THE LOGARITHMIC PICARD GROUP AND ITS TROPICALIZATION

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Abstract. We construct the logarithmic and tropical Picard groups of a family of logarithmic curves and realize the latter as the quotient of the former by the algebraic Jacobian. We show that the logarithmic Jacobian is a proper family of logarithmic abelian varieties over the moduli space of Deligne–Mumford stable curves, but does not possess an underlying algebraic stack. However, the logarithmic Picard group does have logarithmic modifications that are representable by logarithmic schemes, all of which are obtained by pullback from subdivisions of the tropical Picard group.

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1. Introduction

Our concern in this paper is the extension of the universal Picard group to the boundary of the Deligne–Mumford moduli space of stable curves. Over the interior, the Picard group of a smooth, proper, connected curve is well-known to be an extension of the integers by a smooth, proper, connected, commutative group scheme, the Jacobian. These properties do not persist over the boundary, and natural variants sacrifice one or another of them to obtain others.

The Deligne–Mumford compactification of the moduli space of curves admits curves with nodal singularities. As long as the dual graph of the curve is a tree, the Picard group remains an extension of a discrete, free abelian group — the group of multidegrees — by an abelian variety, but it becomes nonseparated in families because the multidegrees do. One can focus here on the component of multidegree zero, which is an abelian variety and is well-behaved in families.

Should a curve degenerate so that its dual graph contains nontrivial loops, the multidegree 0 component of the Picard group remains separated, but fails to be universally closed. The construction of compactifications of this group is the subject of a vast literature [Ish78, D'S79, OS79, AK80, AK79, Kaj93, Cap94, Pan96, Jar00, Est01, Cap08a, Cap08b, Mel11, Chi15], of

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which the above references are only a sample. We must direct the reader to the references for a history of the subject.

While we do not attempt to summarize all of the different approaches to compactifying the Picard group, we emphasize that all operate in the category of schemes, and none produces a proper group scheme. Indeed, it is not possible to produce a proper group scheme, for the multidegree 0 component of the Picard group of a maximally degenerate curve is a torus, and there is no way of completing a torus to a proper group *scheme*.

On the other hand, K. Kato observed that the multiplicative group does have natural compactifications — with group structure — in the category of *logarithmic* schemes [Kat, Section 2.1]. This gives reason to hope that the Picard group might also find a natural compactification in the category of logarithmic schemes, as Kato himself anticipated. Kato proposed a definition for, and then calculated, the logarithmic Picard group of the Tate curve [Kat, Section 2.2.4]. Illusie advanced the natural generalization of Kato's calculation as a definition for the Picard group of an arbitrary logarithmic scheme [Ill94, Section 3.3].

In short, every logarithmic scheme X is equipped with an étale sheaf of groups M_X^{gp}, and Kato's and Illusie's logarithmic line bundles on X are torsors under this group. Kajiwara used this definition to construct toroidal compactifications of the Jacobian of a fixed curve. Olsson also proved that the logarithmic Picard group, as defined by Illusie, is representable on the category of *schemes* (as opposed to logarithmic schemes) by an algebraic stack [Ols04, Theorem 4.4].

In the same work, Illusie also proposed the study of logarithmic abelian varieties [Ill94, Section 3.3]. Kajiwara, Kato, and Nakayama have pursued this problem in a long program [KKN08c, KKN08b, KKN13, KKN15] and constructed a logarithmic Picard variety in the analytic category using Hodge-theoretic methods [KKN08a]. Significantly, they discovered the need to restrict attention to a subfunctor of the one defined by Illusie in order to get the logarithmic Picard group, and logarithmic abelian varieties in general, to vary well geometrically over logarithmic base schemes. In the present work, that condition appears under the heading of *bounded monodromy*, first introduced in Section 3.5 to play an essential role throughout.

In this paper, we define logarithmic line bundles on families of logarithmic curves as torsors under the logarithmic multiplicative group, as Kato and Illusie proposed, that satisfy the additional bounded monodromy condition. In Section 4.18 we explain why this condition is necessary if infinitesimal deformation of logarithmic line bundles is to have the expected relationship to formal families of logarithmic line bundles. However, even with this condition, the logarithmic Picard group is not representable by an algebraic space with a logarithmic structure, essentially because the logarithmic multiplicative group *itself* is not so representable (see Section 2.2.7). However, the logarithmic multiplicative group does have a logarithmically étale cover by a logarithmic scheme, and we prove that the logarithmic Picard group of a logarithmic curve is algebraic in the same sense:

Theorem A. *Let X be a proper, vertical logarithmic curve over S. The logarithmic Picard group* $\operatorname{Log}\operatorname{Pic}(X/S)$ *has a logarithmically smooth cover by a logarithmic scheme, is logarithmically smooth and proper, is a commutative group object, has finite diagonal, and contains* $\operatorname{Pic}^{[0]}(X/S)$ *as a subgroup.*

Proof. See Corollary 4.11.5 for the existence of a logarithmically smooth cover, Theorem 4.13.1 for the logarithmic smoothness, Corollary 4.12.5 for the properness, and Theorem 4.12.1 for the finiteness of the diagonal. The group structure and inclusion of $\mathrm{Pic}^{[0]}(X/S)$ are immediate from the construction in Definition 4.1. □

Corollary B. *The logarithmic Jacobian is a logarithmic abelian variety, in the sense of Kajiwara, Kato, and Nakayama* [KKN08c, KKN08b].

Proof. See Theorem 4.15.7. □

Our results for the logarithmic Picard *stack*, which remembers automorphisms, are similar, but a bit more technical:

Theorem C. *Let X be a proper, vertical logarithmic curve over S. The logarithmic Picard stack $\mathbf{Log\,Pic}(X/S)$ has a logarithmically smooth cover by a logarithmic scheme and its diagonal is representable by logarithmic spaces (sheaves with logarithmically smooth covers by logarithmic schemes). The logarithmic Picard stack is logarithmically smooth and proper, is a commutative group stack, and receives a canonical homomorphism from the algebraic stack $\mathbf{Pic}^{[0]}(X/S)$.*

Proof. See Theorem 4.11.2 for the existence of a logarithmically smooth cover and Corollary 4.11.6 for the claim about the diagonal. The logarithmic smoothness is proved in Theorem 4.13.1 and the properness is Corollary 4.12.5. The group structure and the map from $\mathrm{Pic}^{[0]}(X/S)$ are come directly from Definition 4.1. □

The difference between Theorems A and C and Olsson's result [Ols04, Theorem 4.4] is that Olsson works with a fixed logarithmic structure on the base while we allow the logarithmic structure to vary. This is necessary for the logarithmic Picard group to be proper. Our method of proof also differs from Olsson's: we do not rely on the Artin–Schlessinger representability criteria (for which there is not yet an analogue in logarithmic geometry) and instead construct logarithmically smooth covers directly.

Our proofs of the boundedness properties in Theorems A and C make use of the tropical Picard group, whose relationship to the logarithmic Picard group is the second theme of this paper. Indeed, Foster, Ranganathan, Talpo, and Ulirsch observed that the geometry of the logarithmic Picard group is intimately tied up with the geometry of the tropical Picard group [FRTU16]. The connection can also be seen in Kajiwara's work [Kaj93], albeit without explicit mention of tropical geometry.

A tropical curve is simply a metric graph. Baker and Norine introduced the tropical Jacobian as a quotient of tropical divisors by linear equivalence [BN07]. At first, the tropical Jacobian of a fixed graph (not yet metrized) was only a finite set, but subdivision of the graph suggests the presence of a finer geometric structure. This was explained by Gathmann and Kerber [GK08], who extended Baker and Norine's results to metric graphs, and Amini and Caporaso added a vertex weighting [AC13]. Mikhalkin and Zharkov defined tropical line bundles as torsors under a suitably defined sheaf of linear functions on a tropical curve [MZ08, Definition 4.5]. They gave a separate definition of the tropical Jacobian as a quotient of a vector space by a lattice [MZ08, Section 6.1], and proved an analogue of the Abel–Jacobi theorem, showing that the tropical Jacobian parameterizes tropical line bundles of degree 0. We will recover this result in Corollary 3.4.7.

In order to relate the tropical Picard group and tropical Jacobian to their logarithmic analogues, we require a formalism by which tropical data may vary over a logarithmic base scheme. This formalism is supplied by Cavalieri, Chan, Ulirsch, and the second author [CCUW17, Section 5], who allow an arbitrary partially ordered abelian group to stand in for the real numbers in the definition of a tropical curve as a metric graph. Logarithmic schemes come equipped with sheaves of partially ordered abelian groups and one can therefore speak of tropical curves over logarithmic base schemes. We summarize these ideas in Sections 2.3.1–2.3.3. In Section 3 we use them to define the tropical Jacobian, Picard group, and Picard stack over logarithmic bases.

The following theorem summarizes the relationship between the logarithmic Picard group and its tropical doppelgänger:

Theorem D. *Let X be a proper, vertical logarithmic curve over S and let $\mathfrak{X}$ be its tropicalization. There is an exact sequence:*

$$0 \to \operatorname{Pic}^{[0]}(X/S) \to \operatorname{Log}\operatorname{Pic}(X/S) \to \operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S) \to 0$$

There is also an exact sequence of group stacks:

$$0 \to \mathbf{Pic}^{[0]}(X/S) \to \mathbf{Log}\,\mathbf{Pic}(X/S) \to \mathbf{Tro}\,\mathbf{Pic}(\mathfrak{X}/S) \to 0$$

The tropicalization morphism $\operatorname{Log}\operatorname{Pic}(X/S) \to \operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S)$ allows us to to identify combinatorial data associated with $\operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S)$ necessary to construct proper, schematic compactifications of the Picard group, following Kajiwara, Kato, and Nakayama [Kaj93, KKN15], in Section 4.17.

Theorem E. *Let X be a proper, vertical logarithmic curve over S with tropicalization $\mathfrak{X}$. Polyhedral subdivisions of* $\operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S)$ *correspond to toroidal compactifications of* $\operatorname{Pic}^{[0]}(X/S)$.

Recent work of Abreu and Pacini describes polyhedral subdivisions of $\operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S)$ when $\mathfrak{X}$ is the universal curve over the moduli space of 1-pointed tropical curves (and, for certain degrees, over the moduli space of unpointed tropical curves) [AP18]. They show that the corresponding compactification of the Picard group coincides with Esteves's compactification [Est01].

Section 5 gives a few examples of the logarithmic and tropical Picard groups, and the reader may find it useful to consult these in parallel with the earlier sections.

The provenance of logarithmic geometry. This section is intended to motivate the presence of logarithmic geometry in the compactification of the Picard group. Consider a family of logarithmic curves X over a 1-parameter base S with generic point η and a line bundle L_η on the general fiber of X. Let $i : s \to S$ denote the inclusion of the closed point and also write $i : X_s \to X$ for the inclusion of the closed fiber; write $j : \eta \to S$ and $j : X_\eta \to X$ for the inclusion of the generic point and the generic fiber.

Let $\mathfrak{S}$ denote the ringed space $(s, i^{-1} j_* \mathcal{O}_\eta)$ and let $\mathfrak{X}$ denote the ringed space $(X_s, i^{-1} j_* \mathcal{O}_{X_\eta})$. Then $\mathfrak{L} = i^{-1} j_* L_\eta$ is a line bundle on $\mathfrak{X}$.

We can describe $\mathfrak{L}$ by giving local trivializations and transition functions in $\mathbf{G}_m$. However, these cannot necessarily be restricted to X_s because a unit of $i^{-1} j_* \mathcal{O}^*_{X_\eta}$ may have zeroes or poles along components of the special fiber.

If the dual graph of X_s is a tree then it is possible to modify the local trivializations to ensure that the transition functions have no zeroes or poles, but in general such a modification may not exist.

The degeneration of transition functions suggests we might compactify the Picard group by allowing 'line bundles' whose transition functions are sometimes allowed to vanish or have poles. If transition functions are thus permitted not to take values in a group then the objects assembled from them will no longer have a group structure. However, this leads naturally to the consideration of rank 1, torsion-free sheaves.

Logarithmic geometry takes a different approach to the same idea. Instead of keeping track of only the zeroes and poles of the transition functions, we instead keep track of their orders of vanishing and leading coefficients. Together, order of vanishing and leading coefficient have the structure of a group and therefore the objects glued with transition functions in this group can be organized into a group as well.

The way this is actually done is to take the image of a transition function $f \in i^{-1} j_* \mathcal{O}_{X_\eta}^*$, not in $\mathcal{O}_{X_s} \cup \{\infty\}$, but instead in $M_{X_s}^{\mathrm{gp}}$:

$$M_{X_s}^{\mathrm{gp}} = i^{-1} j_* \mathcal{O}_{X_\eta}^* / \ker(i^{-1}\mathcal{O}_X^* \to \mathcal{O}_{X_s}^*)$$

That is, we obtain a natural limit $M_{X_s}^{\mathrm{gp}}$-torsor P for L_η, whose isomorphism class lies in $H^1(X_s, M_{X_s}^{\mathrm{gp}})$.

Taking transition functions in M_{X_s} has an added benefit, even when the dual graph of the special fiber is a tree. Indeed, if L_η extends to L with limit L_s, one can always produce another limit $L(D)_s$ by twisting L_s by a component D of the special fiber. But the effect of twisting by D on $\mathfrak{L}$ is to modify the local trivializations of $\mathfrak{L}$ by units of $\mathcal{O}_{X_\eta}$. This changes the local trivializations of P by elements of $M_{X_s}^{\mathrm{gp}}$, but that only affects a cocycle representative by a coboundary. In other words, the class of P in $H^1(X_s, M_{X_s}^{\mathrm{gp}})$ is independent of twisting by components of X_s.

Future work. The Picard group (and even the Picard stack) is equipped with a canonical principal polarization. We are mute about the logarithmic analogue in this paper, but we will construct it in a subsequent one.

Our results are limited to relative dimension 1 because we do not have the means yet to study families of tropical varieties of higher dimension over logarithmic bases. We are developing methods towards that end in ongoing work with Gillam [GW].

Neither have we addressed any algebraicity properties of the tropical Picard group in a systematic way. It follows from our results that the tropical Picard group has a logarithmically étale cover by a Kato fan, but it is less clear how one should characterize its diagonal (we prove only that it is quasicompact here), or whether one should demand further properties of a purely tropical cover.

In Section 4.17, we indicate how the tropical Picard group can be used to construct proper schematic models of the logarithmic Picard group over a local base. We are pursuing a global construction over the moduli space of stable curves in collaboration with Melo, Ulirsch, and Viviani.

Conventions. Let X be a curve over S. We use the term 'Picard group' to refer to the sheaf on S of isomorphism classes of line bundles on X, up to isomorphism and denote it $\mathrm{Pic}(X/S)$. The stack of $\mathbf{G}_m$-torsors on X is denoted in boldface: $\mathbf{Pic}(X/S)$. We use a superscript to denote a restriction on degree, and we refer to $\mathrm{Pic}^0(X/S)$ as the Jacobian of X. We apply similar terminology when X is a logarithmic curve or tropical curve over a logarithmic base S.

Throughout, we consider a logarithmic curve X over S. We regularly use $\pi : X \to S$ to denote the projection.

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2. Monoids, logarithmic structures, and tropical geometry

2.1. Monoids.

In this paper, all monoids will be commutative, unital, integral, and saturated, although some results in this section are valid without those assumptions. The monoid operation will be notated additively, unless indicated otherwise. Homomorphisms of monoids are assumed to preserve the unit.

2.1.1. *Partially ordered groups.*

Definition 2.1.1.1. A homomorphism of monoids $f : M \to N$ is called *sharp* if each invertible element of N has a unique preimage under f. A monoid M is called *sharp* if the unique homomorphism $0 \to M$ is sharp.

We write M^* for the subgroup of invertible elements of M and $\overline{M}$ for the quotient M/M^*, which we call the *sharpening* of M. Even when they do not arise as sharpenings of other monoids, we often notate sharp monoids with a bar above them.

Remark 2.1.1.2. A homomorphism $f : M \to N$ of sharp monoids is sharp if and only if $f^{-1}\{0\} = \{0\}$. Note that f^{gp} need not necessarily be injective.

In this situation, sharp homomorphisms are analogous to local homomorphisms of local rings, and some authors prefer to call sharp homomorphisms between sharp monoids *local*. We will favor sharp in order not to create a conflict with connections to topology to be explored in [GW]. Some indications about those connections are given in Section 3.11.

Every monoid M is contained in a smallest associated group M^{gp}, and M determines a partial semiorder on M^{gp} in which M is the subset of elements that are ≥ 0. If M is sharp then the semiorder is a partial order. As M can be recovered from the induced partial order on M^{gp}, we are free to think of monoids as partially (semi)ordered groups, and we frequently shall.

2.1.2. *Valuative monoids.*

Definition 2.1.2.1. A *valuative monoid* is an integral[1] monoid M such that, for all $x \in M^{\mathrm{gp}}$, either $x \in M$ or $-x \in M$.

If M is an integral monoid, and $x, y \in M^{\mathrm{gp}}$, we say that $x \leq y$ if $y - x \in M$. We say that x and y are *comparable* if $x \leq y$ or $y \leq x$.

[1] Despite our convention that all monoids are saturated, we allow valuative monoids not to be saturated a priori, since they are so posteriori.

Lemma 2.1.2.2. *All valuative monoids are saturated.*

Proof. Suppose that M is valuative, $x \in M^{\mathrm{gp}}$, and $nx \in M$. If $x \notin M$ then $-x \in M$. But as $nx \in M$ this means $-x$ is a unit of M, which means that $x \in M$. □

Corollary 2.1.2.3. *All sharp valuative monoids are torsion free.*

Proof. A valuative monoid is integral and saturated, and a sharp, integral, saturated monoid is torsion free. □

Lemma 2.1.2.4. *Suppose that $f : M \to N$ is a sharp homomorphism of monoids and M is valuative. Then f is injective.*

Proof. Suppose that $x \in M^{\mathrm{gp}}$ and $f(x) = 0$. Either $x \in M$ or $-x \in M$. We assume the former without loss of generality. But $0 \in N$ has a unique preimage in M by sharpness, so $x = 0$ and f^{gp} is injective. □

Remark 2.1.2.5. This property is similar to one enjoyed by fields in commutative algebra.

Lemma 2.1.2.6. *Suppose that $f : N \to M$ is a sharp homomorphism of valuative monoids. Then f is an isomorphism if and only if it induces an isomorphism on associated groups.*

Proof. By Lemma 2.1.2.4, we know f is injective, so we replace N by its image and assume f is the inclusion of a submonoid with the same associated group. If $\alpha \in M$ then either α or $-\alpha$ is in N. In the former case we are done, and in the latter, α is an invertible element of M, so $\alpha \in N$ since the inclusion is sharp. □

Definition 2.1.2.7. A homomorphism of monoids $\varrho : N \to M$ is called *relatively valuative* or an *infinitesimal extension* if, whenever $\alpha \in N^{\mathrm{gp}}$ and $\varrho(\alpha) \in M$ either $\alpha \in N$ or $-\alpha \in N$.

Lemma 2.1.2.8. *If $\varrho : N \to M$ is relatively valuative and M is valuative then N is valuative.*

Proof. Suppose that $\alpha \in N^{\mathrm{gp}}$. Either $\varrho(\alpha) \in M$ or $-\varrho(\alpha) \in M$. In either case, either α or $-\alpha$ is in N, by definition. □

Lemma 2.1.2.9. *Any partial order on an abelian group can be extended to a total order.*

Proof. By Zorn's lemma, every partial order on an abelian group has a maximal extension. Assume therefore that $\overline{M}^{\mathrm{gp}}$ is a maximal partially ordered abelian group and let $\overline{M} \subset \overline{M}^{\mathrm{gp}}$ be the submonoid of elements ≥ 0. Let x be an element of $\overline{M}^{\mathrm{gp}}$ that is not in $\overline{M}$. Then $\overline{M}[x]^{\mathrm{sat}}$ is the monoid of elements ≥ 0 in a semiorder on $\overline{M}^{\mathrm{gp}}$ strictly extending the one corresponding to $\overline{M}$. This semiorder cannot be a partial order because $\overline{M}$ was maximal, so $\overline{M}[x]^{\mathrm{sat}}$ cannot be sharp. Therefore there is some $y, z \in \overline{M}$ and some positive integer n and m such that $(y + nx) + (z + mx) = 0$. That is $y + z = -(n + m)x$. As $\overline{M}$ is saturated (by its maximality), this implies that $-x \in \overline{M}$, which shows that every $x \in \overline{M}^{\mathrm{gp}}$ is either ≥ 0 or ≤ 0. □

2.1.3. *Bounded elements of monoids.*

Definition 2.1.3.1. Suppose that α and δ are elements of a partially ordered abelian group, with $\delta \geq 0$. We will say that α is *bounded* by δ if there are integers m and n such that $m\delta \leq \alpha \leq n\delta$. We write $\alpha \prec \delta$ to indicate that α is bounded by δ.

