get functors.

To complete the proof we need two things:

1. exact sequences: given a SES $0 \to G' \xrightarrow{\alpha} G \xrightarrow{\beta} G'' \to 0$ of abelian sheaves, we get
$$0 \to H^0(G') \to H^0(G) \to H^0(G'')$$
and we need the rest of the long exact sequence (up to the H^2s). We get all but the connecting homomorphisms by the functoriality, so it is enough to define the connecting homs (in a functorial way).

   Suppose given $s \in G''(pt)$. want to make a G'-torsor ∂s. Well, just take $\partial s = \beta^{-1}s$. Then $[\partial s] = 0$ iff s is in the image of $G(pt) \to G''(pt)$.

   Now to define the connecting hom $\tilde{H}^1(G'') \to \tilde{H}^2(G')$. Given a G''-torsor $P' \to pt$, we define $\partial P''$ to be the gerbe of liftings of P'' to G (cf. previous lecture). This is banded by G' because G' is central in G: in $\Gamma(S)$ we have a diagram:



   So $G'(S)$ is the set of transformations $f\colon P \to P$ over S such that $\pi \circ f = \pi$. SO $G'(S)$ is the automorphism group in $\Gamma(S)$ of the above diagram.

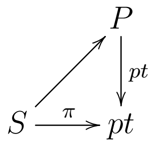
   Define $\partial P'' = \Gamma$.

   The class $[\partial P''] \in \tilde{H}^2(G)$ of the gerbe we just defined is zero iff $\Gamma \cong BG'$ iff $\Gamma(pt) \neq \emptyset$. Also need to check that elements are erasable.

2. If $\xi \in \tilde{H}^1(G)$, then we want to show there exists $G \subset H$ such that
$$\begin{gathered}\tilde{H}^i(G) \to \tilde{H}^i(H)\\ \xi \mapsto 0\end{gathered}$$

If $P \to pt$ is any G-torsor then we want to show there exists $G \subset H$ such that $(P \times^G H)(pt) \neq \emptyset$.

Let $S \to pt$ be a cover such that we have a lift



Define $H = \pi_* \pi^* G = \mathbf{Hom}_{pt}(S, G)$

Adjunction yields a map $G \to H$.

Then $P \times^G H = \pi_* \pi^* P$.

So we deduce $(P \times^G H)(pt) \neq \emptyset$ (because it is push forward of a thing with a section, and so has a section).

□

4 Lecture 4: an application of the theory

4.1 Essential dimension

Introduced in 1993 by Reichstein - Buhler (essential dimension of a finite group).

Let $\Gamma \to \mathbf{Aff}_k$ be a fibred cat, let K/k be a field extension, and let $\xi \in \Gamma(K)$. Given $k \subset L \subset K$, we say ξ *is defined over* L if there exists $\eta \in \Gamma(L)$ such that $\eta_K \cong \xi$.

The essential dimension $\mathrm{ed}_k(\xi)$ of ξ is the minimal transcendence degree of $k \subset L \subset K$ such that ξ is defined over L.

Define $\mathrm{ed}_k(\Gamma)$ to be the supremum of the essential dimensions of elements of $\Gamma(k)$.

If Γ is limit preserving then the essential dimension should be finite (?).

Given an algebraic group $G \to \operatorname{Spec} k$, then define $\mathrm{ed}_k(G) := \mathrm{ed}_k(BG)$.

4.2 Examples

4.2.1 $G = \mathrm{GL}_n$

Well $B\,\mathrm{GL}_n(K)$ is the groupoid of vector spaces over K. S the essential dimension of GL_n is 0 (groups with this property are called special).

