Canonical Heights on Hyperelliptic Curves

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January 18, 2010

Abstract

We describe an algorithm to compute canonical heights of points on hyperelliptic curves over number fields, using Arakelov geometry. We include a worked example for illustration purposes.

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1 Brief introduction to arithmetic intersection theory

It was shown by Faltings ([Fal84]) and Hriljac ([Hri85]) independantly that (under some conditions), given a point on a curve C over a number field k, the self intersection of the closure of that point on an arithmetic surface $\mathscr{C}$

corresponding to C is equal to the canonical height of that point on C with respect to the ϑ divisor.

We use this to compute canonical heights of divisors on hyperelliptic curves. The resulting algorithm works in principal for any genus, though in practice one soon runs onto problems as the search for a sufficiently 'nice' representative in a given linear equivalence class results in a divisor whose naive height is very large, making calculations slow.

Throughout, g will refer to the genus of C.

If D is a divisor, $|D|$ will refer to it's support.

Let X be a scheme over a field k. When we say a divisor D on X is defined over a field $l \supset k$, we mean that D can be cut out by equations with coefficients in l.

1.1 Definitions and normalisations

The precise definitions of the intersection pairings at various places will be recalled in the appropriate sections.

Just as global height can be decomposeed as a sum of local heights, so the global intersection pairing is a sum of local factors.

We are mainly interested in horizontal divisors, that is those which are the Zariski closure of rational points on the generic fibre (this aggrees with Faltings requirement in [Fal84] that horizontal divisors be perpendicular to the fibres of points in the base).

Given two such divisors D and E of degree 0, the local intersection pairing at the place ν (finite or infinite) is written $< D, E >_\nu$ and is required to satisfy the following[1]:

1. $< D, E >_\nu + < D, F >_\nu = < D, E + F >_\nu$.

2. $< D, E >_\nu = < E, D >_\nu$.

3. Given D with support disjoint from that of $div(f)$, $< D, div(f) >_\nu = \log |f[D]|_\nu$ (where $f[\sum n_i p_i] := \prod f(p_i)^{n_i}$).

4. Fix p_0 in $C(k) \setminus |D|$. Then the map $C(k) \setminus |D| \to \mathbb{R}$ given by

$$p \mapsto < D, p - p_0 >_\nu$$

is locally bounded and continuous.

We will define that global height pairing to be the sum of local pairings over a proper set of absolute values satisfying the product formula. Then 3 shows that our global pairing respects linear equivalence.

These properties are almost immediate from the definitions we will give, so it is easy to see we have a quadratic form on $\mathcal{J}(C)$.

What we will not prove here (see Faltings [Fal84], Hriljac [Hri85] or Lang [Lan88]) is that the pairing is non-degenerate - in fact they prove something stronger, namely that the resulting height is that coming from the ϑ divisor.

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[1] This list may also serve as a reference for the choice of normalisations made in this document.

1.2 Why degree zero divisors

It is possible to define an intersection pairing for divisors of arbitrary degree, and indeed this would make life a lot easier when it comes to the main computational problem of finding and working with a nice representative of a linear equivalence class, see later for details. However, the pde that would have to be solved to compute this pairing for infinite places becomes substantially more complicated.

1.3 Mumford coordinates

We use Mumford coordinates to parametrise divisors on the hyperelliptic curve C. We recall the definitions as in [MM84] (we use a slightly simplified form because we are not in general interested in points at infinity):

Suppose C is defined by the equation $y^2 = f_{2g+2}(x)$.

A point on $Jac(C)$ is given by a pair (α, β) where α, β in $k[C]$ such that:

1. α is monic of degree at most g.
2. $deg(\beta) \leq deg(\alpha)$.
3. α divides $\beta^2 - f$.

The pair (α, β) corresponds to the divisor $\alpha = 0, y - \beta = 0$ on C.

The coefficients of such α, β are then coordinates on an affine piece of the Jacobian of C, in particular k-rational points on the Jacobian correspond exactly to such pairs of α, β with coefficients in k.