We say that α is *dominated* by δ, and write $\alpha \ll \delta$, if $n\alpha \leq \delta$ for all integers n.

Lemma 2.1.3.2. *Let M be a saturated monoid, let $\delta \in M$, and let $\alpha \in M^{\mathrm{gp}}$. Then $\alpha \prec \delta$ in M if and only if $\alpha \prec \delta$ in $\mathbf{Q}M$.*

Proof. If $m\delta \leq \alpha \leq n\delta$ in $\mathbf{Q}M$ then there is a positive integer k such that $k(\alpha - m\delta)$ and $k(n\delta - \alpha)$ are both in M. But M is saturated, so this implies $m\delta \leq \alpha \leq n\delta$, as required. □

Lemma 2.1.3.3. *Let M be a monoid. Suppose $\delta \in M$. The elements of M^{gp} that are bounded by δ is precisely $M[-\delta]^*$.*

Proof. If $k\delta \leq \alpha \leq \ell\delta$ then $0 \leq \alpha \leq 0$ in the sharpening $\overline{M[-\delta]}$ of $M[-\delta]$ and therefore $\alpha \in M[-\delta]^*$. Conversely, if $\alpha \in M^{\mathrm{gp}}$ is a unit of $M[-\delta]$ then there is some $\beta \in M$ such that $\alpha + \beta \in \mathbf{Z}\delta$ — in other words, $\alpha \leq \ell\delta$ for some integer ℓ. Applying the same reasoning to $-\alpha$ supplies an integer k such that $-\alpha \leq k\delta \in M$. Therefore $-k\delta \leq \alpha \leq \ell\delta$, as required. □

Definition 2.1.3.4. An *archimedean group* is a totally ordered abelian group M^{gp} such that if $x, y \in M^{\mathrm{gp}}$ with $x > 0$ then y is bounded by x.

Remark 2.1.3.5. A totally ordered abelian group M^{gp} is archimedean if and only M it has no $\prec$-closed submonoids other than 0 and M^{gp}.

Theorem 2.1.3.6 (Hölder)**.** *Every archimedean group can be embedded by an order preserving homomorphism into the real numbers. The homomorphism is unique up to scaling.*

Proof. This is trivial for the zero group, so assume M is a nonzero archimedean group. Choose a nonzero element x of M. It will be equivalent to show that there is a unique order-preserving homomorphism $M \to \mathbf{R}$ sending x to 1.

For any $y \in M$, let S be the set of rational numbers p/q such that $px \leq qy$ in M. Let T be the set of rationals p/q such that $px \geq qy$. Then S and T are a Dedekind cut of $\mathbf{Q}$, hence define a unique real number $f(y)$. This proves the uniqueness part.

All that remains is to show that f is a homomorphism. This amounts to the assertion that if $px \leq qy$ and $p'x \leq q'y'$ then $(pq' + p'q)x \leq qq'(y + y')$, which is an immediate verification. □

Lemma 2.1.3.7. *If x and y are positive elements of a totally ordered abelian group then $x \prec y$ or $y \ll x$.*

Proof. Suppose that y does not bound x. As $x \geq 0$, this means there is no integer such that $x \leq ny$. But the group is totally ordered, so we must therefore have $x \geq ny$ for all n. That is $x \gg y$. □

Proposition 2.1.3.8. *Let M be a valuative monoid. The collection of subsets N of M closed under $\prec$ are submonoids and are totally ordered by inclusion. The graded pieces of this filtration are archimedean.*

Proof. Lemma 2.1.3.3 implies that these subsets are submonoids. Suppose that N and P are $\prec$-closed subgroups and there is some $x \in N$ that is not contained in P. If $y \in P$ then either $y \prec x$ or $x \prec y$ by Lemma 2.1.3.7, but P is $\prec$-closed so it must be the former. Thus $P \prec x$ so $P \subset N$ since N is $\prec$-closed.

Now suppose that $N \subset P$ and there are no intermediate $\prec$-closed submonoids. The image of P in $P^{\mathrm{gp}}/N^{\mathrm{gp}}$ therefore has no $\prec$-closed submonoids other than 0 and itself, so it is archimedean. □

2.2. **Logarithmic structures.** We recall some of the basics of logarithmic geometry. The canonical reference is Kato's original paper [Kat89].

2.2.1. *Systems of invertible sheaves.* We recall a perspective on logarithmic structures favored by Borne and Vistoli [BV12, Definition 3.1].

Definition 2.2.1.1. Let X be a scheme. A *logarithmic structure* on X is an integral, saturated étale sheaf of monoids M_X on X and a sharp homomorphism $\varepsilon : M_X \to \mathcal{O}_X$, the target given its multiplicative monoid structure. The quotient $M_X/\varepsilon^{-1}\mathcal{O}_X^*$ is known as the *characteristic monoid* of M_X and is denoted $\overline{M}_X$.

A morphism of logarithmic structures $M_X \to N_X$ is a homomorphism of monoids commuting with the homomorphisms ε.

Let X be a logarithmic scheme. For each local section α of $\overline{M}_X^{\mathrm{gp}}$, we denote the fiber of M_X over $\overline{M}_X$ by $\mathcal{O}_X^*(-\alpha)$. This is a $\mathcal{O}_X^*$-torsor because M_X^{gp} is an $\mathcal{O}_X^*$-torsor over $\overline{M}_X^{\mathrm{gp}}$. We write $\mathcal{O}_X(-\alpha)$ for the associated invertible sheaf, obtained by contracting $\mathcal{O}_X^*(-\alpha)$ with $\mathcal{O}_X$ using the action of $\mathcal{O}_X^*$.

We can think of the assignment $\alpha \mapsto \mathcal{O}_X(-\alpha)$ as a map $\overline{M}_X^{\mathrm{gp}} \to \mathrm{B}\mathbf{G}_m$. We have canonical isomorphisms $\mathcal{O}_X(\alpha+\beta) \simeq \mathcal{O}_X(\alpha) \otimes \mathcal{O}_X(\beta)$ making the morphism $\overline{M}_X^{\mathrm{gp}} \to \mathrm{B}\mathbf{G}_m$ into a homomorphism of group stacks.

Moreover, if $\alpha \in \overline{M}_X$ then the restriction of ε gives a $\mathcal{O}_X^*$-equivariant map $\mathcal{O}_X^*(-\alpha) \to \mathcal{O}_X$, hence a morphism of invertible sheaves $\mathcal{O}_X(-\alpha) \to \mathcal{O}_X$. If $\beta \geq \alpha$ then $\alpha - \beta \leq 0$ and we obtain $\mathcal{O}_X(\alpha-\beta) \to \mathcal{O}_X$; twisting by $\mathcal{O}_X(\beta)$, we get $\mathcal{O}_X(\alpha) \to \mathcal{O}_X(\beta)$.

If we regard $\overline{M}_X^{\mathrm{gp}}$ as a sheaf of categories over X, with a unique morphism $\alpha \to \beta$ whenever $\alpha \leq \beta$, then the logarithmic structure induces a monoidal functor $\overline{M}_X^{\mathrm{gp}} \to \mathfrak{L}_X$ where $\mathfrak{L}_X$ is the stack of invertible sheaves on X. It is clearly possible to recover the original logarithmic structure on X from this monoidal functor, so we often think of logarithmic strutures in these terms.

2.2.2. *Coherent logarithmic structures.* Let X be a scheme with a logarithmic structure M_X. If $\overline{N}$ is an integral, saturated monoid and $e : \overline{N} \to \Gamma(X, \overline{M}_X)$ is a homomorphism, there is an initial logarithmic structure M'_X and morphism $M'_X \to M_X$ such that e factors through $\Gamma(X, \overline{M}'_X) \to \Gamma(X, \overline{M}_X)$. If $M'_X \to M_X$ is an isomorphism then $\overline{N}$ and e are called a chart of $\overline{M}_X$.

Definition 2.2.2.1. A logarithmic scheme is a scheme equipped with a logarithmic structure that has étale-local charts by integral, saturated monoids. It is said to be *locally of finite type* if the underlying scheme is locally of finite type and the charts can be chosen to come from finitely generated monoids.

A logarithmic scheme that is locally of finite type comes equipped with a stratification, defined as follows. Assume that M_X has a global chart by a finitely generated monoid $\overline{N}$. For each of the finitely many generators α of $\overline{N}$, the image of the homomorphism $\mathcal{O}_X(-\alpha) \to \mathcal{O}_X$ is an ideal, which determines a closed subset of X. All combinations of intersections and complements of these closed subsets stratify X.

To patch this construction into a global one, we must argue that the stratification defined above does not depend on the choice of chart. To see this, it is sufficient to work locally, and therefore to assume X is the spectrum of a henselian local ring with closed point x. Then the strata correspond to the ideals of the characteristic monoid $\overline{M}_{X,x}$.

On each stratum, the characteristic monoid of X is locally constant.

Definition 2.2.2.2. A logarithmic scheme X of finite type is called *atomic* if $\Gamma(X, \overline{M}_X) \to \overline{M}_{X,x}$ is a bijection for all geometric points of the closed stratum and the closed stratum is connected.

Lemma 2.2.2.3. *The closed stratum of an atomic logarithmic scheme X is connected and $\overline{M}_X$ is constant on it.*

Proof. Assume that X is an atomic logarithmic scheme. It is immediate that $\overline{M}_X$ is constant on the closed stratum, for we have a global isomorphism to a constant sheaf there, by definition. $\square$

Proposition 2.2.2.4. *Suppose that X is a logarithmic scheme of finite type. Then X has an étale cover by atomic logarithmic schemes.*

Proof. For each point geometric x of X, choose an étale neighborhood U of x such that $\Gamma(U, \overline{M}_X) \to \overline{M}_{X,x}$ admits a section. This is possible because $\overline{M}_{X,x}$ is finitely generated (because of the existence of charts by finitely generated monoids), hence finitely presented by Rédei's theorem [Gri17, Proposition 9.2]. As $\Gamma(U, \overline{M}_X)$ is finitely generated, $\Gamma(U, \overline{M}_X^{\mathrm{gp}})$ is a finitely generated abelian group, and therefore the kernel of (2.2.2.4.1)

$$\Gamma(U, \overline{M}_X^{\mathrm{gp}}) \to \overline{M}_{X,x}^{\mathrm{gp}} \tag{2.2.2.4.1}$$

is a finitely generated abelian group. By shrinking U, we can therefore ensure it is an isomorphism. Finally, we delete any closed strata of U other than the one containing x. $\square$

2.2.3. *Finite type and finite presentation.* Because we admit logarithmic structures whose underlying monoids are not locally finitely generated, we must adapt the definitions of finite type and finite presentation.

Definition 2.2.3.1. A morphism of logarithmic schemes $f : X \to Y$ is said to be *locally of finite type* if, locally in X and Y, it is possible to construct X relative to Y by adjoining finitely many elements to $\mathcal{O}_Y$ and M_Y, imposing some relations, and then passing to the associated saturated logarithmic structure. It is said to be *locally of finite presentation* if the relations can also be taken to be finite in number.

We say X is of *finite type* over Y if, in addition to being locally of finite type over Y, it is quasicompact over Y. For *finite presentation*, we require local finite presentation, quasicompactness, and quasiseparatedness.

Lemma 2.2.3.2. *(1) A logarithmic scheme of finite type is of finite presentation.*
(2) A logarithmic scheme of finite type over a logarithmic scheme of finite type is itself of finite type.

Remark 2.2.3.3. Because we insist on saturated monoids, some unexpected phenomena can occur when working over bases that are not finitely generated. For example, let Y be a punctual logarithmic scheme whose characteristic monoid is the submonoid of $\mathbf{R}^2_{\geq 0}$ consisting of all (a, b) such that $a+b \in \mathbf{Z}$. Let X be the logarithmic scheme obtained from Y by adjoining $(1, -1)$ to the characteristic monoid. This can be effected by adjoining an element γ to M_Y and imposing the relation $\beta\gamma = \alpha$, where α and β are elements of M_Y whose images in $\overline{M}_Y$ are $(1, 0)$ and $(0, 1)$, respectively. This requires adjoining $\varepsilon(\gamma)$ to $\mathcal{O}_Y$. In the category of not-necessarily-saturated logarithmic schemes, this would suffice to construct X with underlying scheme $\mathbf{A}^1 \times Y$.

The monoid $\overline{M}_Y[(1,-1)]$ is not saturated, and the saturation $\overline{M}_X$ involves the adjunction of infinitely many additional elements. Each of these elements requires an image in $\mathcal{O}_X$, and therefore neither the characteristic monoid nor the underlying scheme of X — when working with saturated logarithmic schemes — is finitely generated over Y. However, as an *saturated logarithmic scheme*, X is finitely generated over Y and therfore deserves to be characterized as of finite type.

For further evidence that X should be considered of finite type over Y, consider that Y admits a morphism to Y' that is an isomorphism on underlying schemes and such that $\overline{M}_{Y'} = \mathbf{N}^2 \subset \mathbf{R}^2$. Then X is the base change of X', which is representable by a logarithic structure $\mathbf{A}^1 \times Y$ (the construction from the first paragraph produces a saturated logarithmic structure when executed over Y'). The morphism $X' \to Y'$ must certainly be considered of finite type. If finite type is to be a property stable under base change to logarithmic schemes whose logarithmic structures are not necessarily locally finitely generated then we must admit that $X \to Y$ be of finite type as well.

Lemma 2.2.3.4. *A morphism of logarithmic schemes $f : X \to Y$ is locally of finite presentation if and only if, for every cofiltered system of affine logarithmic schemes S_i over Y, the map (2.2.3.4.1) is a bijection:*

$$\varinjlim \operatorname{Hom}_Y(S_i, X) \to \operatorname{Hom}_S(\varprojlim S_i, X) \tag{2.2.3.4.1}$$

Proof. First we prove that local finite presentation guarantees that (2.2.3.4.1) is a bijection. We demonstrate only the surjectivity, with the injectivity being similar. Let $S = \varprojlim S_i$ and let $f : S \to X$ be a Y-morphism. Choose covers of X and Y by U_k and V_k such that U_k can be presented with finitely many data and finitely many relations relative to V_k. Since S is affine, it is quasicompact, so finitely many of the U_k suffice to cover the image of S. Let $\{W_k\}$ be a cover S by open affines such that $W_k \subset f^{-1}U_k$ (repeat some of the U_k if $f^{-1}U_k$ is not affine).

The open sets W_k are pulled back from open sets $W_{ik} \subset S_i$ for i sufficiently large, and $S_i = \bigcup W_{ik}$ for i potentially larger. Since U_k can be presented with finitely many data and finitely many relations, the V_k-map $W_k \to U_k$ descends to $W_{ik} \to U_k$ for i sufficiently large. The maps W_{ik} and $W_{i\ell}$ may not agree on their common domain of definition, but we can cover it with finitely many affines (since S_i is quasiseparated) and therefore arrange for agreement when i is sufficiently large. This descends f to S_i.

Now we consider the converse. Working locally in Y and in X, we can assume that X may be presented over Y by finitely many data and finitely many relations. We argue first that $\mathcal{O}_X$ and M_X are generated, up to saturation, relative to $\mathcal{O}_Y$ and M_Y by finitely many elements. Indeed, we can write the pair $(\mathcal{O}_X, M_X)$ as a union of finitely generated sub-logarithmic structures $(\mathcal{O}_{S_i}, M_{S_i})$. These correspond to maps $S_i \to Y$ and their limit is $X \to Y$. By (2.2.3.4.1), $S_i \to Y$ lifts to X for all sufficiently large i and therefore $(\mathcal{O}_X, M_X) = (\mathcal{O}_{S_i}, M_{S_i})$ for all sufficiently large i. This proves that X is locally of finite type over Y.

Now we check that X is locally of finite presentation over Y. Let $(\mathcal{O}_{S_0}, M_{S_0})$ be freely generated over $(\mathcal{O}_Y, M_Y)$ by the finitely many generators of $(\mathcal{O}_X, M_X)$. Every finite subset of the relations among those generators determines a quotient $(\mathcal{O}_{S_i}, M_{S_i})$ and a map $S_i \to Y$. For all sufficiently large i, we get a lift to X by (2.2.3.4.1), which means that $S_i = X$ for all sufficiently large i. This completes the proof. □

2.2.4. *Universal surjectivity.*

Definition 2.2.4.1. Let S be a logarithmic scheme. By a *valuative geometric point* of S we will mean a point of S valued in a logarithmic scheme whose underlying scheme is the spectrum of an algebraically closed field and whose characteristic monoid is valuative.

Proposition 2.2.4.2 (Gillam)**.** *A morphism of logarithmic schemes is universally surjective if and only if it is surjective on valuative geometric points.*

Proof. Certainly if f is universally surjective then it is surjective on valuative geometric points. Suppose that $f : X \to Y$ is surjective on valuative geometric points. Then this is also true universally, so it is sufficient to prove that f is surjective and therefore to assume Y is the spectrum of an algebraically closed field. But any monoid can be embedded in a valuative monoid by Lemma 2.1.2.9, so after embedding $\overline{M}_Y$ in a valuative monoid $\overline{N}$ we can construct a morphism $Y' \to Y$, with $\overline{M}_{Y'} = \overline{N}$ valuative, that is an isomorphism on the underlying schemes. Then $X' = X \times_Y Y'$ surjects onto Y' by assumption. As $Y' \to Y$ is surjective, this implies that $X \to Y$ is surjective, as required. $\square$

2.2.5. *Valuative criteria.*

Lemma 2.2.5.1. *Let S be the spectrum of a valuation ring with generic point η and assume that M_η is a logarithmic structure on η. Then there is a maximal logarithmic structure M on S extending M_η such that $M^{\mathrm{gp}} = M_\eta^{\mathrm{gp}}$. The map $\varrho : M \to M_\eta$ is relatively valuative.*

Proof. Let $\varepsilon : M_\eta \to \mathcal{O}_\eta$ be the structure morphism of M_η. Define $M = \varepsilon^{-1}\mathcal{O}_S$.

Note that ε restricts to a bijection on $\varepsilon^{-1}\mathcal{O}_\eta^*$, so it also restricts to a bijection on $\varepsilon^{-1}\mathcal{O}_S^*$. Therefore $\varepsilon : M \to \mathcal{O}_S$ is a logarithmic structure. In fact, it is the direct image logarithmic structure defined more generally by Kato [Kat89, (1.4)].

The maximality of M is the universal property of the direct image logarithmic structure, which we verify explicitly. If M' also extends M_η then we have a commutative diagram

$$\begin{tikzcd} M' \arrow[r] \arrow[d] & M'_\eta \arrow[d] \\ \mathcal{O}_S \arrow[r] & \mathcal{O}_\eta \end{tikzcd}$$

from which we obtain $M' \to M$ by the universal property of the fiber product.

We argue $M \to M_\eta$ is relatively valuative. Suppose $\alpha \in M^{\mathrm{gp}}$ and $\varrho(\alpha) \in M_\eta$. As $\mathcal{O}_S$ is a valuation ring, either $\varepsilon(\varrho(\alpha)) \in \mathcal{O}_S$ or $\varepsilon(\varrho(\alpha)) \in \mathcal{O}_\eta^*$ and $\varepsilon(\varrho(\alpha))^{-1} \in \mathcal{O}_S$. In the first case $\alpha \in \varepsilon^{-1}\mathcal{O}_S$ and in the latter case, $\varrho(-\alpha) \in M$ and $-\alpha \in \varepsilon^{-1}\mathcal{O}_S$ so $-\alpha \in M$. $\square$

Theorem 2.2.5.2. *The morphism of schemes underlying a morphism of logarithmic schemes $X \to Y$ satisfies the valuative criterion for properness if and only if it has the unique right lifting property with respect to inclusions $S \subset \overline{S}$ where $\overline{S}$ is the spectrum of a valuation ring, S is its generic point, S has a valuative logarithmic structure M_S, and the logarithmic structure of $\overline{S}$ is the maximal extension of M_S.*

Proof. Let $S = \operatorname{Spec} K$ and $\overline{S} = \operatorname{Spec} R$, and let $j : S \to \overline{S}$ be the inclusion. Let M_K be a logarithmic structure on S and let M_R be the maximal logarithmic structure extending M_K to R. Let M'_K be a valuative logarithmic structure on K extending M_K and contained in M_K^{gp} (whose existence is guaranteed by Lemma 2.1.2.9), and let M'_R be its maximal extension to R, which is valuative. We consider a lifting problem (2.2.5.2.1) with M_S pulled back from X:

(2.2.5.2.1)
$$\begin{tikzcd} S' \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & X \arrow[d] \\ \overline{S}' \arrow[r] \arrow[urr, dashed, "f"] & \overline{S} \arrow[r, "h"] \arrow[ur, dashed, "g"'] & Y \end{tikzcd}$$

Note that the valuative criterion for properness for the underlying schemes of X over Y is equivalent to the existence of a unique arrow lifting the square on the right and that the assertion of the theorem is therefore that lifts of the square on the right are in bijection with lifts of the outer rectangle. Let us assume f has been specified and show that there is a unique choice of g.