4.2.2 $G = \mathrm{O}_n$

Then $B\,\mathrm{O}_n$ is the stack of non-degenerate quadratic forms of rank n. We know that every quadratic form q can be diagonalised (written as $q \cong \sum_i a_i x_i^2$). This implies that $\mathrm{ed}_k\,\mathrm{O}_n \leq n$. Can you do better? Well, let $k(a_1, \cdots, a_n)$ and take universal form over it. Can we define this form over a smaller field? Yes, for example we can take $k(a_1^3, \cdots, a_n^3)$. But the transcendence degree does not change. In fact, this will 'always happen', so $\mathrm{ed}_k\,\mathrm{O}_n = n$ [Reichstein, 2000].

4.2.3 $G = \mathrm{PGL}_n$

Well, $B\,\mathrm{PGL}_n(K)$ corresponds to twisted forms of $M_{n,K}$, AKA central simple algebras of degree n. A basic example is that of **cyclic algebras**:

Assume $\mu_n \subset k$, write $\mu_n = \langle w \rangle$. Let $a, b \in k^\times$. Take the algebra generated by x, y with the relations $x^n = a$, $y^n = b$, $yx = wxy$ (cf quaternions). This turns out to be a CSA

For $n = 2$ and $n = 3$, every CSA is cyclic (for $n = 3$ this is a fairly deep theorem of Wedderburn). For $n = 4$ it is not true. A major open question is whether, for n prime, every CSA is cyclic (always assuming $n \in k^\times$ and $\mu_n \subset k$).

It is clear that a CSA has essential dimension at most 2, since there are only two things (a and b) needed to generate it. So if you can show that $\mathrm{ed}_k\,\mathrm{PGL}_n > 2$ for some n then there must be non-cyclic CSAs of degree n.

Theorem 4.1 (Reichstein, ?, ?, ?)**.** *Let n prime and $n \geq 5$. Then*

$$2 \leq \mathrm{ed}_k\,\mathrm{PGL}_n \leq \frac{(n-1)(n-2)}{2}.$$

4.2.4 Other examples

$\mathrm{ed}_k\,\mathrm{O}_n = n$ and $\mathrm{ed}_k\,\mathrm{SO}_n = n - 1$.

4.2.5 Spin group

$$1 \to \mu_n \to \mathrm{Spin}_n \to \mathrm{SO}_n \to 1,$$

Known (Reichstein-?, Serre -?...):

$$\mathrm{ed}_k\,\mathrm{Spin}_n \geq \lfloor n/2 \rfloor.$$

Rost computed it exactly for $n \leq 14$.

Theorem 4.2 (B?, Reichstein, Vistoli)**.** *Let $k = \mathbb{C}$. Then*

$$\mathrm{ed}_k\,\mathrm{Spin}_n \geq 2^{\lfloor \frac{n-1}{2} \rfloor} - \frac{n(n-1)}{2},$$

with equality if ($n \geq 15$ and $4 \nmid n$).

4.3 How proven above theorem?

Proposition 4.3 (BRV)**.** *Let*

$$1 \to Z \to G \to H \to 1$$

be an exact sequence of algebraic groups. Pick an extension K/k and an H-torsor $Q \to \operatorname{Spec} K$. As usual write ∂Q for the gerbe of liftings of Q to G_K. Then

$$\operatorname{ed}_k G \geq \operatorname{ed}_K(\partial Q) - \dim H.$$

The boundary map

$$\partial\colon H^1(K, SO_n) \to H^2(K, \mu_2) \subset \operatorname{Br}(K) \tag{2}$$

is the Hasse-Witt invariant (note $H^1(K, SO_n)$ corresponds to SO_n torsors, equivalently to quadratic forms with discriminant 1).

Proof.

$$\operatorname{ed}_k G \geq \operatorname{ed}_K G_K,$$

so may assume $k = K$.

Let $e := \operatorname{ed}_k(\partial Q)$. Then there exists an extension F/k and a G-torsor $P_F \to Q_F \in \partial Q(F)$ such that $\operatorname{ed}_F(P_F) = e$.

Then there exists $k \subset L \subset F$ with $\operatorname{tr.deg}(L/k) \leq \operatorname{ed}_k G$ and such that there exist some P'/L such that $P'_F \cong P$ (note that in general the map does not descend).