2 Finite intersections

To define and in simple cases compute, intersections at finite places are much easier than those at infinite places. However, while the computational complexity of intersections at infinite places depends mainly on the genus, those at finite places rapidly become hard to compute as the naive height of the divisors involved increases.

Fix a finite prime ν of $\mathcal{O}_k$. Given D, E horizontal divisors of smooth reduction at ν on an arithmetic surface $\mathscr{C}$ over $\mathcal{O}_k$, we define the local intersection pairing at ν :

$$< D, E >_\nu = \sum_{\substack{\mathfrak{p} \text{ a point of} \\ \mathscr{C} \text{ over } \nu}} - \log \# \frac{\mathcal{O}_\mathfrak{p}}{I_{D,\mathfrak{p}} + I_{E,\mathfrak{p}}} = \sum_{\substack{\mathfrak{p} \text{ a point of} \\ \mathscr{C} \text{ over } \nu}} log \left| \# \frac{\mathcal{O}_\mathfrak{p}}{I_{D,\mathfrak{p}} + I_{E,\mathfrak{p}}} \right|_\nu \tag{1}$$

where $I_{D,\mathfrak{p}} + I_{E,\mathfrak{p}}$ are defining ideals for D and E respectively in the local ring $\mathcal{O}_\mathfrak{p}$.

To see this symbol satisfies 1, 2, 3 and 4 above, see [Lan88], chapter 3.

2.1 Resultant calculations

We assume throughout this subsection that every divisor involved has good reduction modulo ν (or modulo every finite prime in the global case).

While these formulae are easy to compute explicitly for a small number of points defined over a number field of small degree, we can do better for divisors in

Mumford representation (see 1.3). Given $D_1 := (\alpha_1, \beta_1), D_2 := (\alpha_2, \beta_2)$ divisors on C defined over k, we can use resultants to rapidly compute $< D_1, D_2 >_\nu$ for ν a finite place. For free we get a simple algorithm to determine the sum of the local intersections over all the finite places of $\mathcal{O}_k$ if the class number of k is 1.

We assume the following conditions on the divisors D_1, D_2:

1. For each i, $|D_i|$ contains no branch points of C.
2. For each i, D_i has degree g.
3. For each i, $\#|D_i| = g$ ie D_i is reduced.
4. $|D_1| \cap |D_2| = \emptyset$, ie they have no common support.

It can be seen from equation (1) that

$$
\begin{aligned}
&< D_1, D_2 >_\nu = \\
&\log(\#\kappa_\nu). \sum_{\substack{p \text{ root of } \alpha_1 \\ \text{q root of } \alpha_2}} \max\left(\min\left(\frac{ord_\nu(N_{l_{p,q}/k}(p-q))}{[l_{p,q}:k]}, \frac{ord_\nu(N_{l_{p,q}/k}(\beta_1(p)-\beta_2(q)))}{[l_{p,q}:k]}\right), 0\right) \\
&= \\
&\log \prod_{\substack{p \text{ root of } \alpha_1 \\ \text{q root of } \alpha_2}} \max\left(\min\left(\left|({}_{l_{p,q}/k}(p-q))\right|_\nu^{1/[l_{p,q}:k]}, \left|N_{l_{p,q}/k}(\beta_1(p)-\beta_2(q))\right|_\nu^{1/[l_{p,q}:k]}\right), 1\right)
\end{aligned}
\tag{2}
$$

where $l_{p,q}$ is a finite extention of k such that both p and q are defined over $l_{p,q}$, and κ_ν is the residue field at ν.

Now recalling that the roots of a univariate polynomial irreducible over k have the same norm in k, we have the following algorithm to calculate $< D_1, D_2 >_\nu$:

We will consider r a factor of α_1 and s a factor of α_2.

We let

$$\varphi_r(x) := Resultant(r(t), x - \beta_1(t), t)$$

which has roots $\beta_1(z)$ for z a root of r,

$$\psi_s(x) := Resultant(s(t), x - \beta_2(t), t)$$

which has roots $\beta_2(z)$ for z a root of s, so that

$$\gamma_{r,s}(t) := Resultant(\varphi_r(x), \psi_s(x-t), x)$$

has roots $\beta_1(z_1) - \beta_2(z_2)$ for z_1 a root of r,z_2 a root of s.