We draw the maps of monoids and rings implied by (2.2.5.2.1):

$$\begin{tikzcd} K & j_* M'_K \arrow[l] & j_* M_K \arrow[l] & f^* M_X \arrow[l] \arrow[dll] \arrow[dl, dashed] \\ R \arrow[u] \arrow[r, phantom, "A"] & M'_R \arrow[l] \arrow[u] \arrow[r, phantom, "B"] & M_R \arrow[l] \arrow[u] & h^* M_Y \arrow[l] \arrow[u] \end{tikzcd}$$

By definition of the maximal extension of a logarithmic structure, the rectangles A and $A \cup B$ are cartesian. Therefore B is cartesian and we get a unique dashed arrow by the universal property of fiber product. $\square$

2.2.6. *Logarithmic modifications and root stacks.*

Definition 2.2.6.1. Let X be a logarithmic scheme. A *logarithmic modification* is a morphism $Y \to X$ that is, locally in X, the base change of a toric modification of toric varieties.

More generally, we say that a morphism of presheaves $G \to F$ on the category of logarithmic schemes is a logarithmic modification if, for every logarithmic scheme X and morphism $X \to F$, the base change $X \times_F G \to X$ is a logarithmic modification.

Let X be a logarithmic scheme and let γ and δ be two sections of $\overline{M}_X^{\mathrm{gp}}$. We say that γ and δ are *locally comparable* on X if, for each geometric point x of X, we have $\gamma \leq \delta$ or $\delta \leq \gamma$ at x — that is, $\delta - \gamma \in \overline{M}_{X,x}$ or $\gamma - \delta \in \overline{M}_{X,x}$.

Given X, γ, and δ, as above, but not necessarily locally comparable, the property of local comparability defines a subfunctor of the one represented by X. That is, we can make the following definition:

$$Y(W) = \{ f : W \to X \mid f^*\gamma \text{ and } f^*\delta \text{ are locally comparable} \}$$

Then Y is representable by a logarithmic modification of X. Indeed, locally in X, we can find a morphism $X \to \mathbf{A}^2$, with the target given its standard logarithmic structure. Then Y is the pullback of the blowup of $\mathbf{A}^2$ at the origin.

Definition 2.2.6.2. Let X be a logarithmic scheme and let $\overline{N}$ be a locally finitely generated extension of $\overline{M}_X$ such that $\overline{N}^{\mathrm{gp}} / \overline{M}_X^{\mathrm{gp}}$ is torsion (a *Kummer morphism*). Define Y to be the following subfunctor of the one represented by X:

$$Y(U) = \{ f : W \to X \mid f^* \overline{M}_X \to \overline{M}_W \text{ factors through } f^* \overline{N} \}$$

An algebraic stack with a logarithmic structure that represents Y is called the *root stack* of X along $\overline{N}$.

2.2.7. *The logarithmic multiplicative group.*

Definition 2.2.7.1. Define functors $\mathbf{LogSch}^{\mathrm{op}} \to \mathbf{Sets}$ by the following formulas:

$$\mathbf{G}_m^{\log}(S) = \Gamma(S, M_S^{\mathrm{gp}})$$
$$\overline{\mathbf{G}}_m^{\log}(S) = \Gamma(S, \overline{M}_S^{\mathrm{gp}})$$

We call the first of these the *logarithmic multiplicative group.*

Proposition 2.2.7.2. *Neither $\mathbf{G}_m^{\log}$ nor $\overline{\mathbf{G}}_m^{\log}$ is representable by an algebraic stack with a logarithmic structure.*

Proof. We will treat $\mathbf{G}_m^{\log}$. The argument is essentially the same with $\overline{\mathbf{G}}_m^{\log}$.

Suppose that there is an algebraic stack X with a logarithmic structure representing $\mathbf{G}_m^{\log}$. Let S_0 be the spectrum of a field k, equipped with a logarithmic structure $k^* \times (\mathbf{N}e_1 + \mathbf{N}e_2)$. The element e_2 gives a map $f : S_0 \to X$, hence $f^*\overline{M}_X \to \overline{M}_{S_0}$.

Now, for each $t \in \mathbf{Z}$, let S_t have the same underlying scheme as S_0, with the logarithmic structure $k^* \times (\mathbf{N}e_1 + \mathbf{N}(e_2 + te_1))$. Then $M_{S_t}^{\mathrm{gp}} = M_{S_0}^{\mathrm{gp}}$ for all t, so the map $S_0 \to X$ factors through $S_0 \subset S_t$ for all $t \geq 0$. Therefore the map $f^*\overline{M}_X \to \overline{M}_{S_0}$ factors through $\overline{M}_{S_t}$ for all $t \geq 0$. Thus $\overline{M}_X \to \overline{M}_{S_0}$ factors throught $\bigcap_t \overline{M}_{S_t} = \mathbf{N}e_1$. But the element $e_2 \in \Gamma(S_0, M_{S_0}^{\mathrm{gp}})$ is clearly not induced from an element of $\mathbf{Z}e_1$. $\square$

Lemma 2.2.7.3. *Let $\mathbf{P}$ be the subfunctor of $\mathbf{G}_m^{\log}$ whose S points consist of those $\alpha \in \mathbf{G}_m^{\log}(S)$ that are locally (in S) comparable to 0.*

(1) $\mathbf{P}$ is isomorphic to $\mathbf{P}^1$ with its toric logarithmic structure.
(2) $\mathbf{P}$ is a logarithmic modification of $\mathbf{G}_m^{\log}$.

Proof. Note that the logarithmic structure $M_{\mathbf{A}^2}$ has two tautological sections, α and β, coming from the two projections to $\mathbf{A}^1$. The difference of these sections determines a map $\mathbf{A}^2 \to \mathbf{G}_m^{\log}$. The open subset $\mathbf{A}^2 - \{0\}$ may be presented as the union of the loci where $\alpha \geq 0$ and where $\beta \geq 0$, which coincide, respectively, with the loci where $\alpha - \beta \leq 0$ or $\alpha - \beta \geq 0$. We note that adjusting α and β simultaneously by the same unit leaves $\alpha - \beta$ unchanged, so that we have constructed a map $\mathbf{P}^1 \to \mathbf{P}$.

To see that this is an isomorphism, consider the open subfunctors of $\mathbf{P}$ where $\alpha \geq 0$ and where $\alpha \leq 0$. These are each isomorphic to $\mathbf{A}^1$ and their preimages under $\mathbf{P}^1 \to \mathbf{P}$ are the two standard charts of $\mathbf{P}^1$.

Finally, we verify that $\mathbf{P} \to \mathbf{G}_m^{\log}$ is a logarithmic modification. We need to show that if Z is a logarithmic scheme and $Z \to \mathbf{G}_m^{\log}$ is any morphism then $Z \times_{\mathbf{G}_m^{\log}} \mathbf{P} \to Z$ is a logarithmic modification. This is a local assertion in Z, and section in M_Z^{gp} is locally pulled back from a logarithmic map to an affine toric variety, so we can assume Z is an affine toric variety with cone σ.

Let $\overline{\alpha}$ be the image of α in $\overline{M}_Z^{\mathrm{gp}}$. We can regard sections of $\overline{M}_Z^{\mathrm{gp}}$ as linear functions with integer slope on the ambient vector space of σ. Then $Z \times_{\mathbf{G}_m^{\log}} \mathbf{P}$ is representable by the subdivision of σ along the hyperplane where $\overline{\alpha}$ vanishes. $\square$

Corollary 2.2.7.4. *Both $\mathbf{G}_m^{\log}$ and $\overline{\mathbf{G}}_m^{\log}$ have logarithmically smooth covers by logarithmic schemes.*

Proof. We have just seen that $\mathbf{G}_m^{\log}$ has a logarithmically étale cover by $\mathbf{P}^1$, and therefore $\overline{\mathbf{G}}_m^{\log} = \mathbf{G}_m^{\log}/\mathbf{G}_m$ has a logarithmically étale cover by $[\mathbf{P}^1/\mathbf{G}_m]$. $\square$

Proposition 2.2.7.5. *The inclusion of the origin in $\overline{\mathbf{G}}_m^{\log}$ is representable by affine logarithmic schemes of finite type.*

Proof. Suppose that S is a logarithmic scheme and $S \to \overline{\mathbf{G}}_m^{\log}$ is a morphism corresponding to a section $\alpha \in \Gamma(S, \overline{M}_S^{\mathrm{gp}})$. Let $\overline{N}$ be the image of $\overline{M}_S$ in $\overline{N}^{\mathrm{gp}} = \overline{M}_S^{\mathrm{gp}}/\mathbf{Z}\alpha$. Each choice of $\tilde{\alpha} \in \mathcal{O}_S^*(\alpha)$ induces an extension $M_S^{\mathrm{gp}}/\mathbf{Z}\tilde{\alpha}$ of $\overline{N}^{\mathrm{gp}}$ by $\mathcal{O}_S^*$. Let T be the total space of $\mathcal{O}_S^*(\alpha)$, so that it has a universal $\tilde{\alpha}$, and define N to be preimage of $\overline{N}$ in $M_S^{\mathrm{gp}}/\mathbf{Z}\tilde{\alpha}$ on T.

Let $q : T \to S$ denote the projection. In order to make N into a logarithmic structure, we must factor the map $q^* M_S \to \mathcal{O}_T$ through N. The condition to do so is that if β and γ are local sections of N that differ by a multiple of α then $\varepsilon(\beta) - \varepsilon(\gamma) = 0$ in $\mathcal{O}_T$. These differences generate an ideal, which defines a universal closed subscheme $i : U \to T$ over which the factorization exists. Then $\varepsilon : i^* N \to \mathcal{O}_U$ is sharp because N is the image of $i^* q^* M_S$ and $i^* q^* M_S \to \mathcal{O}_U$ is sharp. This makes U into a (not necessarily saturated) logarithmic scheme over S that represents the pullback of the origin of $\overline{\mathbf{G}}_m^{\log}$, by construction. Passing to the saturation completes the proof. □

2.3. **Tropical geometry.**

2.3.1. *Tropical moduli problems.* We summarize [CCUW17]. For the purposes of this paper, a tropical moduli problem is a covariant functor on, or category covariantly fibered in groupoids over, the category of integral, saturated, sharp, commutative monoids. Such a moduli problem extends automatically, in a canonical fashion, to one defined on all integral, saturated, sharp, commutative monoidal spaces, and even all such monoidal topoi. In particular, it extends to logarithmic schemes, by regarding logarithmic schemes as monoidal topoi by way of the characteristic monoid.

There are two ways to produce this extension of the moduli problem. The first, and perhaps simpler, of the two is to extend a moduli problem F on commutative monoids to one defined on monoidal spaces (or topoi) by setting $F(S) = F(\Gamma(S, \overline{M}_S))$ and then sheafifying (or stackifying) the result.

An equivalent construction, when working over logarithmic schemes with coherent logarithmic structures, is to define $F(S)$ to be the set of systems of data $\xi_s \in F(\overline{M}_{S,s})$, one for each geometric point s of S, such that $\xi_t \mapsto \xi_s$ under the morphism $F(\overline{M}_{S,t}) \to F(\overline{M}_{S,s})$ associated to a geometric specialization $s \rightsquigarrow t$. This has the effect of building stackification into the definition, but either construction is adequate for our needs.

In practice, when formulating a tropical moduli problem, the difficult part seems to tend to lie in describing the functoriality with respect to monoid homomorphisms. More specifically, any homomorphism of commutative monoids can be factored into a localization homomorphism followed by a sharp homomorphism. Functoriality with respect to sharp homomorphisms is straightforward, but localizations tend to involve changes of topology that are more difficult to control. For the tropical Picard group and tropical Jacobian, the notion that makes this work is called *bounded monodromy*, and is first discussed in Section 3.5.

The principal concern of [CCUW17] was the question of algebraicity of tropical moduli problems, meaning possession of a well-behaved cover by rational polyhedral cones. None of the moduli problems we consider here is algebraic in this sense, although they often do have logarithmically smooth covers by logarithmic schemes. This suggests the subject of algebraicity should be revisited with a more inclusive perspective. To do so will require a less chaotic topology than the one introduced in [CCUW17], such as is currently under development by Gillam and the second author [GW], which appears implicitly in Section 3.1, and a bit more explicitly in Section 3.11 of this paper.

2.3.2. *Tropical topology.* We introduce a tropical topology that does not appear in [CCUW17]. This material will be needed in Section 3.11 and nowhere else, so we develop only the few facts we will need there. A thorough treatment will be taken up elsewhere.

Definition 2.3.2.1. Let $\overline{M}$ be an integral, saturated monoid. A *sharp valuation* of $\overline{M}$ is an isomorphism class of surjective homomorphisms from $\overline{M}^{\mathrm{gp}}$ to totally ordered abelian groups that preserve the *strict* order of $\overline{M}^{\mathrm{gp}}$.

Remark 2.3.2.2. Equivalently, a sharp valuation of $\overline{M}$ is an isomorphism class of sharp homomorphisms $\overline{M} \to V$ where V is a valuative monoid such that $\overline{M}^{\mathrm{gp}} \to V^{\mathrm{gp}}$ is surjective.

Proposition 2.3.2.3. *Let $\overline{M}$ be a sharp (integral, commutative, unital) monoid and let $\mathbf{Cone}^{\circ}(\overline{M})$ be its set of sharp valuations. Give $\mathbf{Cone}^{\circ}(\overline{M})$ the coarsest topology in which a subset defined by a finite set of strict inequalities among elements of $\overline{M}^{\mathrm{gp}}$ is open. Then $\mathbf{Cone}^{\circ}(\overline{M})$ is quasicompact.*

Proof. Consider a descending sequence of closed subsets $\mathbf{Cone}^{\circ}(\overline{M}) = Z_0 \supset Z_1 \supset Z_2 \supset \cdots$, with Z_i defined relative to Z_{i-1} by an inequality $\alpha_i \geq 0$, with $\alpha_i \in \overline{M}^{\mathrm{gp}}$. Then Z_i is represented by the monoid $\overline{M}[\alpha_1, \ldots, \alpha_i]$ in the sense that a valuation of $\overline{M}$ with valuation monoid V lies in Z_i if and only if the homomorphism $\overline{M} \to V$ factors (not necessarily sharply) through $\overline{M}[\alpha_1, \ldots, \alpha_i]$. By Lemma 2.1.2.9, the condition that $\bigcap Z_i = \varnothing$ means that $\overline{M}[\alpha_1, \alpha_2, \ldots]$ contains the inverse $-\beta$ of some element $\beta \in \overline{M}$. But then $-\beta$ is a finite combination of the α_i and elements of $\overline{M}$ and lies therefore in $\overline{M}[\alpha_1, \ldots, \alpha_i]$ for some i. We conclude that $Z_i = \varnothing$. ∎

Remark 2.3.2.4. The basic open subsets of $\mathbf{Cone}^{\circ}(\overline{M})$ are the subsets representable as $\mathbf{Cone}^{\circ}(\overline{N})$ where $\overline{N} \subset \overline{M}^{\mathrm{gp}}$ is a finitely generated extension.

Remark 2.3.2.5. Suppose that $\delta \in \mathbf{Q}\overline{M}$. Then there is some positive integer n such that $n\delta \in \overline{M}$ and the inequality $n\delta > 0$ determines an open subset of $\mathrm{Cone}^{\circ}(\overline{M})$. Since valuative monoids are always saturated, this open subset does not depend on the choice of n. We can therefore construct open subsets of $\mathbf{Cone}^{\circ}(\overline{M})$ from inequalities in $\mathbf{Q}\overline{M}$.

2.3.3. *Tropical curves.* The main example in [CCUW17] is the moduli space of tropical curves. We recall the main definition here, with small modifications, one of which is significant: first, we have no use for marked points here (which appear as unbounded legs in the graphs of tropical curves), so we omit them below; second, we allow unrooted edges that are not attached at any vertex. This second modification is essential for the definition of the topology in Section 3.1.

Definition 2.3.3.1. Let $\overline{M}$ be a commutative monoid. A *tropical curve* metrized by $\overline{M}$ is a tuple $\mathfrak{X} = (G, r, i, \ell)$ where

(1) G is a set,
(2) $r : G \to G$ is a partially defined idempotent function,
(3) $i : G \to G$ is an involution, and
(4) $\ell : G \to \overline{M}$ is a function

such that

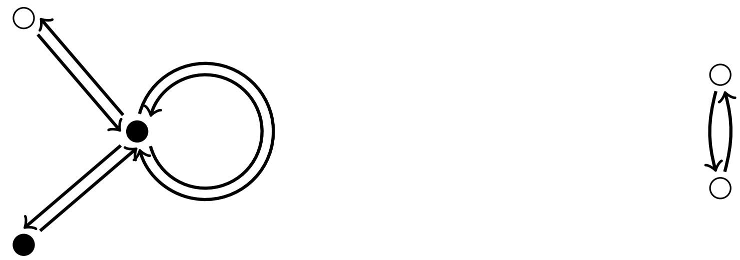

FIGURE 1. Graphical representations of tropical curves. Filled circles are vertices while open circles are endpoints of edges with absent vertices.

(5) $\ell(i(x)) = \ell(x)$ for all x, and
(6) $r(x) = x$ if and only if $i(x) = x$ if and only if $\ell(x) = 0$.

We often abuse notation and write $x \in \mathfrak{X}$ to mean that $x \in G$.

If $x \in \mathfrak{X}$ then $\ell(x)$ is called its *length*. The elements of $\mathfrak{X}$ of length 0 are called *vertices*. The remaining elements are called *flags* or *oriented edges*. An unordered pair of flags exchanged by i is called an *edge*. We call $\mathfrak{X}$ *compact* if r is defined on all of G.

Remark 2.3.3.2. It is customary to include a weighting by non-negative integers on the vertices in the definition of a tropical curve, standing for the genus of a component of a stable curve. Such a weighting could be added to Definition 2.3.3.1 with no significant change to the rest of the paper. As the weighting has no effect on the definition of the tropical Picard group, we have omitted it to keep the notation as light as possible.

The work of Amini and Caporaso on the Riemann–Roch for tropical curves with vertex weights [AC13] suggests that a vertex with positive weight g can be imagined as a vertex of weight 0 with g phantom loops attached, all of length 0. They prove Riemann–Roch by endowing these loops with positive length ϵ and then allowing ϵ to shrink to zero. The most naive application of the same approach would yield a different tropical Picard group than the one we consider, and would not have the same relationship to the logarithmic Picard group.

If $f : \overline{M} \to \overline{N}$ is a homomorphism of commutative monoids, and $\mathfrak{X}$ is a tropical curve metrized by $\overline{M}$, and r is defined on every flag x of $\mathfrak{X}$ such that $f(\ell(x)) = 0$, then f induces an *edge contraction* of $\mathfrak{X}$. Let $\mathfrak{Y}$ be the quotient of $\mathfrak{X}$ in which a flag x is identified with $r(x)$ if $f(\ell(x)) = 0$. Note that if $f(\ell(x)) = 0$, this identification also identifies $r(x) \sim r(i(x))$ since $i^2(r(x)) = x$. Then ℓ descends to a well-defined function on $\mathfrak{Y}$, valued in $\overline{N}$ and makes $\mathfrak{Y}$ into a tropical curve.

Following the procedure outlined in Section 2.3.1, we can now think of tropical curves as a tropical moduli problem: for any sharp monoid $\overline{P}$, we define $\mathcal{M}^{\mathrm{trop}}(\overline{P})$ to be the groupoid of tropical curves metrized by $\overline{P}$. Note, however, that Definition 2.3.3.1 is slightly different from the one considered in [CCUW17].