Then define $I \to \operatorname{Spec} L$ by $I(\operatorname{Spec} S) = \{P'_A \to Q_A : G-\text{equivariant}\}$. Then $\dim I = \dim H$.

We have a point $\operatorname{Spec} F \to I$ over $\operatorname{Spec} L$. This must factor through some $\operatorname{Spec} E$ where $\operatorname{tr.deg}(E/L) \leq \dim H$. Then by definition the map $P_F \to Q_F$ is defined over E. Then can complete proof by collecting together the inequalities. □

We want to apply this to spin groups. If Γ is a gerbe banded by μ_n, then what can I say about $\operatorname{ed}_k \Gamma$?

Back to general theory of the Brauer group. Given $[\Gamma] \in H^2_{fppf}(k, \mu_n) \subset H^2(k, \mathbb{G}_m)$ (étale OK in char 0) we get an exact sequence (use Hilbert 90)

$$1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

$$0 = H^1(k, \mathbb{G}_m) \to H^2(k, \mu_n) \to H^2(k, \mathbb{G}_m) \xrightarrow{n} H^2(k, \mathbb{G}_m)$$

(recall $H^2(k, \mathbb{G}_m) = \operatorname{Br}(k)$. Then

$$H^2(k, \mu_n) = \operatorname{Br}(k)[n].$$

If $a \in \operatorname{Br}(k)$ then define

1. $\exp(a)$ is the order of a in $\mathrm{Br}(k)$
2. $\mathrm{ind}(a)$ as in Max's lectures.

Have

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_n & \longrightarrow & SL_n & \longrightarrow & PGL_n & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & GL_n & \longrightarrow & PGL_n & \longrightarrow & 1 \end{array}$$

We have that $\exp(a) \mid \mathrm{ind}(a)$ and that $\exp(a)$ and $\mathrm{ind}(a)$ have the same prime factors.

Have $\partial Q \in H^2(k, \mu_n)$ and map $\partial\colon H^1(SO_n) \to H^2(k, \mu_2) \subset \mathrm{Br}(k)[2]$.

If Q is 'generic' then $\mathrm{ind}[\partial Q] = 2^{\lfloor(\frac{n-1}{2})\rfloor}$.

4.4 Canonical dimension

Definition 4.4. Let X a smooth projective geom con variety $/k$. Then the canonical dimension

$$\mathrm{cd}(X) = \min\{\dim\left(\overline{\mathrm{im}\, X \overset{rat}{\dashrightarrow} X}\right)\} \tag{3}$$

Exercise: canonical dimension $\mathrm{cd}(X) = 0$ iff X has a rational point.

Theorem 4.5 (BRV)**.** *Suppose that Γ is a gerbe banded by μ_n (so $[\Gamma] \in \mathrm{Br}(k)$). Let X be a Brauer-Severi variety whose Brauer class is $[\Gamma]$ (this X is not unique, just pick one). Then*

$$\mathrm{ed}_k \Gamma = \mathrm{cd}(X) + 1.$$

Theorem 4.6 (Karpenko)**.** *Let X be a Brauer-Severi variety with class $a \in \mathrm{Br}(X)$. If $\exp(a)$ is a prime power (this is the same as saying that $\mathrm{ind}(a)$ is a prime power) then*

$$\mathrm{cd}(X) = \mathrm{ind}(a) - 1.$$

Putting together these results we get that if $\mathrm{ind}(a) \mid \exp(a)$ is a prime power then

$$\mathrm{ed}_k \Gamma = \mathrm{ind}[\Gamma]. \tag{4}$$

Putting all these together proves the above theorem giving a lower bound on the essential dimension of the spin group. For the upper bounds you have to do some more work, 'much more classical'.

References

[Sta13] The Stacks Project Authors. Stacks project. http://stacks.math.columbia.edu, 2013.