From these we can deduce the following expression for $< D_1, D_2 >_\nu$:

$$
\begin{aligned}
&< D_1, D_2 >_\nu = \\
&\log \prod_{\substack{r|\alpha_1 \\ s|\alpha_2 \\ u|\gamma_{r,s}(t)}} \max\left[\min\left[|u(0)|_\nu, |Resultant(r(x), s(x), x)|_\nu\right], 1\right]^{ord_r(\alpha_1) \times ord_s(\alpha_2) \times ord_u(\gamma_{r,s})}
\end{aligned}
\tag{3}
$$

To obtain the corresponding global expressions in the class number one case, simply replace $| - |_\nu$ with the usual absolute value, replace 'min' with 'gcd' and 'max$[-, 0]$' with '*norm*'.

These functions have been implemented in magma.

2.2 Regularity

At finite places we have to consider singularities on the special fibre. Our current approach is to use linear equivalence to move (some positive multiples of) divisors away from points with singular reduction so that we can assume divisors only intersect at regular points. This is the approach attempted in the worked example.

However, there is another alternative, namely blowing up (as in the 'Tamagawa numbers' notes at http://math.scu.edu/ eschaefe/nt.html) and then intersecting on the resulting regular model. At a bad prime a horizontal divisor is perpendicular to the vertical fibers, so if the two divisors meet transversely at the non-regular point then it suffices to compute the intersection of each divisor with the exceptional locus. However, to do this in practice is time consuming, and to automate the process you first have to automate the blowups. Apparently Qing Liu is working on this in some cases, but I know no details.

The big advantage of this approach is that you may get away with a smaller field extention, and it is not necessary to find a (sometimes large) multiple of a divisor that is Cartier. However, we avoid this (for now) due to the difficulty of automating blowups.

3 Infinite intersections

See [Lan88] for the definitions of intersections at infinite places.

3.1 The pde we need to solve

We will make use of the following observation of B. Gross in 'Arithmetic Geometry' [CSA86]:

Fix an infinite place '∞' of k.

Given a pair of divisors $a, b = \sum_i n_i.b_i$ on C of degree zero with disjoint support, let ω be a differential form on C such that the residue divisor $res(\omega)$ equals a (such an ω can always be found using the Riemann-Roch theorem).

Normalise ω by adding on holomorphic forms until the periods of ω are purely imaginary.

Let

$$dg_a := \omega + \bar{\omega} \tag{4}$$

(this g_a is a Green's function for a). Then:

$$< a, b >_\infty = \frac{1}{2}. \sum_i n_i.g_a(b_i) \tag{5}$$

Thus it remains to find, normalise and integrate such an ω.

3.2 Other families of curves

So far in this section we have not used that C is hyperelliptic, and indeed the formulae given remain valid. Thus it is reasonable to consider applications to other families of curves. In all cases the finite intersections are relatively straightforward.

3.2.1 Smooth plane quartics ($g = 3$)

I have been unable to make this work, the problem being to normalise ω such that all periods are purely imaginary. I attempted a numerical approach but was unable to find a homology basis in even one example.

3.2.2 Modular curves

Here we have the advantage of a map from a fundamental domain in the upper half plane, so this can be made to work using numerical integration techniques, see section 4 for an example. The problem is to prove bounds on the errors of the numerical integrals involved.

3.3 Application of theta functions to the function theory of hyperelliptic curves

Now we return to the case of hyperelliptic curves, where we can use ϑ-functions to solve the pde (4) of section 3.1, in a very simple way. For background on ϑ-functions we refer to the first two books of the 'Tata lectures on theta' trilogy, [Mum83], [MM84]. ϑ-functions are complex analytic functions on $\mathbb{C}^g$ which satisfy some quasi-periodicity conditions, thus they are an excellent source of differential forms on the (analytic) Jacobian of C. To get from this a differential form on C we simply use that C is canonically embedded in $Jac^0(C)$ by the Abel-Jacobi map, so we can pull back forms from $Jac^0(C)$ to C. .