Definition 2.3.3.3. Let S be a logarithmic scheme. A *tropical curve* over S is the choice of a tropical curve $\mathfrak{X}_s$ for each geometric point s of S and an edge contraction $\mathfrak{X}_s \to \mathfrak{X}_t$ for each geometric specialization $t \rightsquigarrow s$ such that the edges of $\mathfrak{X}_s$ contracted in $\mathfrak{X}_t$ are precisely the ones whose lengths lie in the kernel of $\overline{M}_s^{\mathrm{gp}} \to \overline{M}_t^{\mathrm{gp}}$.

Definition 2.3.3.4. Let $\mathfrak{X}$ be tropical curve metrized by a monoid $\overline{M}$ with vertex set V. We define $\mathfrak{P}(\mathfrak{X})$ to be the set of functions $\lambda = (\alpha, \mu) : G_{\mathfrak{X}} \to \overline{M}^{\mathrm{gp}} \times \mathbf{Z}$ satisfying the following conditions:

(1) if x is a vertex then $\mu(x) = 0$,
(2) we have $\alpha(r(x)) = \alpha(x)$ for all x on which r is defined, and
(3) we have $\alpha(i(x)) = \alpha(x) + \mu(x)\ell(x)$.

Note that the third condition implies that $\mu(x) = -\mu(i(x))$ since

$$\alpha(x) = \alpha(i^2(x)) = \alpha(i(x))) + \mu(i(x))\ell(x)$$

and $\ell(x)$ is nonzero. We define $\mathfrak{L}(\mathfrak{X})$ to be the subset of $\mathfrak{P}(\mathfrak{X})$ where the following additional condition is satisfied:

(4) (*balancing*) for each vertex x of $\mathfrak{X}$, we have $\sum_{r(y)=x} \mu(y) = 0$.

Elements of $\mathfrak{P}(\mathfrak{X})$ are called *piecewise linear functions* on $\mathfrak{X}$ and elements of $\mathfrak{L}(\mathfrak{X})$ are called *linear functions*.

If $\mathfrak{X}$ is a tropical curve metrized by $\overline{M}$ and $\overline{M} \to \overline{N}$ is a monoid homomorphism inducing a tropical curve $\mathfrak{Y}$ metrized by $\overline{N}$ then there are natural homomorphisms $\mathfrak{P}(\mathfrak{X}) \to \mathfrak{P}(\mathfrak{Y})$ and $\mathfrak{L}(\mathfrak{X}) \to \mathfrak{L}(\mathfrak{Y})$. Thus tropical curves equipped with piecewise linear functions are a tropical moduli problem. See Proposition 3.7.3 for further details.

2.3.4. *Subdivision of tropical curves.*

Definition 2.3.4.1. Let $\mathfrak{Y}$ be a tropical curve metrized by a commutative monoid $\overline{M}$. Let y be a 2-valent vertex of $\mathfrak{Y}$. We construct a new tropical curve $\mathfrak{X}$ by removing y from $\mathfrak{Y}$ along with the two flags e and f incident to y and defining

$$i_{\mathfrak{X}}(i_{\mathfrak{Y}}(e)) = i_{\mathfrak{Y}}(f)$$
$$\ell_{\mathfrak{X}}(i_{\mathfrak{Y}}(e)) = \ell_{\mathfrak{X}}(i_{\mathfrak{Y}}(f)) = \ell_{\mathfrak{Y}}(e) + \ell_{\mathfrak{Y}}(f).$$

We call $\mathfrak{Y}$ a *basic subdivision* of $\mathfrak{X}$ at the edge $\{i_{\mathfrak{Y}}(e), i_{\mathfrak{Y}}(f)\}$. If $\mathfrak{X}$ is obtained from $\mathfrak{Y}$ by a sequence of basic subdivisions, we call $\mathfrak{Y}$ a *subdivision* of $\mathfrak{X}$.

If $\mathfrak{Y}$ is a subdivision of $\mathfrak{X}$ then $G_{\mathfrak{Y}}$ contains a copy of $G_{\mathfrak{X}}$. An isomorphism of subdivisions is an isomorphism of tropical curves that respects this copy of the underlying set.

Lemma 2.3.4.2. *If $\mathfrak{X}'$ is a subdivision of a tropical curve $\mathfrak{X}$ metrized by $\overline{M}$, and $\overline{M} \to \overline{N}$ is a localization homomorphism, then the edge contractions $\mathfrak{Y}'$ of $\mathfrak{X}'$ is naturally a subdivision of the edge contraction $\mathfrak{Y}$ of $\mathfrak{X}$.*

Proof. It is sufficient to assume that $\mathfrak{X}'$ is a basic subdivision of $\mathfrak{X}$ at an edge e into edges e' and e''. The main point is that if $\ell(e)$ maps to 0 in $\overline{N}$ then $\ell(e')$ and $\ell(e'')$ do as well, since $0 \leq \ell(e') \leq \ell(e)$ and $0 \leq \ell(e'') \leq \ell(e)$, which implies that e' and e'' are both contracted if e is. $\square$

2.4. **Logarithmic curves.**

2.4.1. *Local structure.*

Definition 2.4.1.1. Let S be a logarithmic scheme. A *logarithmic curve* over S is an integral, saturated, logarithmically smooth morphism $\pi : X \to S$ of relative dimension 1.

Theorem 2.4.1.2 (F. Kato)**.** *Let X be a logarithmic curve over S. Then the underlying scheme of X is a flat family of nodal curve over S and, for a geometric point x of X lying above the geometric point s of S, one of the following applies:*

(1) x is a smooth point of its fiber in the underlying schematic curve of X, and $\overline{M}_{S,s} \to \overline{M}_{X,x}$ is an isomorphism.
(2) x is a marked point, and there is an isomorphism $\overline{M}_{S,s} + \mathbf{N}\alpha \to \overline{M}_{X,x}$ where $\mathcal{O}_X(-\alpha)$ is the ideal of the marking.
(3) x is a node of its fiber and there is an isomorphism $\overline{M}_{S,s} + \mathbf{N}\alpha + \mathbf{N}\beta/(\alpha + \beta = \delta) \to \overline{M}_{X,x}$, with $\delta \in \overline{M}_{S,s}$. The invertible sheaf $\mathcal{O}_S(-\delta)$ is the pullback of the ideal sheaf of the boundary divisor corresponding to the node x from the moduli space of curves, and $\mathcal{O}_X(-\alpha)$ and $\mathcal{O}_X(-\beta)$ are the pullbacks ideal sheaves of the two branches of the universal curve at x.

If X is vertical *over S then the second possibility does not occur.*

We write $\mathcal{M}_{g,n}^{\log}$ for the moduli space of logarithmic curves of genus g with n marked points.

2.4.2. *Logarithmic curves over valuative bases.*

Theorem 2.4.2.1. *Let S be the spectrum of a valuation ring with generic point η and let X be a family of nodal curves over S. Assume that X_η and η have been given logarithmic structures M_{X_η} and M_η making X_η into a logarithmic curve over η, with M_η valuative. Let M_X and M_S be the maximal extensions, respectively, of M_{X_η} and M_η to X and to S. Then X is a logarithmic curve over S.*

Lemma 2.4.2.2. *The conclusion of the theorem holds when M_η is the trivial logarithmic structure.*

Proof. Under this assumption, the underlying curve of X_η is smooth over η. Let U_η denote the complement of the marked points in X_η. The logarithmic structure of X_η is the maximal extension of the (trivial) logarithmic structure on U_η, so M_X is the maximal extension to X of M_{U_η}. Let $j : U_\eta \to X$ denote the inclusion.

By definition, $\overline{M}_X^{\mathrm{gp}} = j_*(\mathcal{O}_{U_\eta}^*)/\mathcal{O}_X^*$ is the group of Cartier divisors on X whose support does not meet U_η and $\overline{M}_X$ is the submonoid of effective divisors. To complete the proof, we must show that this monoid has the local form required by Theorem 2.4.1.2 in an étale neighborhood of a geometric point x with image s in S. There are three possibilities to consider:

(1) x is a smooth point of its schematic fiber. Then any Cartier divisor of X supported at x and not meeting the general fiber must be pulled back from a Cartier divisor of S not meeting the generic point, and therefore $\overline{M}_{X,x} = \overline{M}_{S,s}$.
(2) x is a marked point of its schematic fiber. Let ξ be the corresponding marked point of the general fiber. Let γ be a section of $\overline{M}_{X,x}$ and let u be a local parameter for X along x with image $\alpha \in \overline{M}_{X,x}$. Then the is an integer n such that $\gamma - n\alpha$ is not supported at ξ. This recovers the situation of the previous case and we find $\overline{M}_{X,x}^{\mathrm{gp}} = \mathbf{Z}\alpha + \overline{M}_{S,s}^{\mathrm{gp}}$. It is immediate that the submonoid of effective elements is $\mathbf{N}\alpha + \overline{M}_{S,s}$, which is the required local form.
(3) x is a node of its schematic fiber. Let y and z be the two distinct generizations of x in its fiber.

   We have étale-local parameters $uv = t$ at the node, with $t \in \mathcal{O}_S \cap \mathcal{O}_\eta^*$. Let α and β be the elements of $\overline{M}_{X,x}$ corresponding to u and v, and let δ be the element sof $\overline{M}_{S,s}$ corresponding to t. Inverting u corresponds to generization to y, and we have already computed $\overline{M}_{X,y} = \overline{M}_{S,s}$. Therefore $\overline{M}_{X,x}^{\mathrm{gp}} = \mathbf{Z}\alpha + \overline{M}_{S,s}^{\mathrm{gp}}$.

Since generization to y sends α to 0, the effective elements must be contained in $\mathbf{Z}\alpha + \overline{M}_{S,s}$. On the other hand, generizing the other way sends α to δ, which means that for $n\alpha + \gamma$ to be effective, we must have $n\delta + \gamma \in \overline{M}_{S,s}$. That is, $\overline{M}_{X,x}$ is contained in the submonoid of $\mathbf{Z}\alpha + \overline{M}^{\mathrm{gp}}_{S,s}$ generated by $\overline{M}_{S,s}$, by α, and by $\beta = \delta - \alpha$. Since all such elements are easily seen to be effective, we conclude that $\overline{M}_{X,x} = \overline{M}_{S,s} + \mathbf{N}\alpha + \mathbf{N}\beta/(\alpha + \beta = \delta)$, as required by Theorem 2.4.1.2 (3).

□

Proof of Theorem 2.4.2.1. We will reduce to the special case of the lemma. Let A be a henselian valuation ring with residue field $\mathcal{O}_\eta$ and valuation monoid $\overline{M}_\eta$. Let $B = A \times_{\mathcal{O}_\eta} \mathcal{O}_S$. Then B is a valuation ring. Indeed, B is a subring of A, so it is an integral domain, and if x is contained in the fraction field of B then either x or x^{-1} is contained in A. If x is not a unit of A then it reduces to 0 in $\mathcal{O}_\eta$, so it is contained in $\mathcal{O}_S$ and hence is contained in B. If x is a unit of A then both x and x^{-1} reduce to nonzero elements of $\mathcal{O}_\eta$ and at least one of them is contained in $\mathcal{O}_S$ since $\mathcal{O}_S$ is a valuation ring. Assuming it is x we then have $x \in B$, as required for B to be a valuation ring.

Let $T = \operatorname{Spec} B$. Since deformations of logarithmic curves are unobstructed, we can extend X to a logarithmic curve Y over T. By Lemma 2.4.2.2, the maximal extensions of the (trivial) logarithmic structure from the generic fiber to Y is a logarithmic curve over T. This restricts to a logarithmic structure M'_X on X making X into a logarithmic curve over S. By the universal property of the maximal extension, we have a morphism of logarithmic structures $M_X \to M'_X$ giving a morphism of logarithmic schemes $(X, M'_X) \to (X, M_X)$ over S. By consideration of the local structure of M'_{X_η} guaranteed by Lemma 2.4.2.2 and of M_{X_η} required by Theorem 2.4.1.2, we deduce that $M_{X_\eta} \to M'_{X_\eta}$ is an isomorphism. Now, by the universal property of the maximal extension from η to S, we obtain a morphism $M'_X \to M_X$. The composition $M_X \to M'_X \to M_X$ must be the identity by the universal property of the maximal extension, and the composition $M'_X \to M_X \to M'_X$ must be the identity because the logarithmic structure of a logarithmic curve admits no non-identity maps to itself commuting with the projection to the base.

We conclude that $M_X = M'_X$. But M'_X was the restriction of the maximal extension of the trivial logarithmic structure from the generic fiber of Y and therefore makes X into a logarithmic curve over S, by Lemma 2.4.2.2. □

2.4.3. *Tropicalizing logarithmic curves.* Theorem 2.4.1.2 allows us to construct a family of tropical curves over S from a family X of logarithmic curves over S. For each geometric point s of S, let $\mathfrak{X}_s$ be the dual graph of X_s, metrized by $\overline{M}_{S,s}$ with $\ell(e) = \delta$ when e is the edge associated to the node x in the notation of Theorem 2.4.1.2 (3).

If $s \rightsquigarrow t$ is a geometric specialization, then $\mathfrak{X}_s$ is obtained from $\mathfrak{X}_t$ by contracting the edges of $\mathfrak{X}_t$ that correspond to nodes of X_t smoothed in X_s. Therefore the association $X_s \mapsto \mathfrak{X}_s$ commutes with the geometric generization maps and defines a morphism $\mathcal{M}^{\log}_{g,n} \to \mathcal{M}^{\mathrm{trop}}$ from the moduli space of logarithmic curves to the moduli space of tropical curves. See [CCUW17, Section 5] for further details.

The essence of the following lemma comes from Gross and Siebert [GS13, Section 1.4]. It allows us to relate the characteristic monoid of a logarithmic curve to piecewise linear functions on the tropicalization.

Lemma 2.4.3.1. *Let $\overline{M}$ be a commutative monoid. Then*
$$\overline{M} + \mathbf{N}\alpha + \mathbf{N}\beta / (\alpha + \beta = \delta) \xrightarrow{\sim} \{(a,b) \in \overline{M} \times \overline{M} \mid a - b \in \mathbf{Z}\delta\}$$
where $\alpha \mapsto (0,\delta)$, $\beta \mapsto (\delta, 0)$, and $\gamma \mapsto (\gamma,\gamma)$ for all $\gamma \in \overline{M}$.

Proof. The map is well-defined by the universal property of the pushout. The following formula gives the inverse:
$$(a,b) \mapsto \begin{cases} a + \frac{b-a}{\delta}\alpha & b \geq a \\ b + \frac{a-b}{\delta}\beta & a \geq b \end{cases}$$
□

Corollary 2.4.3.2. *Let S be a the spectrum of an algebraically closed field and let X be a logarithmic curve over S with tropicalization $\mathfrak{X}$. Then $\Gamma(X, \overline{M}_X^{\mathrm{gp}})$ and $\Gamma(\mathfrak{X}, \mathfrak{P})$ are naturally identified.*

Proof. Lemma 2.4.3.1 identifies the stalk of $\overline{M}_X^{\mathrm{gp}}$ at a node of X with the linear functions of integer slope on the corresponding edge of $\mathfrak{X}$. Generizing to one branch or the other of the node corresponds to evaluating the function at one endpoint or the other of the edge. Therefore a global section of $\overline{M}_X^{\mathrm{gp}}$ amounts to a function on $\mathfrak{X}$ taking values in $\overline{M}_S^{\mathrm{gp}}$ that is linear along the edges with integer slopes. □

We give a more local version of this corollary.

Let S be a logarithmic scheme whose underlying scheme is the spectrum of an algebraically closed field, let X be a logarithmic curve over S, and let $\mathfrak{X}$ be its tropicalization. Suppose that $p : \mathfrak{U} \to \mathfrak{X}$ is a tropical local isomorphism. Each v vertex of $\mathfrak{X}$ corresponds to a component X_v of the normalization of X and each edge v of $\mathfrak{X}$ corresponds to a node X_v of X. Let $U = \varinjlim_{u \in U} X_{p(u)}$. Effectively, U is the union of components of the normalization of X indexed by the vertices of $\mathfrak{U}$, joined along nodes indexed by the edges of $\mathfrak{U}$, together with some disjoint nodes corresponding to unattached edges of $\mathfrak{U}$.

There is a canonical projection $U \to X$ that is étale except at the points corresponding to 0- and 1-sided edges. We give U the logarithmic structure pulled back from X.

Remark 2.4.3.3. This construction extends to families with locally constant dual graph, but no further. Should $\mathfrak{U}$ be a covering space of $\mathfrak{X}$ then U will be étale over X and therefore this construction extends infinitesimally, but no further. If $\mathfrak{U}$ is in addition *finite* over $\mathfrak{X}$ then the construction can be extended to an aribtrary base.

The construction described above gives a functor t^{-1} from the category of local isomorphisms $\mathfrak{U} \to \mathfrak{X}$ to the category of finite strict X-schemes. We refer to this as an *anticontinuous morphism* from X to $\mathfrak{X}$, but we make no attempt to develop a general theory of anticontinuous maps here.

Lemma 2.4.3.4. *We have $t_* \overline{M}_X^{\mathrm{gp}} = \mathfrak{P}_{\mathfrak{X}}$. That is, for any open subset $\mathfrak{U}$ of $\mathfrak{X}$, we have $\Gamma(\mathfrak{U}, \mathfrak{P}) = \Gamma(t^{-1}\mathfrak{U}, \overline{M}_X^{\mathrm{gp}})$.*

2.4.4. *Subdivision of logarithmic curves.* Let X be a logarithmic curve over S and let $\mathfrak{X}$ be its tropicalization. Suppose that $\mathfrak{Y} \to \mathfrak{X}$ is a subdivision. We construct an associated logarithmic modification $Y \to X$ such that the tropicalization of Y is $\mathfrak{X}$.

We may make this construction étale-locally on S, provided we do so in a manner compatible with further localization. Every subdivision of tropical curves is locally an iterate of basic subdivisions, so we may assume that $\mathfrak{Y}$ is a basic subdivision of $\mathfrak{X}$. We now describe $Y \to X$ locally in X.

Suppose that e is the edge of $\mathfrak{X}$ subdivided in $\mathfrak{Y}$, and that Z is the corresponding node of X. Note that Z is a closed subset of X, not necessarily a point unless S is a point. Over the complement of Z, we take the map $Y \to X$ to be an isomorphism. It remains to describe Y on an étale neighborhood of Z.

We may work étale-locally in X, again provided that our construction is compatible with further étale localization. We can therefore work in an étale neighborhood U of a geometric point $x \in Z$ and an étale neighborhood T of its image in S, and we can assume that

(1) $\overline{M}_{X,x} = \overline{M}_{S,s} + \mathbf{N}\alpha + \mathbf{N}\beta / (\alpha + \beta = \delta)$ for some $\delta \in \overline{M}_{S,s}$,
(2) α and β come from global sections of $\overline{M}_X$ over U, and
(3) δ comes from a global section of $\overline{M}_S$ over T.

Now, recall we may think of α and β as barycentric coordinates on the edge e of $\mathfrak{X}$ that was subdivided in $\mathfrak{Y}$. Suppose that this edge was subdivided at the point where $\alpha = \gamma$ (and therefore $\beta = \delta - \gamma$) for some $\gamma \in \Gamma(T, \overline{M}_S)$. We ask V to represent the subfunctor of the functor represented by U where α and γ are locally comparable. Then V is a logarithmic modification of U.

We leave it to the reader to verify that this construction is compatible with further étale localizations in the places where we used them and that the logarithmic modifications $V \to U$ therefore patch to a logarithmic modification $Y \to X$.

Remark 2.4.4.1. It is possible to understand $Y \to X$ as the pullback of $\mathfrak{Y} \to \mathfrak{X}$ along the tropicalization map $t : X \to \mathfrak{X}$. This point of view will be developed in [GW].

3. The tropical Picard group and the tropical Jacobian

3.1. The topology of a tropical curve.

Definition 3.1.1. Let $\mathfrak{X}$ be a tropical curve and let x be a vertex of $\mathfrak{X}$. The *star* of x is the set of all $y \in \mathfrak{X}$ such that $r(y) = x$.

Definition 3.1.2. Let $\mathfrak{Y}$ and $\mathfrak{X}$ be tropical curves metrized by the same monoid $\overline{M}$. A function $f : \mathfrak{Y} \to \mathfrak{X}$ is called a *local isomorphism* if it commutes with all of the functions r, ℓ, and i and it restricts to a bijection on the star of each vertex.

A local isomorphism is called an *open embedding* if it is also injective. The image of an open embedding of tropical curves is called an *open subcurve*.

In Figure 1, there are 6 distinct local isomorphisms from the curve on the right to the curve on the left, assuming that all edges have the same length.