Fix a point ∞_C at infinity on C.

We let α denote the map from the group $Div(C)$ of divisors on C to $Jac^0(C)(\mathbb{C})$ thought of as a complex torus, sending a divisor D to the point corresponding to the equivalence class of $D - \deg(D).\infty_C$. Thus the restriction of α to effective divisors of degree one (points!) can be thought of as the Abel-Jacobi map.

Fix a symplectic homology basis A_i, B_i on the Riemann surface corresponding to $C(\mathbb{C})$ as in [MM84]; if $i(-,-)$ denotes the intersection of paths, then we require that the A_i, B_i form a basis of $H_1(C, \mathbb{Z})$ such that

$$i(A_i, A_j) = i(B_i, B_j) = 0 \text{ for } i \neq j$$

and

$$i(A_i, B_j) = \delta_{ij}$$

.

We also choose a normalised basis $\omega_1, \ldots \omega_g$ of holomorphic 1-forms on C, normalised such that

$$\int_{A_i} \omega_j = \delta_{ij}$$

We recall the definition and basic properties of the multivariate ϑ-function:

$$\vartheta(z;\Omega) = \sum_{\underline{n} \text{ in } \mathbb{Z}^g} \exp(\pi i \underline{n}\Omega\underline{n}^{\mathrm{T}} + 2\pi i \underline{n}.z) \tag{6}$$

which converges for z in $\mathbb{C}^g$ and Ω a $g \times g$ complex matrix with positive definite imaginary part.

ϑ satisfies the following periodicity conditions for $\underline{m}, \underline{n}$ in $\mathbb{Z}^g$:

$$\vartheta(z + \underline{m};\Omega) = \vartheta(z;\Omega). \tag{7}$$

$$\vartheta(z + \underline{n}.\Omega;\Omega) = \exp(-\pi i \underline{n}\Omega\underline{n}^{\mathrm{T}} - 2\pi i \underline{n} z)\,\vartheta(z;\Omega). \tag{8}$$

We will set Ω to be the period matrix of the analytic Jacobian of C with respect to the fixed symplectic homology basis (as in [MM84]), and z will be a coordinate on the analytic Jacobian. This means that

$$\Omega_{ij} = \int_{B_i} \omega_j.$$

Let

$$\delta' := (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}) \in \frac{1}{2}\mathbb{Z}^g$$

$$\delta'' := (\frac{g}{2}, \frac{g-1}{2}, \ldots, 1, \frac{1}{2}) \in \frac{1}{2}\mathbb{Z}^g$$

$$\Delta := \Omega.\delta' + \delta''.$$

Then part 1 of theorem 5.3 of [MM84] tells us that

$$\vartheta(\Delta - z) = 0 \Leftrightarrow \left[\exists P_1, \ldots P_{g-1} \in C \text{ such that } z = \sum_{i=1}^{g-1} \int_{\infty}^{P_i} \underline{\omega} \bmod \Omega \right]$$

.

This is a crucial result which allows us to construct a quasifunction on $Jac^0(C)(\mathbb{C})$ with prescribed zeros, and from this obtain the Green's function we seek.

3.4 Solution of the pde

Let D,D_0 be two reduced divisors of degree g on C with disjoint support, containing no branch points or points at infinity.

For z in $Jac^0(C)(\mathbb{C})$ we set

$$G(z) = \frac{\vartheta(z + \Delta - \alpha(D))}{\vartheta(z + \Delta - \alpha(D_0))}.$$

Then for p in $C(\mathbb{C})$ we set $F(p) = G(\alpha(p))$ so

$$F(p) = \frac{\vartheta(\alpha(p) + \Delta - \alpha(D))}{\vartheta(\alpha(p) + \Delta - \alpha(D_0))}. \tag{9}$$