Lemma 3.1.3. *An open subcurve of $\mathfrak{X}$ is a subset of $\mathfrak{X}$ that is stable under i and r^{-1}.*

Proof. This is immediate. □

Example 3.1.4. Let $\mathfrak{X}$ be a tropical curve with one vertex, x, and one edge $\{e, i(e)\}$, of length δ, connecting that vertex to itself. Let $\mathfrak{Y}$ by a tropical curve with one vertex, y, and two edges $\{f, i(f)\}$ and $\{g, i(g)\}$, both of length δ, with $r(e) = r(f) = y$ and with $r(i(e))$ and $r(i(f))$ both undefined. There is a local isomorphism $\mathfrak{Y} \to \mathfrak{X}$ sending y to x, sending f to e, and sending g to $i(e)$. This local isomorphism does not restrict to open embeddings on any open cover of $\mathfrak{Y}$.

Lemma 3.1.5. *Any logarithmic curve $\mathfrak{X}$ has a minimal cover by a local isomorphism $\mathfrak{Y} \to \mathfrak{X}$. That is, for any cover $\mathfrak{Z} \to \mathfrak{X}$, there is a (not necessarily unique) factorization $\mathfrak{Y} \to \mathfrak{Z}$ of the projection from $\mathfrak{Y}$ to $\mathfrak{X}$.*

Proof. Let $\mathfrak{X}$ be a tropical curve and let $\mathfrak{Y}_0$ be the disjoint union of the stars of the vertices. Construct $\mathfrak{Y}$ by adjoining a new flag $i(x)$ for each non-vertex flag x of $\mathfrak{Y}_0$. □

Definition 3.1.6. A collection of local isomorphisms $p_i : \mathfrak{U}_i \to \mathfrak{X}$ of a tropical curve $\mathfrak{X}$ is called a *cover* if $\mathfrak{X} = \bigcup p_i(\mathfrak{U}_i)$. We call this the *tropical topology* of $\mathfrak{X}$.

Let $\mathfrak{Y}$ be a subdivision of $\mathfrak{X}$. We construct an associated morphism of sites $\rho : \mathfrak{Y} \to \mathfrak{X}$. Let $\tau : \mathfrak{U} \to \mathfrak{X}$ be a local isomorphism. For each edge e of $\mathfrak{U}$, the restriction of τ to e is a bijection. Form $\rho^{-1}\mathfrak{U}$ by subdividing e in precisely the same way $\tau(e)$ is subdivided in $\mathfrak{Y}$. Then we have an evident local isomorphism $\rho^{-1}\mathfrak{U} \to \mathfrak{Y}$.

Proposition 3.1.7. *The construction outlined above determines a morphism of sites ρ from that $\mathfrak{Y}$ to that of $\mathfrak{X}$.*

Proof. One must verify that the construction respects covers and fiber products of local isomorphisms. Both are immediate. □

Suppose that $\mathfrak{X}$ is a tropical curve over a logarithmic scheme S. This construction makes it possible to organize the sites of the fibers of $\mathfrak{X}$ over S into a fibered site [?, 7.2.1] over $\text{ét}(S)^{\text{op}}$, the *opposite* of the étale site of S.

3.2. **The sheaves of linear and piecewise linear functions.** If $\mathfrak{U} \to \mathfrak{X}$ is a local isomorphism then we have maps $\mathfrak{P}(\mathfrak{X}) \to \mathfrak{P}(\mathfrak{U})$ and $\mathfrak{L}(\mathfrak{X}) \to \mathfrak{L}(\mathfrak{U})$ by restriction. This makes $\mathfrak{P}$ and $\mathfrak{L}$ into presheaves on the category of tropical curves with local isomorphisms to $\mathfrak{X}$.

Proposition 3.2.1. *The presheaves $\mathfrak{L}$ and $\mathfrak{P}$ are sheaves in the tropical topology.*

Proof. Since piecewise linear functions are functions defined on the underlying set of a tropical curve, and tropical covers are set-theoretic covers, it is immediate that $\mathfrak{P}$ forms a sheaf. The subpresheaf $\mathfrak{L}$ is defined by the balancing condition at each vertex of the underlying graph, which depends only on the star of that vertex. By definition, a tropical cover induces a bijection on the star of each vertex, and therefore the balancing condition is visible locally in a tropical cover. □

Proposition 3.2.2. *Let $\rho : \mathfrak{Y} \to \mathfrak{X}$ be a subdivision of tropical curves. Then $\mathfrak{L}_{\mathfrak{X}} \to \mathrm{R}\rho_*\mathfrak{L}_{\mathfrak{Y}}$ is an isomorphism.*

Proof. By induction, we can also assume that $\mathfrak{Y}$ is a basic subdivision of $\mathfrak{X}$. The assertion is local on $\mathfrak{X}$ so we can assume that $\mathfrak{X}$ is a bare edge with no vertices. In that case, $\mathfrak{Y}$ is a vertex with two edges. The claim is now a straightforward calculation. □

Suppose that $\mathfrak{X}$ is a tropical curve over S. On each stratum Z of S, the tropical curve $\mathfrak{X}$ is locally constant, so the cohomology $H^*(\mathfrak{X}_Z, \mathfrak{L})$ can be represented by a complex of locally constant abelian groups. If $s \rightsquigarrow t$ is a geometric generization of S there is a map $\mathfrak{L}(\mathfrak{X}_t) \to \mathfrak{L}(\mathfrak{X}_s)$, but there is no guarantee of a generization map $H^1(\mathfrak{X}_t, \mathfrak{L}) \to H^1(\mathfrak{X}_s, \mathfrak{L})$ if s and t are in different strata. To get the generization map, we will need to impose the bounded monodromy condition in Section 3.5.

Proposition 3.2.3. *Let $\mathfrak{X}$ be a logarithmic curve metrized by $\overline{M}$, let $\overline{M} \to \overline{N}$ be a homomorphism such that $\overline{M}^{\mathrm{gp}} = \overline{N}^{\mathrm{gp}}$, and let $\mathfrak{Y}$ be the induced tropical curve metrized by $\overline{N}$. Then $\mathrm{B}\mathfrak{L}_{\mathfrak{X}} \to \mathrm{B}\mathfrak{L}_{\mathfrak{Y}}$ is an isomorphism of stacks on $\mathfrak{X}$.*

Proof. Since $\overline{M}^{\mathrm{gp}} = \overline{N}^{\mathrm{gp}}$, the sheaves $\mathfrak{L}_{\mathfrak{X}}$ and $\mathfrak{L}_{\mathfrak{Y}}$ are the same when we identify the underlying graphs of $\mathfrak{X}$ and $\mathfrak{Y}$. $\square$

3.3. The intersection pairing on a tropical curve.

Definition 3.3.1. Let $\mathfrak{X}$ be a tropical curve metrized by a monoid $\overline{M}$, and let $\ell(e) \in \overline{M}$ denote the length of an edge e of $\mathfrak{X}$. If e and f are oriented edges of $\mathfrak{X}$, we define

$$e.f = \begin{cases} \ell(e) & f = e \\ -\ell(e) & f = e' \\ 0 & \text{else} \end{cases}$$

and extend by linearity to an *intersection pairing* on the free abelian group generated by the oriented edges of $\mathfrak{X}$. By restriction it also gives a pairing on the first homology of $\mathfrak{X}$.

Lemma 3.3.2. *Suppose that $\mathfrak{X}$ is a tropical curve metrized by a monoid $\overline{M}$ and $u : \overline{M} \to \overline{N}$ is a homomorphism inducing an edge contraction $\mathfrak{Y}$ of $\mathfrak{X}$. Then the intersection pairing is compatible with u, in the sense that diagram (3.3.2.1) commutes:*

$$\begin{tikzcd} H_1(\mathfrak{X}) \times H_1(\mathfrak{Y}) \subset \mathbf{Z}^{E(\mathfrak{X})} \times \mathbf{Z}^{E(\mathfrak{X})} \arrow[r] \arrow[d] & \overline{M}^{\mathrm{gp}} \arrow[d] \\ H_1(\mathfrak{Y}) \times H_1(\mathfrak{Y}) \subset \mathbf{Z}^{E(\mathfrak{Y})} \times \mathbf{Z}^{E(\mathfrak{Y})} \arrow[r] & \overline{N}^{\mathrm{gp}} \end{tikzcd} \tag{3.3.2.1}$$

Proof. The proof is immediate. $\square$

3.4. The tropical degree.

Let $\mathscr{V}$ denote the quotient $\mathfrak{P}/\mathfrak{L}$. Then $\mathscr{V}(\mathfrak{U})$ is the free abelian group generated by the vertices of $\mathfrak{U}$.

Let $\mathfrak{X}$ be a tropical curve metrized by $\overline{M}$. There is an embedding of the constant sheaf $\overline{M}^{\mathrm{gp}}$ inside $\mathfrak{L}_{\mathfrak{X}}$ as the constant functions. We write $\mathscr{H}$ for the quotient of $\mathfrak{L}$ by $\overline{M}^{\mathrm{gp}}$ and $\mathscr{E}$ for the quotient of $\mathfrak{P}$ by $\overline{M}^{\mathrm{gp}}$. This yields a commutative diagram (3.4.1) with exact rows and columns:

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \overline{M}^{\mathrm{gp}} \arrow[r] \arrow[d, equal] & \mathfrak{L} \arrow[r] \arrow[d] & \mathscr{H} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \overline{M}^{\mathrm{gp}} \arrow[r] & \mathfrak{P} \arrow[r] \arrow[d] & \mathscr{E} \arrow[r] \arrow[d] & 0 \\ & & \mathscr{V} \arrow[r, equal] \arrow[d] & \mathscr{V} \arrow[d] & \\ & & 0 & 0 & \end{tikzcd} \tag{3.4.1}$$

We note that $\mathscr{E}$ is the sheaf freely generated by the edges and that $\mathscr{E} \to \mathscr{V}$ is the coboundary map in homology. Therefore $\mathscr{H}$ is the sheaf whose value on $\mathfrak{U}$ is the first Borel–Moore homology of $\mathfrak{U}$. Note that because $\mathfrak{X}$ can have 1-sided or even 0-sided edges, the Borel–Moore homology is not locally trivial.

Lemma 3.4.2. *Let $\mathfrak{X}$ be a tropical curve. Then $H^p(\mathfrak{X}, \mathscr{E}) = 0$ for all $p > 0$ and $H^p(\mathfrak{X}, \mathscr{V}) = 0$ for all $p > 0$.*

Proof. Note first that $\mathscr{V}$ is the pushforward along the closed embedding of the vertices of $\mathfrak{X}$ of the constant sheaf $\mathbf{Z}$. Therefore $H^p(\mathfrak{X}, \mathscr{V}) = H^p(V(\mathfrak{X}), \mathbf{Z}) = 0$ for all $p > 0$.

Next, note that $\mathscr{E}$ is the direct sum of sheaves $\mathscr{E}_i$ supported on each of the edges of $\mathfrak{X}$. Then $\mathscr{E}_i$ is the pushforward along the closed embedding of either an interval or a circle. We can therefore assume that $\mathfrak{X}$ is either an interval or a circle.

If $\mathfrak{X}$ is an interval then its topology is generated by open subsets and $\mathscr{E}$ is flasque, hence has no higher cohomology. If $\mathfrak{X}$ is a circle then its universal cover $\mathfrak{Y}$ has no self-loops, so $\mathscr{E}$ is flasque on $\mathfrak{Y}$. Therefore $H^p(\mathfrak{X}, \mathscr{E})$ can be identified with the group cohomology $H^p(\mathbf{Z}, \mathscr{E}(\mathfrak{Y}))$. The group cohomology of $\mathbf{Z}$ vanishes for $p > 1$ and for $p = 1$ it coincides with the coinvariants of $\mathscr{E}(\mathfrak{Y})$. We identify $\mathscr{E}(\mathfrak{Y})$ with a countable product of copies of $\mathbf{Z}$, with $\mathbf{Z}$ acting by shift. The coinvariants are therefore zero and the lemma is proved. □

Corollary 3.4.3. *If $\mathfrak{X}$ is a compact tropical curve then $H^0(\mathfrak{X}, \mathscr{H}) = H_1(\mathfrak{X})$ and $H^1(\mathfrak{X}, \mathscr{H}) = H_0(\mathfrak{X})$.*

Proof. This is immediate, as $\mathscr{E}(\mathfrak{X})$ is the free abelian group generated by the edges of $\mathfrak{X}$ and $\mathscr{V}(\mathfrak{X})$ is the free abelian group generated by the vertices, and the map between them is the boundary map. □

Lemma 3.4.4. *If $\mathfrak{X}$ is a compact tropical curve then $H^0(\mathfrak{X}, \overline{M}^{\mathrm{gp}}) \to H^0(\mathfrak{X}, \mathfrak{L})$ is an isomorphism.*

Proof. We want to show that on a compact tropical curve, every globally defined linear function is locally constant. Let $\mathfrak{Z}$ be a maximal connected subgraph where f takes its minimum value. Then if e is a flag of $\mathfrak{X}$ exiting $\mathfrak{Z}$, the slope of f along e must be positive. But by the balancing condition, $\sum_e \mu(e) = 0$, when the sum is taken over all edges exiting $\mathfrak{Z}$. The only way a sum of positive numbers can be zero is if it is empty, so we conclude $\mathfrak{Z} = \mathfrak{X}$ and that f is constant. □

Using Corollary 3.4.3 and Lemma 3.4.4, we write down the long exact sequence in cohomology associated to the short exact sequence in the first row two rows of (3.4.1):

$$
\begin{gathered}
0 \to H_1(\mathfrak{X}) \to H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}}) \to H^1(\mathfrak{X}, \mathfrak{L}) \xrightarrow{\deg} H_0(\mathfrak{X}) \to 0 \\
0 \to \mathscr{E}(\mathfrak{X}) \to H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}}) \to H^1(\mathfrak{X}, \mathfrak{P}) \to 0
\end{gathered} \tag{3.4.5}
$$

The homomorphism $H^1(\mathfrak{X}, \mathfrak{L}) \to H_0(\mathfrak{X})$ is called the *degree*. We can also identify $H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}}) = \mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}})$.

Lemma 3.4.6. *The homomorphisms (3.4.6.1)*

$$
\begin{gathered}
H_1(\mathfrak{X}) = H^0(\mathfrak{X}, \mathscr{H}) \to H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}}) = \mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}}) \\
\mathscr{E}(\mathfrak{X}) = H^0(\mathfrak{X}, \mathscr{E}) \to H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}}) = \mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}})
\end{gathered} \tag{3.4.6.1}
$$

in the exact sequences (3.4.5) are the intersection pairing on $\mathfrak{X}$.

Proof. The first homomorphism is induced from the second by restriction to $H_1(\mathfrak{X}) \subset \mathscr{E}(\mathfrak{X})$, so it suffices to consider the second. Suppose that $\alpha \in H^0(\mathfrak{X}, \mathscr{E})$. We can regard α as a section of $\mathscr{E}$, and therefore as an integer-valued function on the edges of $\mathfrak{X}$. The class of its coboundary in $H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}})$ is the $\overline{M}^{\mathrm{gp}}$-torsor on $\mathfrak{X}$ of piecewise linear functions having slopes α along the edges.

Such a torsor is classified by its failure to be representable by a well-defined, piecewise linear function, in the form of its monodromy around the loops of $\mathfrak{X}$. In other words, we may make the identification (3.4.6.2):

$$H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}}) = \mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}}) \tag{3.4.6.2}$$

Given $\alpha \in H^0(\mathfrak{X}, \mathscr{E})$ and a $\gamma \in H_1(\mathfrak{X})$, represented as a sum of oriented edges of $\mathfrak{X}$, the monodromy of α around γ is

$$\sum_{e \in \gamma} \alpha(e)$$

which is exactly the same as $\alpha.\gamma$. □

We summarize our results in the following corollary, which may be viewed as a tropical Abel theorem:

Corollary 3.4.7. *Let $\mathfrak{X}$ be a compact tropical curve metrized by $\overline{M}$. Then there are exact sequences* (3.4.7.1)*, where ∂ is the intersection pairing:*

$$\begin{gathered} 0 \to H_1(\mathfrak{X}) \xrightarrow{\partial} \mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}}) \to H^1(\mathfrak{X}, \mathfrak{L}) \xrightarrow{\deg} H_0(\mathfrak{X}) \to 0 \\ 0 \to \mathscr{E}(\mathfrak{X}) \xrightarrow{\partial} \mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}}) \to H^1(\mathfrak{X}, \mathfrak{P}) \to 0 \end{gathered} \tag{3.4.7.1}$$

3.5. **Monodromy.** Let $\mathfrak{X}$ be a tropical curve metrized by $\overline{M}$. Let Q be a $\mathfrak{P}$-torsor on $\mathfrak{X}$. By Corollary 3.4.7, there is an $\alpha \in \mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}})$ inducing Q, uniquely determined by Q up to addition of $\partial(e)$, for an edge e of $\mathfrak{X}$. We refer to α as a *monodromy representative* of Q.

Proposition 3.5.1. *Let $\mathfrak{X}$ be a compact, connected tropical curve metrized by a* valuative *monoid $\overline{M}$. The following conditions are equivalent of $Q \in H^1(X, \overline{M}_X^{\mathrm{gp}})$:*

(1) There exists a subdivision $\mathfrak{Y}$ of $\mathfrak{X}$ such that the restriction of Q to $\mathfrak{Y}$ is trivial.
(2) For any monodromy representative α of Q and any $\gamma \in H_1(\mathfrak{X})$, the monodromy of α around γ is bounded by the length of γ.

Before we begin the proof, we note that to verify the monodromy condition, it is sufficient to consider a single monodromy representative:

Lemma 3.5.2. *Suppose that α and β are monodromy representatives of the same $\mathfrak{P}$-torsor on a tropical curve $\mathfrak{X}$, and let $\gamma \in H_1(\mathfrak{X})$. The monodromy of α around γ is bounded by the length of γ if and only if the monodromy of β around γ is bounded by the length of γ.*

Proof. Since α and β differ by a linear combination of $\partial(e)$, for e among the edges of $\mathfrak{X}$, it is sufficient by Lemma 2.1.3.3 to show that $\partial(e)$ has bounded monodromy around each $\gamma \in H_1(\mathfrak{X})$. But the monodromy of $\partial(e)$ around γ is $e.\gamma$. If e is not contained in γ then $e.\gamma = 0$, which is obviously bounded by $\ell(\gamma)$. If e is contained in γ then $e.\gamma = \pm\ell(e)$, and $\ell(e)$ is bounded by γ because e is contained in γ. □

Lemma 3.5.3. *Suppose that $\tau : \mathfrak{Y} \to \mathfrak{X}$ is a subdivision of tropical curves. Let α be an element of $H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}})$. The monodromy of α around the loops of $\mathfrak{X}$ is bounded by their lengths if and only if same holds of the monodromy of $\tau^*\alpha$ around the loops of $\mathfrak{Y}$.*

Proof. The lengths of the loops of $\mathfrak{Y}$ is the same as the length of the loops in $\mathfrak{X}$ and the monodromy around them is the same as the monodromy around the loops in X. □

Proof of Proposition 3.5.1. Suppose first that Q can be trivialized on a subdivision $\tau : \mathfrak{Y} \to \mathfrak{X}$. Let $\mu : H_1(\mathfrak{X}) \to \overline{M}^{\mathrm{gp}}$ be a monodromy representative of Q. Then μ lies in the image of $\partial : \mathscr{E}(\mathfrak{Y}) \to \mathrm{Hom}(H_1(\mathfrak{Y}), \overline{M}^{\mathrm{gp}}) = \mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}})$, so by Lemma 3.5.2, its monodromy around the loops of $\mathfrak{Y}$ is certainly bounded by the lengths of the loops. But by Lemma 3.5.3, this implies that μ has the same property.

Now assume that the monodromy of μ around the loops of $\mathfrak{X}$ is bounded by their lengths. We construct a subdivision $\tau : \mathfrak{Y} \to \mathfrak{X}$ such that $\tau^*\mu$ is in the image of $\partial : \mathscr{E}(\mathfrak{Y}) \to \mathrm{Hom}(H_1(\mathfrak{Y}), \overline{M}_S^{\mathrm{gp}})$.

The proof will be by induction on the rank of the image of the monodromy homomorphism (3.5.3.1):

$$\mu : H_1(\mathfrak{X}) \to \overline{M}^{\mathrm{gp}} \tag{3.5.3.1}$$

Our strategy will be to subdivide $\mathfrak{X}$ so that $\mathscr{E}(\mathfrak{Y})$ enlarges and adjust μ by the addition of elements in its image so that the rank of the image of μ decreases. We therefore permit ourselves to adjust μ as necessary by elements of the image of ∂.

By Proposition 2.1.3.8, there is a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset \overline{M}_S^{\mathrm{gp}}$$

of ordered subgroups such that each V_n/V_{n-1} may be embedded in $\mathbf{R}$, preserving the ordering. Let n be the largest index such that the image of μ is contained in V_n. If $n = 0$ we are done. Otherwise, choose an embedding of V_n/V_{n-1} in $\mathbf{R}$.

This induces a metric on $\mathfrak{X}$ with lengths in $\mathbf{R}$. We write $\overline{\mathfrak{X}}$ for the tropical curve obtained by collapsing those edges in $\mathfrak{X}$ whose lengths in $\mathbf{R}$ are zero. Note that there is a well-defined monodromy function

$$\overline{\mu} : H_1(\overline{\mathfrak{X}}) \to \mathbf{R}$$

precisely because the monodromy around $\gamma \in H_1(\overline{\mathfrak{X}}, \mathbf{Z})$ is bounded by the length of γ. Indeed, $\gamma \in H_1(\mathfrak{X})$ has length δ, and the image of δ in $\mathbf{R}$ is zero then the boundedness of the mondromy around γ implies that $\overline{\mu}(\gamma) = 0$ as well.

Choose a spanning tree of $\overline{\mathfrak{X}}$ and let E be a set of edges of $\overline{\mathfrak{X}}$ not in the spanning tree. Each of these edges corresponds uniquely to an edge of $\mathfrak{X}$, so we will also think of E as a set of edges of $\mathfrak{X}$.

For each $e \in E$, let γ_e be the corresponding basis element of $H_1(\overline{\mathfrak{X}})$. Let δ_e be the length of γ_e and let $\mu_e = \overline{\mu}(\gamma_e)$ be the monodromy around it, both valued in $\mathbf{R}$. Since $\delta_e \neq 0$, there is some integer k such that $k\delta_e \leq \mu_e \leq (k+1)\delta_e$. We replace μ by $\mu - k\partial(e)$ so that we may assume that $0 \leq \mu_e \leq \delta_e$. Let $\tau : \mathfrak{Y} \to \mathfrak{X}$ be the subdivision of $\mathfrak{X}$ that divides the edge e into edges e' and e'' of lengths μ_e and $\delta_e - \mu_e$, respectively. Then $\tau^*\mu - \partial(e')$ has no monodromy around γ_e.

Repeating this procedure for all e in E, we arrive at a representative for the monodromy of Q such that the image of μ in V_n/V_{n-1} is zero. Now we repeat the process with n replaced by $n-1$ until we have replaced μ by 0. □

Definition 3.5.4. We say that a homomorphism $H_1(\mathfrak{X}) \to \overline{M}^{\mathrm{gp}}$ on $\mathfrak{X}$ has *bounded monodromy* if it satisfies the equivalent conditions of Proposition 3.5.1. We indicate a subgroup of bounded monodromy by decoration with a dagger (†).

3.6. **The tropical Jacobian.** Let $\mathfrak{X}$ be a tropical curve metrized by a monoid $\overline{M}$. We construct the tropical Jacobian of $\mathfrak{X}$ in a manner covariantly functorial in $\overline{M}$. This effectively constructs the tropical Jacobian relative to the moduli space of tropical curves.

Definition 3.6.1. We define the tropical Jacobian by (3.6.1.1), where the dagger (†) indicates the subgroup of elements with bounded monodromy (Definition 3.5.4):

$$\operatorname{Tro Jac}(\mathfrak{X}) = \operatorname{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}})^{\dagger} / H_1(\mathfrak{X}) \tag{3.6.1.1}$$

Now suppose that we have a monoid homomorphism $\overline{M} \to \overline{N}$. This induces an edge contraction $\mathfrak{Y}$ of $\mathfrak{X}$. We wish to produce a morphism (3.6.2):

$$\operatorname{Tro Jac}(\mathfrak{X}) \to \operatorname{Tro Jac}(\mathfrak{Y}) \tag{3.6.2}$$

The edge contraction $\mathfrak{X} \to \mathfrak{Y}$ induces a homomorphism $H_1(\mathfrak{X}) \to H_1(\mathfrak{Y})$. Note that if $\mu \in \operatorname{Tro Jac}(\mathfrak{X})$ has bounded monodromy then, by definition, the composition (3.6.3)

$$H_1(\mathfrak{X}) \to \overline{M}^{\mathrm{gp}} \to \overline{N}^{\mathrm{gp}} \tag{3.6.3}$$

takes the value zero on all loops of $\mathfrak{X}$ contracted in $\mathfrak{Y}$. Therefore the homomorphism factors through $H_1(\mathfrak{Y})$, and does so uniquely because $H_1(\mathfrak{X}) \to H_1(\mathfrak{Y})$ is surjective. The factorization still has bounded monodromy, since if α is bounded by δ in $\overline{M}^{\mathrm{gp}}$ then its image in $\overline{N}^{\mathrm{gp}}$ is bounded by the image of δ. We obtain the desired morphism 3.6.2.

It is clear from the construction that it respects compositions of monoid homomorphisms. Following the procedure described in Section 2.3.1, we may extend the definition of $\operatorname{Tro Jac}(\mathfrak{X})$ to families. That is, given a family of tropical curves $\mathfrak{X}$ over a logarithmic scheme S we obtain an étale sheaf on the category of logarithmic schemes over S by either of the following equivalent procedures:

1. If T is an atomic neighborhood of a geometric point t, then we define $\operatorname{Tro Jac}(\mathfrak{X}/S)(T) = \operatorname{Tro Jac}(\mathfrak{X}_t)$ and sheafify the resulting presheaf.
2. If T is a logarithmic scheme over S of finite type then an object of $\operatorname{Tro Jac}(\mathfrak{X}/S)(T)$ consists of objects of $\operatorname{Tro Jac}(\mathfrak{X}_t)$ for each geometric point t that are compatible along geometric generizations. We extend from logarithmic schemes that are of finite type to all logarithmic schemes by the approximation procedure of [GD67, IV.8].

If X is a logarithmic proper, vertical logarithmic curve over S with tropicalization $\mathfrak{X}$ then we pose $\operatorname{Tro Jac}(X/S) = \operatorname{Tro Jac}(\mathfrak{X}/S)$.

Proposition 3.6.4. $\operatorname{Tro Jac}(X)$ *is unchanged by replacing* S *with a logarithmic modification and* X *with a different logarithmic model.*

Proof. This follows from Lemma 3.5.3, Proposition 3.2.3, and the fact that $\overline{M}_S^{\mathrm{gp}}$ is unchanged by a logarithmic modification. □

3.7. **The tropical Picard group.**

Definition 3.7.1. Let $\mathfrak{X}$ be a tropical curve metrized by a monoid $\overline{M}$. We say that an element of $H^1(\mathfrak{X}, \mathfrak{L})$ has bounded monodromy if its image in $H^1(\mathfrak{X}, \mathfrak{P})$ has bounded monodromy (which means that it is the image of a class of bounded monodromy in $H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}}) =$

$\mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}}))$. For each $d \in H_0(\mathfrak{X})$, we write $\mathrm{Tro\,Pic}^d(\mathfrak{X})$ for the preimage of d under the degree homomorphism $\mathrm{Tro\,Pic}(\mathfrak{X}) \subset H^1(\mathfrak{X}, \mathfrak{L}) \xrightarrow{\deg} H_0(\mathfrak{X})$ from Corollary 3.4.7.

We define $\mathbf{Tro\,Pic}(\mathfrak{X})$ to be category of $\mathfrak{L}$-torsors whose classes in $H^1(\mathfrak{X}, \mathfrak{L})$ have bounded monodromy, and we define $\mathrm{Tro\,Pic}(\mathfrak{X})$ to be the set of isomorphism classes of $\mathrm{Tro\,Pic}(\mathfrak{X})$. Objects of $\mathbf{Tro\,Pic}(\mathfrak{X})$ are called *tropical line bundles* on $\mathfrak{X}$.

The main task of this section is to describe the functoriality of $\mathrm{Tro\,Pic}(\mathfrak{X})$ in the monoid $\overline{M}$.

Lemma 3.7.2. *A class in $H^1(\mathfrak{X}, \mathfrak{L})$ has bounded monodromy if and only if it is the sum of a class in the image of $H^0(\mathfrak{X}, \mathscr{V})$ and a class of bounded monodromy in $H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}}) = \mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}})$ (under the maps induced from diagram (3.4.1)).*

Proof. This follows from the commutativity of the diagram (3.7.4.3), below, and its exactness in the second row (which is the long exact sequence associated to the middle column of (3.4.1)):

(3.7.2.1)
$$\begin{tikzcd}
 & H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}}) \arrow[r, equal] \arrow[d] & H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}}) \arrow[d] \\
H^0(\mathfrak{X}, \mathscr{V}) \arrow[r] & H^1(\mathfrak{X}, \mathfrak{L}) \arrow[r] & H^1(\mathfrak{X}, \mathfrak{P})
\end{tikzcd}$$

□

We will obtain the functoriality of $\mathrm{Tro\,Pic}(\mathfrak{X})$ from naturally defined functorial operations on $\mathfrak{P}$ and $\mathscr{V}$. We begin by summarizing these.

Proposition 3.7.3. *Let $\mathfrak{X}$ be a tropical curve metrized by $\overline{M}$, let $u : \overline{M} \to \overline{N}$ be a homomorphism of monoids, and let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be the induced edge contraction.*

- *(i) There is a unique homomorphism $\mathfrak{P}(\mathfrak{X}) \to \mathfrak{P}(\mathfrak{Y})$, sending $f \in \mathfrak{P}(\mathfrak{X})$ to $\overline{f} \in \mathfrak{P}(\mathfrak{Y})$ such that $\overline{f}(\sigma(x)) = f(x)$ whenever x is not contracted in $\mathfrak{Y}$.*
- *(ii) There is a unique homomorphism $\mathscr{V}(\mathfrak{X}) \to \mathscr{V}(\mathfrak{Y})$ by sending the basis vector $[x]$ to $[\sigma(x)]$.*
- *(iii) The homomorphisms above commute with the quotient map $\mathfrak{P} \to \mathscr{V}$.*

Proof.

- (i) The uniqueness is evident, since every $y \in \mathfrak{Y}$ is the image of some $x \in \mathfrak{X}$ that is not contracted. To check the existence, assume that x is a flag of $\mathfrak{X}$ that is contracted in $\mathfrak{Y}$ and $f(x) = (\alpha, \mu)$. Then $\alpha(i(x)) - \alpha(x) \in \mathbf{Z}\ell(x)$ and $\ell(x)$ lies in the kernel of u (because x is contracted), so $u(\alpha(i(x)) = u(\alpha(x))$. Therefore $u \circ f$ is constant on the regions contracted by σ and descends to $\mathfrak{Y}$.
- (ii) Immediate.
- (iii) We argue that the diagram (3.7.3.1) commutes:

(3.7.3.1)
$$\begin{tikzcd}
\mathfrak{P}(\mathfrak{X}) \arrow[r] \arrow[d] & \mathscr{V}(\mathfrak{X}) \arrow[d] \\
\mathfrak{P}(\mathfrak{Y}) \arrow[r] & \mathscr{V}(\mathfrak{Y})
\end{tikzcd}$$

Let $f = (\alpha, \mu)$ be a piecewise linear function on $\mathfrak{X}$. The coefficient of v in the image of f in $\mathscr{V}(\mathfrak{X})$ is $\sum_{r(e)=v} \mu(e)$. Therefore the image of f in $\mathscr{V}(\mathfrak{Y})$, going around the top and right of (3.7.3.1), is $\sum_{f(v)=w}\sum_{r(e)=v}\mu(e)$. In this sum, each edge of the contracted locus appears twice, with opposite orientations, and each edge exiting the contracted locus appears once, oriented out. The sum therefore reduces to $\sum_{r(e)=w}\mu(e)$, which is what we get from following f around the bottom and left of the diagram.

□

Proposition 3.7.4. *Let $\overline{M} \to \overline{N}$ be a homomorphism of commutative monoids inducing an edge contraction $\mathfrak{X} \to \mathfrak{Y}$ of tropical curves. Then the maps* (3.7.4.1)

$$H^0(\mathfrak{X}, \mathscr{V}) \to H^0(\mathfrak{Y}, \mathscr{V}) \qquad (3.7.4.1)$$
$$H^1(\mathfrak{X}, \overline{M})^\dagger \to H^0(\mathfrak{Y}, \overline{N})^\dagger$$

agree on their common domain of definition and define a map (3.7.4.2)*:*

$$H^1(\mathfrak{X}, \mathfrak{L})^\dagger \to H^1(\mathfrak{Y}, \mathfrak{L})^\dagger \qquad (3.7.4.2)$$

Proof. Diagram (3.4.1) induces a commutative square (3.7.4.3), below:

$$\begin{tikzcd} & H^0(\mathfrak{X}, \mathscr{E}) \arrow[r] \arrow[d] & H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}})^\dagger \arrow[r] \arrow[d] & H^1(\mathfrak{X}, \mathfrak{P})^\dagger \arrow[d, equal] \\ H^0(\mathfrak{X}, \mathfrak{P}) \arrow[r] & H^0(\mathfrak{X}, \mathscr{V}) \arrow[r] & H^1(\mathfrak{X}, \mathfrak{L})^\dagger \arrow[r] & H^1(\mathfrak{X}, \mathfrak{P})^\dagger \end{tikzcd} \qquad (3.7.4.3)$$

Suppose that $u \in H^1(\mathfrak{X}, \mathfrak{L})$ is the image of some $v \in H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}})^\dagger$ and $w \in H^0(\mathfrak{X}, \mathscr{V})$. Then the image of u in $H^1(\mathfrak{X}, \mathfrak{P})$ must vanish. This is also the image of v, so that v is the image of some $x \in H^0(\mathfrak{X}, \mathscr{E})$. The difference between w and the image of x maps to 0 in $H^1(\mathfrak{X}, \mathfrak{L})$, hence is the image of some $y \in H^0(\mathfrak{X}, \mathfrak{P})$. Replacing w by $w - y$ we discover that we must show the two maps in question agree on $H^0(\mathfrak{X}, \mathscr{E})$.

We can define a map (3.7.4.4) sending an edge x to itself if it is not contracted in $\mathfrak{Y}$ and to 0 if it is contracted.

$$H^0(\mathfrak{X}, \mathscr{E}) \to H^0(\mathfrak{Y}, \mathscr{E}) \qquad (3.7.4.4)$$

This commutes with the maps to $H^0(\mathfrak{X}, \mathscr{V})$ and $H^1(\mathfrak{X}, \overline{M}^{\mathrm{gp}}) = \mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}^{\mathrm{gp}})$. □

3.8. **The tropical Picard stack.** The construction in Proposition 3.7.4 can be categorified to operate on $\mathfrak{L}$-torsors with bounded monodromy, and not merely their isomorphism classes. Given an edge contraction $\sigma : \mathfrak{X} \to \mathfrak{Y}$ associated to a homomorphism of monoids $\overline{M} \to \overline{N}$ and an $\mathfrak{L}$-torsor Q on $\mathfrak{X}$ with bounded monodromy, we wish to produce an $\mathfrak{L}$-torsor on $\mathfrak{Y}$ in a canonical way.

Using the following lemma, we may promote σ to be a morphism of sites.

Lemma 3.8.1. *Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be an edge contraction induced from a homomorphism $\overline{M} \to \overline{N}$. Let $\mathfrak{V} \to \mathfrak{Y}$ be a local isomorphism. Then the set-theoretic fiber product $\mathfrak{U} = \mathfrak{V} \times_{\mathfrak{Y}} \mathfrak{X}$ is naturally equipped with the structure of a tropical curve and the projection $\mathfrak{U} \to \mathfrak{X}$ is a local isomorphism.*

Proof. The involution i and the partially defined function r on $\mathfrak{U}$ are induced from those on $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{V}$ and their compatibility. The metric ℓ is induced from the projection to $\mathfrak{X}$. We must verify that $r(u) = u$ if and only if $i(u) = u$ if and only if $\ell(u) = 0$ for all $u \in \mathfrak{U}$. Indeed, because $\sigma : \mathfrak{X} \to \mathfrak{Y}$ is surjective, each of these conditions is equivalent to the corresponding condition in $\mathfrak{X}$, which is a tropical curve by assumption.

To see that $\mathfrak{U} \to \mathfrak{X}$ is a local isomorphism, let u be a vertex of $\mathfrak{U}$ and denote by x, y, and v its images in $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{V}$. Let $\mathfrak{S}_x$, $\mathfrak{S}_y$, $\mathfrak{S}_u$, and $\mathfrak{S}_v$ denote their stars. Then $\mathfrak{S}_v$ bijects onto $\mathfrak{S}_y$, by the assumption that $\mathfrak{V} \to \mathfrak{Y}$ is a local isomorphism. Therefore $\sigma^{-1}\mathfrak{S}_v$ maps isomoprhically onto $\sigma^{-1}\mathfrak{S}_y$. But $\mathfrak{S}_u \subset \sigma^{-1}\mathfrak{S}_v$ and $\mathfrak{S}_x \subset \sigma^{-1}\mathfrak{S}_y$, so that $\mathfrak{S}_u$ maps isomorphically onto $\mathfrak{S}_x$, as required. □

The lemma shows that if $\mathfrak{V} \to \mathfrak{Y}$ is a local isomorphism, then $\sigma^{-1}\mathfrak{V} \to \mathfrak{X}$ is also a local isomorphism. It is immediate that σ^{-1} respects fiber products and covers so we obtain a morphism of sites $\sigma : \mathfrak{X} \to \mathfrak{Y}$.

We construct the desired functor $\mathbf{Tro\,Pic}(\mathfrak{X}) \to \mathbf{Tro\,Pic}(\mathfrak{Y})$ by working locally. We write $\sigma_* \mathrm{B}\mathfrak{L}^\dagger_{\mathfrak{X}}$ for the substack of those $Q \in \sigma_* \mathrm{B}\mathfrak{L}_{\mathfrak{X}}(\mathfrak{V})$ such that Q has bounded monodromy on $\sigma^{-1}\mathfrak{U}$ for each local isomorphism $\mathfrak{U} \to \mathfrak{Y}$.

Proposition 3.8.2. *There is a morphism* (3.8.2.1)

$$\sigma_* \mathrm{B}\mathfrak{L}^\dagger_{\mathfrak{X}} \to \mathrm{B}\mathfrak{L}^\dagger_{\mathfrak{Y}} \tag{3.8.2.1}$$

inducing the morphisms in Proposition 3.7.4.