If we let $\omega = d \log F(p)$ then it is clear that $res(\omega) = D - D_0$. It then remains to normalise ω to make it's periods purely imaginary, and then integrate it. We have a homology basis A_i, B_i, and we find:

$$\int_{A_k} \omega = \int_{A_k} d \log F(p) = \log G(\alpha(p) + e_k) - \log G(\alpha(p)) = 0$$

(where $e_k = (0,0,\dots 0, \underbrace{1}_{\text{in } k^{th} \text{ position}}, 0\dots,0))$, and

$$\begin{aligned}\int_{B_k}\omega = \int_{B_k} d\log F(p) &= \log G(\alpha(p)+\Omega.e_k) - \log G(\alpha(p)) \\ &= 2\pi i e_k^T.(\alpha(D)-\alpha(D_0)) = 2\pi i \underbrace{[\alpha(D)-\alpha(D_0)]_k}_{k^{th} \text{ component}}\end{aligned} \tag{10}$$

From this we can deduce that the normalisation is

$$\omega = d\log\left[\frac{\vartheta(\alpha(p)+\Delta-\alpha(D))}{\vartheta(\alpha(p)+\Delta-\alpha(D_0))}\right] - 2\pi i\left[(Im\Omega)^{-1}Im(\alpha(D)-\alpha(D_0))\right].\begin{bmatrix}\omega_1\\ \omega_2\\ \vdots\\ \omega_g\end{bmatrix} \tag{11}$$

where p is a coordinate on C.

Now we integrate to get the Green's function $g_{D-D_0}(p) = \int_{\infty_C}^p \omega + \overline{\omega}$:

$$g_{D-D_0}(p) = 2\log\left|\frac{\vartheta(\alpha(p)+\Delta-\alpha(D))}{\vartheta(\alpha(p)+\Delta-\alpha(D_0))}\right|$$

$$+ 4\pi i\left[(Im\Omega)^{-1}Im(\alpha(D)-\alpha(D_0))\right].Im\left(\int_{\infty_C}^p\begin{bmatrix}\omega_1\\ \omega_2\\ \vdots\\ \omega_g\end{bmatrix}\right)$$

$$= 2\log\left|\frac{\vartheta(\alpha(p)+\Delta-\alpha(D))}{\vartheta(\alpha(p)+\Delta-\alpha(D_0))}\right| + 4\pi i(Im\Omega)^{-1}.Im(\alpha(D)-\alpha(D_0)).Im\left(\alpha(p)\right) \tag{12}$$

Again, this has been implemented in magma, so given divisors D,D_0 and E,E_0 as above (ie reduced, containing no branch points or infinite points and having disjoint support) then we have

$$< D - D_0, E - E_0 >_\infty = g_{D-D_0}[E-E_0]$$

where $g_{D-D_0}[E-E_0]$ is interpreted as in section (1.1). So we are done.

3.5 A worked example

WARNING: This example is included to give an idea of how to apply the algorithm in practice. There are very many opportunities for errors in sign and normalisation, it is very likely I have made some of these. Hopefully in time this algorithm will be entirely automated.

We consider the genus 3 curve

$$y^2 = f_7(x) = x^7 + 12x^5 + 1. \tag{13}$$

This has good reduction away from the primes 2 and 447898423543, it has no rational 2-torsion, and has 2-Selmer group $\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$.

We notice the rational point $(0, 1)$, and a quick computation suggests it is not torsion in the Jacobian. We will confirm this by computing it's canonical height.

We set $D = (0, 1) - \infty$ (where ∞ is the unique point at infinity on the curve defined by $s = t = 0$ if s, t are coordinates chosen as usual such that $s.x = 1, t = s^4 y)$), see previous comment on degree. We wish to compute it's self intersection globally, so we start by moving it by linear equivalence, to minus the divisor $D' = (0, -1) - \frac{1}{2}(1, \pm\sqrt{f(1)})$, so D and $-D'$ are linearly equivalent and have no common support.

3.5.1 Finite good primes

Here the degree is so small, we don't really need the resultants described above. It is easy to see (using the other affine piece of a smooth model for C that

$$< D, D' >_v = 0$$

for all good finite primes, and also at 447898423543.