Proof. Provided we do so compatibly with restriction, it is sufficient to work locally in $\mathfrak{Y}$. We can therefore assume that $\mathfrak{Y}$ is either a single edge or has a single vertex with a number of edges attached to it at only one side. In the former case, $\mathfrak{X}$ is also a single edge and $\sigma : \mathfrak{X} \to \mathfrak{Y}$ is an isomorphism, because σ is an edge contraction. We therefore assume that $\mathfrak{Y}$ is a single vertex with edges radiating from it. We note that in this case, $\mathfrak{Y}$ has no nontrivial covers, so that we only need to construct (3.8.2.1) on global sections:

$$\mathbf{Tro\,Pic}(\mathfrak{X}) \to \mathbf{Tro\,Pic}(\mathfrak{Y}) \tag{3.8.2.2}$$

Let us write $\mathfrak{L}'$ for the sheaf of $\overline{N}$-valued linear functions on $\mathfrak{X}$ and $\mathbf{Tro\,Pic}(\mathfrak{X})'$ for the category of $\mathfrak{L}'$-torsors on $\mathfrak{X}$ of bounded monodromy. Since every edge of $\mathfrak{X}$ that is contracted by σ has length 0 in $\overline{N}$ (by definition), $\mathfrak{L}' = \sigma^* \mathfrak{L}_{\mathfrak{Y}}$. In particular, $\mathfrak{L}'$ is constant with value $\overline{N}^{\mathrm{gp}}$ on the preimage of the vertex of $\mathfrak{Y}$. The quotient $\mathfrak{L}'/\overline{N}^{\mathrm{gp}}$ is a constant $\mathbf{Z}$ on the edges of $\mathfrak{X}$ not contracted by σ and therefore has vanishing H^1. We may therefore make the identifications (3.8.2.3):

$$H^1(\mathfrak{X}, \mathfrak{L}') = H^1(\mathfrak{X}, \overline{N}^{\mathrm{gp}}) = \mathrm{Hom}(H_1(\mathfrak{X}), \overline{N}^{\mathrm{gp}}) \tag{3.8.2.3}$$

But every loop of $\mathfrak{X}$ has length 0 when measured in $\overline{N}$, so that a homomorphism $H_1(\mathfrak{X}) \to \overline{N}^{\mathrm{gp}}$ of bounded monodromy must be 0. Therefore $H^1(\mathfrak{X}, \mathfrak{L}')^\dagger = 0$ and $\mathbf{Tro\,Pic}(\mathfrak{X})' = \mathrm{B}\Gamma(\mathfrak{X}, \mathfrak{L}')$. Observing now that $\mathfrak{L}'(\mathfrak{X}) = \mathfrak{L}(\mathfrak{Y})$, we conclude that $\mathbf{Tro\,Pic}(\mathfrak{X})' = \mathbf{Tro\,Pic}(\mathfrak{Y})$. The sought after morphism (3.8.2.2) now arises as the composition (3.8.2.4):

$$\begin{tikzcd}
\mathbf{Tro\,Pic}(\mathfrak{X}) \arrow[r] \arrow[d, equal] & \mathbf{Tro\,Pic}(\mathfrak{X})' \arrow[d, equal] & \mathbf{Tro\,Pic}(\mathfrak{Y}) \arrow[l, "\sim"'] \arrow[d, equal] \\
\Gamma(\mathfrak{X}, \mathrm{B}\mathfrak{L})^\dagger \arrow[r] & \Gamma(\mathfrak{X}, \mathrm{B}\sigma^*\mathfrak{L})^\dagger & \Gamma(\mathfrak{Y}, \mathrm{B}\mathfrak{L})^\dagger \arrow[l]
\end{tikzcd} \tag{3.8.2.4}$$

□

We leave it to the reader to verify that the morphism of Proposition 3.8.2 is compatible with composition of homomorphisms of monoids. We can now define the tropical Picard group in families, using the process described in Section 2.3.1. If $\mathfrak{X}$ is a family of tropical curves over a logarithmic scheme S, we obtain a stack $\mathbf{Tro\,Pic}(\mathfrak{X}/S)$ on the large étale site of S characterized by either of the following two descriptions:

(1) If T is an atomic neighborhood of a geometric point t, then $\mathbf{Tro\,Pic}(\mathfrak{X}/S)(T) = \mathbf{Tro\,Pic}(\mathfrak{X}_t)$.
(2) If T is a logarithmic scheme over S and T is of finite type then an object of $\mathbf{Tro\,Pic}(\mathfrak{X}/S)(T)$ consists of a tropical line bundle Q_t on $\mathfrak{X}_t$ for each geometric point t such that, for any geometric specialization $t \rightsquigarrow t'$, the line bundle $Q_{t'}$ induces Q_t by way of the edge contraction $\mathfrak{X}_{t'} \to \mathfrak{X}_t$ and Proposition 3.8.2.

3.9. **Prorepresentability and subdivisions.** Let $\mathfrak{X}$ be a tropical curve metrized by a monoid $\overline{M}$. We saw in Section 3.6 that the tropical Jacobian can be regarded as a functor of pairs $(\overline{N}, u)$ where $u : \overline{M} \to \overline{N}$ is a homomorphism of monoids. This functor is not representable, as we saw in Section 2.2.7. However, it is not that far from being representable: it is the quotient of a prorepresentable functor by a discrete group.

Lemma 3.9.1. *The functor* $\mathrm{Hom}(H_1(\mathfrak{X}), \mathfrak{P})^\dagger$ *is prorepresentable on* $\overline{M}/\mathbf{Mon}$ *by the system of all submonoids* $\overline{P}$ *of* $\overline{M}^{\mathrm{gp}} + H_1(\mathfrak{X})$ *(direct sum) with the following properties:*

(1) $\overline{P}$ *is finitely generated over* $\overline{M}$*;*
(2) for each $\gamma \in H_1(\mathfrak{X})$ *we have* $\gamma \prec \ell(\gamma)$ *in* $\overline{P}$*.*

Proof. Note that the second property implies that $\overline{P}$ generates $\overline{M}^{\mathrm{gp}} + H_1(\mathfrak{X})$ as a group. Indeed, if $\gamma \prec \ell(\gamma)$ then $\gamma - n\ell(\gamma) \in \overline{P}$ for some integer n; as $\ell(\gamma) \in \overline{M} \subset \overline{P}$, this implies $\gamma \in \overline{P}$.

Let I be the diagram of all $\overline{P}$ with the indicated properties. Let $F = \text{“}\varprojlim\text{”}_{\overline{P} \in I} \mathrm{Hom}(\overline{P}, -)$ be the pro-object they represent. Certainly, if $\overline{P} \in I$ then a homomorphism $\overline{P} \to \overline{N}$ commuting with the morphisms from $\overline{M}$ induces an object of $\mathrm{Hom}(H_1(\mathfrak{X}), \overline{N})^\dagger$ by passing to the associated group. This gives us a morphism $F \to \mathrm{Hom}(H_1(\mathfrak{X}), \mathfrak{P})^\dagger$ that we would like to show is an isomorphism.

Suppose that $\mu : H_1(\mathfrak{X}) \to \overline{N}^{\mathrm{gp}}$ is a homomorphism with bounded monodromy. Combining this with the structural homomorphism $\overline{M} \to \overline{N}$ we get a homomorphism of monoids $\nu : \overline{M} + H_1(\mathfrak{X}) \to \overline{N}^{\mathrm{gp}}$. Choose a basis $e_1, \ldots, e_g$ of $H_1(\mathfrak{X})$. For each i, there are integers n and m such that $n\nu(\ell(e_i)) \leq \nu(e_i) \leq m\nu(\ell(e_i))$ in $\overline{N}^{\mathrm{gp}}$. That is $e_i - n\ell(e_i)$ and $m\ell(e_i) - e_i$ both lie in the preimage of $\overline{N}$ under ν. We take $\overline{P}$ to be the submonoid of $\overline{M} + H_1(\mathfrak{X})$ generated by $\overline{M}$ and the $e_i - n\ell(e_i)$ and $m\ell(e_i) - e_i$. Then, by construction, $\overline{P}$ is finitely generated over $\overline{M}$, generates $\overline{M} + H_1(\mathfrak{X})$ as a group, has bounded monodromy, and induces μ via ν.

This shows that $F \to \mathrm{Hom}(H_1(\mathfrak{X}), \mathfrak{P})^\dagger$ is surjective. To see that it is also injective, consider a second map $\overline{Q} \to \overline{N}$ inducing μ as above, with $\overline{Q} \in I$. Then $\overline{Q} \cap \overline{P}$ is also in I and the map $\overline{Q} \cap \overline{P} \to \overline{N}$ induced from either $\overline{Q} \to \overline{N}$ or $\overline{P} \to \overline{N}$ — they must be the same because the induced maps on associated groups is the same — represents the same object of $F(\overline{N})$. This proves the injectivity and completes the proof. □

Let us now assume that $\overline{M}$ is finitely generated. There is no loss of generality in doing so, since we only care about the set of lengths of the edges of $\mathfrak{X}$, which is in any case a finitely generated submonoid.

It is then dual to a rational polyhedral cone σ, and the category of monoids that are finitely generated relative to $\overline{M}$ is contravariantly equivalent to the category $\mathbf{RPC}/\sigma$ of rational polyhedral cones over σ. These observations permit us to reinterpret Lemma 3.9.1 dually, to the effect that $\mathrm{Hom}(H_1(\mathfrak{X}), \mathfrak{P})^\dagger$ is representable by an ind-object of $\mathbf{RPC}/\sigma$.

Rational polyhedral cones are finitely generated, saturated, convex regions in lattices, so we can interpret ind-rational polyhedral cones as not-necessarily-finitely generated, saturated, convex regions in torsion-free abelian groups. Actually, Lemma 3.9.1 gives a pro-object of $\overline{M}/\mathbf{Mon}$ whose associated group is constant, so that it is represented dually by a saturated, convex region in the lattice $\mathrm{Hom}(\overline{M}^{\mathrm{gp}}, \mathbf{Z}) \times H^1(\mathfrak{X})$. The following corollary specifies which:

Corollary 3.9.2. *The functor* $\mathrm{Hom}(H_1(\mathfrak{X}), \mathfrak{P})^\dagger$ *is ind-representable by the collection* τ *of pairs* $(u, v) \in \mathrm{Hom}(\overline{M}^{\mathrm{gp}}, \mathbf{Z}) \times \mathrm{Hom}(H_1(\mathfrak{X}), \mathbf{Z})$ *such that* $u(\overline{M}) \geq 0$ *and whenever* $u(\ell(\gamma)) = 0$ *for some* $\gamma \in H_1(\mathfrak{X})$*, we also have* $v(\gamma) = 0$.

Proof. Let I denote the pro-$\overline{M}$-monoid consisting of all $\overline{P} \subset \overline{M}^{\mathrm{gp}} \times H_1(\mathfrak{X})$ such that $\overline{P}$ is finitely generated over $\overline{M}$ and $\gamma \prec \ell(\gamma)$ in $\overline{P}$ for all $\gamma \in H_1(\mathfrak{X})$ (as in Lemma 3.9.1). Let J denote the ind-rational polyhedral cone consisting of all (u, v) such that $u(\overline{M}) \geq 0$ and $u(\ell(\gamma)) = 0$ implies $v(\gamma) = 0$. We wish to show that I and J are dual.

Since I is closed under finite intersections and J is closed under finite unions, it is sufficient to demonstrate the duality on the level of rays in $\mathrm{Hom}(\overline{M}^{\mathrm{gp}} \times H_1(\mathfrak{X}), \mathbf{Z})$ and the corresponding half-spaces in $\overline{M}^{\mathrm{gp}} \times H_1(\mathfrak{X})$. That is, we need to show that $(u, v) \in \mathrm{Hom}(\overline{M}^{\mathrm{gp}} \times H_1(\mathfrak{X}), \mathbf{Z})$ has the properties $u(\overline{M}) \geq 0$ and $u(\ell(\gamma)) = 0$ implies $v(\gamma) = 0$ if and only if $\overline{M} \subset (u, v)^\vee$ and $\gamma \prec \ell(\gamma)$ in $(u, v)^\vee$. But this is immediate: $u(\overline{M}) \geq 0$ means precisely that $\overline{M} \subset (u, v)^\vee$; likewise, $\gamma \prec \ell(\gamma)$ in the half-space $(u, v)^\vee$ means either that $u(\ell(\gamma)) = v(\gamma) = 0$ or that $u(\ell(\gamma)) > 0$, which is equivalent to the property that $u(\ell(\gamma)) = 0$ implies $v(\gamma) = 0$. □

The advantage of working with cones instead of monoids is that we can see subdivisions rather explicitly.

Corollary 3.9.3. *Subdivisions of* $\mathrm{Hom}(H_1(\mathfrak{X}), \mathfrak{P})^\dagger$ *by representable functors are in bijection with subdivisions of the cone* τ *defined in Corollary 3.9.2.*

Proof. This is entirely a matter of unwinding definitions. Suppose first that T is a subdivision of τ by rational polyhedral cones. Then for every rational polyhedral cone σ and morphism $\sigma \to \tau$, the fiber product $T \times_\tau \sigma$ is a subdivision of σ. But τ represents $\mathrm{Hom}(H_1(\mathfrak{X}), \mathfrak{P})^\dagger$, so that if h_T is the functor represented by T then $h_T \to \mathrm{Hom}(H_1(\mathfrak{X}), \mathfrak{P})^\dagger$ is representable by subdivisions.

Suppose conversely that $h_T \to \mathrm{Hom}(H_1(\mathfrak{X}), \mathfrak{P})^\dagger$ is representable by subdivisions, where T is a cone complex. For any finitely generated subcone σ of τ, the fiber product $h_\sigma \times_{h_\tau} h_T$ is representable by a subdivision of σ. It is immediate from this that T is a subdivision of τ. □

Corollary 3.9.4. *Subdivisions of* $\mathrm{Tro\,Jac}(\mathfrak{X})$ *by cone spaces* [CCUW17] *correspond to* $H_1(\mathfrak{X})$*-equivariant subdivisions of the cone* τ *of Corollary 3.9.2.*

Proof. This is immediate, since subdivisions of $\mathrm{Tro\,Jac}(\mathfrak{X})$ are the same as $H_1(\mathfrak{X})$-equivariant subdivisions of $\mathrm{Hom}(H_1(\mathfrak{X}), \mathfrak{P})^\dagger$. □

3.10. **Boundedness of moduli.** Our definition of boundedness is a natural adaptation to logarithmic schemes of the schematic notion [Gro95, Définition 1.1].

Definition 3.10.1. A moduli problem F over logarithmic schemes over S is said to be *bounded* if, locally in S, there is a logarithmic scheme T of finite type over S and a morphism $T \to F$ that is surjective on valuative geometric points.

Theorem 3.10.2. *If* $\mathfrak{X}$ *is a compact tropical curve over* S *then* $\mathrm{Tro\,Jac}(\mathfrak{X}/S)$ *is bounded over* S*.*

Proof. Let $\mathfrak{X}$ be the tropicalization of X. The assertion is local to the constructible topology and to the étale topology on S, so we can assume that the logarithmic structure on S has constant characteristic monoid and that the dual graph of X is also constant. After these restrictions, we have the exact sequence (3.10.2.1) by definition of the tropical Jacobian:

$$0 \to H_1(\mathfrak{X}) \xrightarrow{\partial} \operatorname{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})^\dagger \to \operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S) \to 0 \tag{3.10.2.1}$$

Since ∂ requires only a finite number of elements of $\overline{M}_S$ to describe, we may assume that $\overline{M}_S^{\mathrm{gp}}$ is a finitely generated abelian group. We identify a bounded subfunctor of $\operatorname{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})^\dagger$ that surjects onto $\operatorname{Tro}\operatorname{Jac}(X)$.

Choose a basis $e_1, \ldots, e_g$ for $H_1(\mathfrak{X})$. Let $\ell(e_i) = \partial(e_i).e_i$ denote the length of e_i. Take $Z \subset \operatorname{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})^\dagger$ to be the locus of those μ where (3.10.2.2) holds

$$-r(g+1)\ell(e_i) \le \mu(e_i) \le r(g+1)\ell(e_i) \tag{3.10.2.2}$$

for all i, with r denoting the rank of $\overline{M}_S^{\mathrm{gp}}$.

Lemma 3.10.2.3. *Z is bounded.*

Proof. Let W be the base change of the toric map $\mathbf{A}^{2g} \to \mathbf{A}^g : (x_1, \ldots, x_g, y_1, \ldots, y_g) \mapsto (x_1y_1, \ldots, x_gy_g)$ along a map $f : S \to \mathbf{A}^g$ such that $f^*\varepsilon^{-1}t_i$ represents $2r(g+1)\ell(e_i)$ in $\overline{M}_S$. Let $\mu \in \operatorname{Hom}(H_1(\mathfrak{X}), \Gamma(W, \overline{M}_W^{\mathrm{gp}}))$ be the homomorphism such that $\mu(e_i) = \varepsilon^{-1}(x_i) - r(g+1)\ell(e_i)$. This gives a map $W \to Z$ over S.

To see that $W \to Z$ is surjective on valuative geometric points, suppose that $\mu : T \to \operatorname{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})$ be a valuative geometric point of Z. Then let x_i be a section of $\overline{M}_T$ such that $\overline{x}_i = \mu(e_i) + r(g+1)\ell(e_i)$ (where $\overline{x}_i$ is the image of x_i in $\overline{M}_T$). Let $y_i = t_i x_i^{-1}$, which lies a priori in M_T^{gp}, but in fact in M_T because $\overline{x}_i \le 2r(g+1)\ell(e_i)$ by the choice of $f : S \to \mathbf{A}^g$. This lifts $T \to S \to \mathbf{A}^g$ to $\mathbf{A}^{2g}$ and therefore lifts $T \to Z$ to $T \to W$, by definition of the fiber product. $\square$

To complete the proof, we need to show that the valuative geometric points of Z surject onto those of $\operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S)$ under the projection $\operatorname{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})^\dagger \to \operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S)$. We shall therefore assume that $\overline{M}_S$ is valuative. Note that passing to a valuation of $\overline{M}_S$ does not change $\overline{M}_S^{\mathrm{gp}}$, so $\overline{M}_S^{\mathrm{gp}}$ is still finitely generated. Let

$$0 = \overline{N}_0 \subset \overline{N}_1 \subset \cdots \subset \overline{N}_k = \overline{M}_S$$

be the filtration guaranteed by Proposition 2.1.3.8. It is finite because Lemma 2.1.2.6 implies that each $\overline{N}_i$ is determined by its associated subgroup of $\overline{M}_S^{\mathrm{gp}}$, and $\overline{M}_S^{\mathrm{gp}}$ is a finitely generated abelian group, hence noetherian. In fact, the length, k, of this filtration is bounded by the rank, r, of $\overline{M}_S^{\mathrm{gp}}$.

We now proceed by induction on the length of this filtration. We argue that *if μ is an element of $\operatorname{Hom}(H_1(\mathfrak{X}), \overline{N}_i^{\mathrm{gp}})$ with bounded monodromy then there is some $\gamma \in H_1(\mathfrak{X})$ and some ζ such that $-(g+1)\ell(e_j) \le \zeta(e_j) \le (g+1)\ell(e_j)$ for all j and $\mu - \zeta - \partial(\gamma)$ takes values in $\overline{N}_{i-1}$.*

By composition with the homomorphism $q : \overline{N}_i \to \overline{N}_i/\overline{N}_{i-1}$, we obtain a map

$$\operatorname{Hom}(H_1(\mathfrak{X}), \overline{N}_i^{\mathrm{gp}})^\dagger \to \operatorname{Hom}(H_1(\mathfrak{X}), \overline{N}_i^{\mathrm{gp}}/\overline{N}_{i-1}^{\mathrm{gp}})^\dagger. \tag{3.10.2.4}$$

The important point here is that if $\mu \in \operatorname{Hom}(H_1(\mathfrak{X}), \overline{N}_i^{\mathrm{gp}})$ has bounded monodromy then $q\mu$ also has bounded monodromy, in the sense that $q\mu(\alpha) \prec q\ell(\alpha)$ where $\ell(\alpha) \in \overline{M}_S$ denotes the length of α. Let us write $\overline{\mu}$ for the image of μ in $\overline{N}_i^{\mathrm{gp}}/\overline{N}_{i-1}^{\mathrm{gp}}$ and $\overline{\ell}$ for the length function taking values in $\overline{N}_i^{\mathrm{gp}}/\overline{N}_{i-1}^{\mathrm{gp}}$.

By Proposition 2.1.3.8, the totally ordered abelian group $\overline{N}_i^{\mathrm{gp}}/\overline{N}_{i-1}^{\mathrm{gp}}$ is archimedean, hence admits an order preserving inclusion in $\mathbf{R}$ by Theorem 2.1.3.6. Since $\overline{\mu} \in \mathrm{Hom}(H_1(\mathfrak{X}), \mathbf{R})$, it can be written $\overline{\mu} = \alpha + \partial(\gamma)$ for some $\gamma \in H_1(\mathfrak{X})$ and some $\alpha = \sum a_i \partial(e_i)$ with $0 \leq a_i \leq 1$ for all i. Now, evaluating α on e_j, we get (3.10.2.5):

$$\alpha(e_j) = \sum_{i=1}^{g} a_i \partial(e_i).e_j \tag{3.10.2.5}$$

But we have $-\ell(e_j) \leq \partial(e_i).e_j \leq \ell(e_j)$ for all i and j, so we obtain

$$-g\overline{\ell}(e_j) \leq \alpha(e_j) \leq g\overline{\ell}(e_j) \tag{3.10.2.6}$$

for all j.

Note now that $\alpha = \overline{\mu} - \partial(\gamma)$, which is in $\mathrm{Hom}(H_1(\mathfrak{X}), \mathbf{R})$ by construction, is actually in the image of $\mathrm{Hom}(H_1(\mathfrak{X}), \overline{N}_i^{\mathrm{gp}}/\overline{N}_{i-1}^{\mathrm{gp}})$. We can lift α to some $\zeta \in \mathrm{Hom}(H_1(\mathfrak{X}), \overline{N}_i^{\mathrm{gp}}) \subset \mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}_S^{\mathrm{gp}})$. In fact, we can arrange for ζ to lie in $\mathrm{Hom}(H_1(\mathfrak{X}), \overline{N}_i^{\mathrm{gp}})^\dagger$ by defining $\zeta(e_j) = 0$ if $\ell(e_j)$ lies in $\overline{N}_{i-1}$; if $\ell(e_j)$ does not lie in $\overline{N}_{i-1}^{\mathrm{gp}}$ then all of $\overline{N}_i^{\mathrm{gp}}$ is bounded by $\ell(e_j)$ and there is nothing to check.