3.5.2 2

This calculation is only done in outline, we do not here compute the exact answer. We observe that the answer will be an integer multiple of $\frac{\log 2}{49}$.

Here we move by linear equivalence. The smallest multiple of D that is locally principal at the bad places over 2 is $7D$. We write $\underline{E}$ for the mumford representative of the divisor E. We let $D'_7 = V(36{*}x^3 - 1, y - \frac{x^2}{6} - 1) - \frac{3}{2}.V(x - 1)$. We find ψ, ψ' such that

$$div\ \psi = 7D - \underline{7D} \tag{14}$$

$$div\ \psi' = 7D' - D'_7 \tag{15}$$

so

$$\begin{aligned} 49 < D, D' >_2 =< 7D, 7D' >_2 =< div\ \psi + 7 * D, div\ \psi' + 7 * D' >_2 \\ =< div\ \psi, 7.D' >_2 + < div\ \psi', \underline{7D} >_2 + < \underline{7D}, D'_7 >_2 \end{aligned} \tag{16}$$

It turns out that

$$\psi = \frac{x^7}{y + 6x^5 + 1} \text{ (a magma calculation)}$$

so

$$< div\ \psi, 7.D' >_2 = 0,$$

another magma calculation using $< div\ F, E >_v = \log |F[E]|_v$.

Similarly,

$$\psi = \frac{x^5}{y - 6x^5 - 1}$$

so

$$< div\ \psi', \underline{7D} >_2 = \log 2.$$

Now we compute the final term using resultants in magma:

$$< \underline{7D}, D'_7 >_2 = 0$$

We add these up to get

$$< D, D' >_2 = \frac{\log 2}{49},$$

of the form we expected.

3.5.3 Infinite place

We are working over $\mathbb{Q}$, so there is only one infinite place, denoted $\infty_{\mathbb{Q}}$ (or just ∞ where this is unambiguous).

Our theta-function approach requires that we input reduced divisors of degree 3 with affine support etc. To obtain such, we note that $7D$ and $11D$ have mumford representations with the required properties, and we find ζ, ζ' such that

$$\begin{aligned} div\ \zeta &= 4D - (\underline{11D} - \underline{7D}) \\ div\ \zeta' &= 4D' - (\underline{11D'} - \underline{7D'}) \end{aligned}$$

.

We can use our magma implementation of infinite intersections to compute

$$< \underline{11D} - \underline{7D}, \underline{11D'} - \underline{7D'} >_{\infty_{\mathbb{C}}} = -1617.135082281702952\ldots$$

-this takes around 8 seconds, depending on your computer.

We find

$$\zeta = \frac{(432 * X - 1) * Y + 6 * X^5 - 31104 * X^4 + 432 * X - 1}{432 * X^7 - 12 * X^4},$$

unfortunately ζ' will not fit onto the width of the page, but can easily be computed. From these we can get

$$< div\ \zeta, D' >_\infty = 3.5708818899\ldots$$

and

$$< div\ \zeta', \underline{11D'} - \underline{7D'} >_\infty = 22.429172244\ldots$$

Thus

$$\begin{aligned} < D, D' >_\infty &= \frac{1}{16}\left[4 < div\ \zeta, D' >_\infty + < div\ \zeta', \underline{11D'} - \underline{7D'} >_\infty + < \underline{11D} - \underline{7D}, \underline{11D'} - \underline{7D'} >_{\infty_{\mathbb{Q}}}\right] \\ &= -406.320181741675\ldots \end{aligned}$$

3.5.4 Canonical height

We can thus obtain the global canonical height of the divisor D by summing local contributions, to get:

$$< D, D >= -406.320181741675\ldots + \frac{\log 2}{49} = -406.30603588084\ldots,$$

in particular D is not torsion.

It is interesting to note that in this case, the contribution to the height from finite places was relatively tiny. This fits with the small number of similar such calculations I have performed, but I know no good theoretical reason for it as yet.

4 Appendix A: A modular curve

Coming soon!

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