The inequalities (3.10.2.6) lift to (3.10.2.7)

$$-g\ell(e_j) - u \leq \zeta(e_j) \leq g\ell(e_j) + v \tag{3.10.2.7}$$

for some $u, v \in \overline{N}_{i-1}$. Now, $\overline{\ell}(e_j)$ is a positive element of $\overline{N}_i^{\mathrm{gp}}/\overline{N}_{i-1}^{\mathrm{gp}}$ and both u and v lie in $\overline{N}_{i-1}$, so u and v are both dominated by $\ell(e_j)$ by Lemma 2.1.3.7. In particular, $u \leq \ell(e_j)$ and $v \leq \ell(e_j)$. Substituting this into (3.10.2.7), we obtain (3.10.2.8),

$$-(g+1)\ell(e_j) \leq \zeta(e_j) \leq (g+1)\ell(e_j) \tag{3.10.2.8}$$

as desired.

We have now shown that $\mu - \partial(\gamma) - \zeta$ takes values in $\overline{N}_{i-1}^{\mathrm{gp}}$. Repeating this process once for each of the k steps of the filtration 3.10.2.4, we obtain $\mu - \sum \partial(\gamma_i) - \sum_{i=1}^k \zeta_i = 0$. Thus $\zeta = \sum \zeta_i$ represents μ in $\mathrm{Tro\,Jac}(\mathfrak{X}/S)$ and, as each ζ_i satisfies (3.10.2.8) and $k \leq r$, their sum satisfies (3.10.2.2), so $\zeta \in Z(S)$. □

Corollary 3.10.3. *Let $\mathfrak{X}$ be a compact tropical curve over S. Then* $\mathrm{Tro\,Pic}^d(\mathfrak{X}/S)$ *is quasi-compact over S for all $d \in H_0(\mathfrak{X})$.*

Proof. As $\mathrm{Tro\,Pic}^d(\mathfrak{X}/S)$ is a torsor under $\mathrm{Tro\,Pic}^0(\mathfrak{X}/S)$ by Corollary 3.4.7, it is sufficient to assume $d = 0$. But $\mathrm{Tro\,Pic}^0(\mathfrak{X}/S) = \mathrm{Tro\,Jac}(\mathfrak{X}/S)$ by Corollary 3.10.3, so the conclusion follows from Theorem 3.10.2. □

3.11. **Boundedness of the diagonal.** The main point of this section is to demonstrate that the lattice defined by a positive definite matrix of real numbers is discrete and that this is also valid as the lattice varies in a tropical family. We make use of the tropical topology defined in Section 2.3.2.

These results are also demonstrated method as part of the proof of [KKN08b, Proposition 4.5]. The proof appears in [KKN08c, Lemma 5.2.7] and [KKN08b, Section 9.4]. Unlike the present proof, that proof does not rely on the tropical topology, but it ultimately comes down to a compactness argument, as this one does.

Definition 3.11.1. Let $\partial : A \to \mathrm{Hom}(A, \overline{M}^{\mathrm{gp}})$ be a pairing on a finitely generated free abelian group A, valued in a partially ordered abelian group $\overline{M}^{\mathrm{gp}}$. We say that ∂ is *positive definite* if, for all $\gamma, \gamma' \in A$, we have $\partial(\gamma).\gamma' \prec \partial(\gamma).\gamma$.

Lemma 3.11.2. *Let A be a finitely generated free abelian group, let V be an totally ordered abelian group, and let $\partial : A \to \mathrm{Hom}(A, V)$ be a positive definite pairing. Then there is a basis $e_1, \ldots, e_g$ of A, a positive $\delta \in \mathbf{Q}V$ and a cover of $\mathbf{R}A$ by rational polyhedral cones σ_i and $-\sigma_i$ such that, for each i, we have $\partial(e_i).\gamma > \delta$ for all nonzero $\gamma \in \sigma_i \cap A$.*

Proof. We assume first that V is archimedean, so that it admits an order prereserving embedding in $\mathbf{R}$. Then ∂ defines a positive definite pairing on A valued in $\mathbf{R}$. The image of (3.11.2.1) is a discrete lattice.

$$\partial : A \to \mathrm{Hom}(A, \mathbf{R}) \tag{3.11.2.1}$$

We choose $\epsilon \in \mathbf{R}$ such that, for every $\gamma \in A$, we have $|\partial(\gamma).e_i| > \epsilon$ for some index i. We may find a positive $\delta \in \mathbf{Q}V$ such that $\delta < \epsilon$.

Now we consider the possibility that V is merely totally ordered. If necessary, we may replace V by the subgroup generated by the image of the pairing $A \times A \to V$, and thus assume that V is finitely generated. It therefore has a finite filtration (3.11.2.2) with archimedean quotients V_p/V_{p+1}.

$$V = V_0 \supset V_1 \supset \cdots \supset V_n \supset V_{n+1} = 0 \tag{3.11.2.2}$$

For each p, let A_p be the subgroup of $\gamma \in A$ such that $\partial(\gamma).\gamma \in V_p$. Since ∂ is positive definite, this implies that $\partial(\gamma).\gamma' \in V_p$ for all $\gamma' \in A$. Therefore ∂ descends to a positive definite pairing ∂_p on A_p/A_{p+1}, valued in the archimedean group V_p/V_{p+1}, for each p:

$$\partial_p : A_p/A_{p+1} \times A_p/A_{p+1} \to V_p/V_{p+1} \tag{3.11.2.3}$$

Choose a basis $e_1, \ldots, e_g$ of A such that the e_i in $A_p - A_{p+1}$ induce a basis of A_p/A_{p+1} for each p. For each p, choose a $\delta_p \in V_p - V_{p+1}$ such that, for every $\gamma \in A_p - A_{p+1}$, there is an $e_i \in A_p - A_{p+1}$ such that $|\partial_p(\gamma).e_i| > \delta_p \pmod{V_{p+1}}$. This implies that, for any nonzero $\gamma \in A_p$, there is some e_i such that

$$\partial(\gamma).e_i > \delta_p - \epsilon$$

for some $\epsilon \in V_{p+1}$. But both δ_{p+1} and ϵ lie in V_{p+1}, so $\delta_{p+1} + \epsilon < \delta_p$. Therefore $\partial(\gamma).e_i > \delta_{p+1}$ as well. By induction, we find that there is some i such that $\partial(\gamma).e_i > \delta_n$. We take $\delta = \delta_n$.

We construct the σ_i by induction, beginning with $\sigma_i^{(0)} = 0$ for all i. We construct $\sigma_i^{(k+1)}$ from $\sigma_i^{(k)}$ by selecting a vector $\gamma \in A$ that is not in the interior of any $\pm\sigma_i^{(k)}$ and adding γ to $\sigma_i^{(k)}$ for some i such that $|\partial(\gamma).e_i| > \delta$ (replacing γ with $-\gamma$ if necessary). Then the real projective space $P = (\mathbf{R}A - \{0\})/\mathbf{R}^*$ is the union of the interiors of the subsets $\big((\sigma_i^{(k)} \smallsetminus \{0\}) \cup (-\sigma_i^{(k)} \smallsetminus \{0\})\big)/\mathbf{R}^*$. But P is quasicompact, so finitely many of these $\sigma_i^{(k)}$ suffice and therefore A is the union of the lattice points in the interiors of the $\pm\sigma_i^{(k)}$, for any sufficiently large k. $\square$

Suppose that $\partial : A \to \mathrm{Hom}(A, \overline{M}^{\mathrm{gp}})$. If $\overline{M} \to \overline{N}$ is a monoid homomorphism, we denote by $\partial_{\overline{N}}$ the induced pairing valued in $\overline{N}^{\mathrm{gp}}$. Likewise, if $\phi : A \to \overline{M}^{\mathrm{gp}}$ is a homomorphism, we denote by $\phi_{\overline{N}}$ the induced homomorphism valued in $\overline{N}^{\mathrm{gp}}$. when $V = \overline{N}^{\mathrm{gp}}$, we also write $\partial_V = \partial_{\overline{N}}$ and $\phi_V = \phi_{\overline{N}}$.

Corollary 3.11.3. *Let A be a finitely generated abelian group, let $\overline{M}$ be a sharp monoid, let $\partial : A \to \mathrm{Hom}(A, \overline{M}^{\mathrm{gp}})$ be a positive definite pairing, and let $\phi : A \to \overline{M}^{\mathrm{gp}}$ be a homomorphism. Let V be a valuation of $\overline{M}$ such that $\phi_V = 0$. Then there is an open neighborhood U of V in $\mathbf{Cone}^\circ(\overline{M})$ such that if $\phi_W = \partial_W(\gamma)$ for some $W \in U$ then $\gamma = 0$.*

Proof. We apply Lemma 3.11.2 to obtain a basis $e_1, \ldots, e_g$ of A, a positive $\delta \in \mathbf{Q}V = \mathbf{Q}\overline{M}^{\mathrm{gp}}$, and rational polyhedral cones $\sigma_i \subset \mathbf{R}A$ such that $\partial(\gamma).e_i > \delta$ for all nonzero $\gamma \in \sigma_i \cap A$. Note that, for each i, the monoid $\sigma_i \cap A$ is finitely generated. We choose a finite set of generators B_i. Then we define U by the inequalities (3.11.3.1) and (3.11.3.2):

$$\delta > 0 \tag{3.11.3.1}$$

$$\partial(\beta).e_i < \delta \qquad \text{for all } \beta \in B_i \tag{3.11.3.2}$$

By definition, these inequalities hold in V. Moreover, if $W \in U$ and $\gamma \in A$ is nonzero then $\pm\gamma$ lies in some σ_i. We assume that $\gamma \in \sigma_i$, the other possibility being similar. Then γ is a finite, nonnegative linear combination of the elements of B_i, with at least one positive coefficient (since $\gamma \neq 0$), say of β. Therefore $\partial(\gamma).e_i \geq \partial(\beta).e_i > \delta$. We conclude that the only solution over W is $\gamma = 0$, as required. $\square$

Corollary 3.11.4. *Let $\overline{M}$ be a sharp monoid, let A be a finitely generated free abelian group, let $\partial : A \to \mathrm{Hom}(A, \overline{M}^{\mathrm{gp}})$ be a positive definite pairing, and let $\phi : A \to \overline{M}^{\mathrm{gp}}$ be a homomorphism. Suppose that V is the value group of a valuation of $\overline{M}$ such that ϕ_V is not in the image of $\partial_V : A \to \mathrm{Hom}(A, V)$. Then there is an open neighborhood U of V in $\mathbf{Cone}^\circ(\overline{M})$ such that $\partial^{-1}(U) = \varnothing$ (that is, there is no $W \in U$ such that ϕ_W is in the image of ∂_W).*

Proof. Choose a finite set of generators B_i for each of the N finitely generated semigroups $\sigma_i \cap A$ guaranteed by Lemma 3.11.2. For notational convenience, set $\sigma_{N+i} = -\sigma_i$ and $B_{N+i} = -B_i$ so that the σ_i cover A. Choose a partition of $\mathbf{R}A$ into regions τ_i, and an $\epsilon \in \mathbf{Q}V = \mathbf{Q}\overline{M}$, with the following properties:

(1) $\epsilon > 0$;
(2) for each i, there is a finite subset $C_i \subset \tau_i \cap A$ such that every element of $\tau_i \cap A$ can be written as a sum of an element of C_i and a finite number of elements of B_i;
(3) for all $\gamma \in \tau_i \cap A$, we have $|\partial_V(\gamma).e_i - \phi_V(e_i)| > \epsilon$.

These data may be constructed by the same inductive procedure used in the proof of Lemma 3.11.2 to construct the σ_i.

As γ ranges over C_i, we obtain finitely many inequalities $|\partial(\gamma).e_i - \phi(e_i)| > \epsilon$. These, together with $\epsilon > 0$, determine an open neighborhood U of V in $\mathbf{Cone}^\circ(\overline{M})$.

Moreover, if $W \in U$, and $\gamma \in A$, then γ lies in some $\tau_i \cap A$, by definition. Therefore $\gamma = c + \sum b_j$ for some $c \in C_i$ and $b_j \in B_i$. In particular,

$$|\partial_W(\gamma).e_i - \phi_W(e_i)| \geq |\partial_W(c).e_i - \phi_W(e_i)| > \epsilon$$

so $\partial_W(\gamma) \neq \phi_W$. This holds for all $\gamma \in A$ and all $W \in U$. $\square$

Corollary 3.11.5. *Let A be a finitely generated abelian group, let $\overline{M}$ be a sharp monoid, and let $\partial : A \to \mathrm{Hom}(A, \overline{M}^{\mathrm{gp}})$ be a positive definite pairing. Then $A \to \mathrm{Hom}(A, \overline{\mathbf{G}}_m^{\log})$ is of finite type.*

Proof. The assertion is, in other words, that for any logarithmic scheme of finite type and any morphism $\phi : S \to \mathrm{Hom}(A, \overline{\mathbf{G}}_m^{\log})$, the fiber product $A \times_{\mathrm{Hom}(A, \overline{\mathbf{G}}_m^{\log})} S$ is quasicompact.

This assertion is local in the constructible and étale topologies on S, so we can assume S is connected and has a constant logarithmic structure, the stalk of whose characteristic monoid is $\overline{M}$. We therefore regard ϕ as a homomorphism $A \to \overline{M}^{\mathrm{gp}}$.

Let V be a sharp valuation of $\overline{M}$. By Corollary 3.11.4, there is a maximal open subset U of $\mathbf{Cone}^\circ(\overline{M})$ such that ϕ_W is not in $\partial_W(A)$ for any $W \in U$. By Corollary 3.11.3, there is, for each $\gamma \in A$, an open subset U_γ such that, for all $W \in U_\gamma$, if ϕ_W is in the image of ∂ then $\phi_W = \partial(\gamma)$. Since $\mathbf{Cone}^\circ(\overline{M})$ is quasicompact, finitely many of the open subsets U_γ and U suffice to cover it. We label these U_i. Refining them if necessary, we can assume they are all basic open sets, meaning they are defined by finitely many strict inequalities.

Each open set U_i determines a subfunctor of the functor represented by $\overline{M}$ on logarithmic schemes, and this subfunctor is representable by a logarithmic S-scheme T of finite type. In fact, the underlying scheme of T is isomorphic to that of S. Letting $\overline{N} \subset \overline{M}^{\mathrm{gp}}$ be the finitely generated extension of $\overline{M}$ that represents U, we may construct $M_T = \overline{N} \times \mathcal{O}_T^*$ and define $\varepsilon(\alpha, \lambda) = 0$ for $\alpha \neq 0$.

Since the disjoint union of the logarithmic schemes T covers S and is quasicompact, it is sufficient to demonstrate the corollary after replacing S by one of the T. We can therefore assume that $\mathbf{Cone}^\circ(\overline{M}) = U$ or that $\mathbf{Cone}^\circ(\overline{M}) = U_\gamma$ for some γ. In the first case, $S \times_{\mathrm{Hom}(A, \overline{\mathbf{G}}_m^{\log})} A = \varnothing$, which is certainly bounded.

In the second case, $S \times_{\mathrm{Hom}(A, \overline{\mathbf{G}}_m^{\log})} A$ is defined as a subfunctor of S by the finitely many conditions $\phi(e_i) - \partial(\gamma).e_i = 0$, as the e_i run over a basis of A. By Proposition 2.2.7.5, each of these finitely many conditions is satisfied on a universal S-scheme of finite type. $\square$

Corollary 3.11.6. *Let $\mathfrak{X}$ be a tropical curve metrized by $\overline{M}$. Then the intersection pairing* (3.11.6.1) *is bounded.*

$$\partial : H_1(\mathfrak{X}) \to \mathrm{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log}) \tag{3.11.6.1}$$

Proof. The intersection pairing is positive definite. $\square$

4. The logarithmic Picard group

Suppose that X is a proper, vertical logarithmic curve over S where the underlying scheme of S is the spectrum of an algebraically closed field, and let $\mathfrak{X}$ be the tropicalization of X. Then $H^1(X, \overline{\mathbf{G}}_m^{\log}) = H^1(X, \overline{M}_X^{\mathrm{gp}}) = H^1(\mathfrak{X}, \mathfrak{P})$ because $\overline{M}_X^{\mathrm{gp}}$ is a sheaf of torsion-free abelian groups. If $\overline{Q}$ is an $\overline{M}_X^{\mathrm{gp}}$-torsor on X then we say $\overline{Q}$ has *bounded monodromy* if the corresponding $\mathfrak{P}$-torsor on $\mathfrak{X}$ does. If Q is an $\mathbf{G}_m^{\log}$-torsor on X then we say Q has bounded monodromy if its induced $\overline{M}_X^{\mathrm{gp}}$-torsor has bounded monodromy.

Definition 4.1. Let X be a proper, vertical logarithmic curve over S. A *logarithmic line bundle* on X is a $\mathbf{G}_m^{\log}$-torsor on X in the strict étale topology whose fibers have bounded monodromy. Let $\mathbf{Log\,Pic}(X/S)$ be the category fibered in groupoids on logarithmic schemes over S whose T-points are the logarithmic line bundles on X_T. We write $\mathrm{Log\,Pic}(X/S)$ for its associated sheaf of isomorphism classes.

4.2. Local finite presentation.

Definition 4.2.1. A category fibered in groupoids F over logarithmic schemes is said to be *locally of finite presentation* if, for any cofiltered system of affine logarithmic schemes S_i, the map

$$\varinjlim F(S_i) \to F(\varprojlim S_i)$$

is an equvialence of categories.

Local finite presentation is important because it allows us to limit our attention to logarithmic schemes of finite type.

Proposition 4.2.2. *Suppose X is a proper, vertical logarithmic curve over S. Then* $\mathbf{Log\,Pic}(X/S)$ *is locally of finite presentation over S.*

Proof. We begin by proving the essential surjectivity part of Definition 4.2.1 for the functor $\pi_* \mathrm{B}\mathbf{G}_m^{\log}$. The full faithfulness is similar but easier, and we omit it. Then we prove that the bounded monodromy condition defining $\mathbf{Log\,Pic}(X/S)$ inside $\pi_* \mathrm{B}\mathbf{G}_m^{\log}$ is locally of finite presentation.

The assertion is local in S, so we assume S is quasicompact and quasiseparated. Consider a cofiltered inverse system of affine logarithmic schemes S_i over S. Let X_i be the base change of X to S_i. Let L be a logarithmic line bundle over $Y = \varprojlim X_i$. Then there is an étale cover U_j of Y over which L can be trivialized. We can assume that the U_j are all quasicompact and quasiseparated. We note that Y is quasicompact and quasiseparted because all the X_i were. In particular, we can arrange for the U_j to be finite in number. By [GD67, Théorème IV.8.8.2], they are induced by maps $U_{ij} \to X_i$ for some index i. These maps can be assumed étale by [GD67, Proposition IV.17.7.8] and surjective by [GD67, Théorème IV.8.10.5].

The transition functions defining L come from $\Gamma(U_{jk}, M_Y^{\mathrm{gp}}) = \varinjlim_i \Gamma(U_{ijk}, M_{X_i}^{\mathrm{gp}})$, so are induced from transition functions over U_{ijk} for some sufficiently large i. Likewise, the cocycle condition is checked in $\Gamma(U_{jk\ell}, M_Y^{\mathrm{gp}}) = \varinjlim_i \Gamma(U_{ijk\ell}, M_{X_i}^{\mathrm{gp}})$ and is therefore valid for a sufficiently large i. Then L is induced from X_i.

It remains to verify that the bounded monodromy condition is locally of finite presentation. That is, we assume that we have a cofiltered inverse system of affine logarithmic schemes S_i over S, as before, and that $\alpha_i \in H^1(X_i, \overline{M}_{X_i}^{\mathrm{gp}})$. We assume that their limit $\beta \in H^1(Y, \overline{M}_Y^{\mathrm{gp}})$ has bounded monodromy and we prove the same for a sufficiently large α_i.

There is a finite stratification of S into locally closed subschemes such that $\overline{M}_S$ is locally constant on each stratum. Since the bounded monodromy condition is checked on geometric points, we can replace S with one of its strata and assume $\overline{M}_S$ is constant. Now replacing S by an étale cover, we can assume $\overline{M}_S$ is constant and that the dual graph $\mathfrak{X}$ of X is constant as well.

Using the exact sequence (4.2.2.1)

$$R^1\pi_*\pi^*\overline{M}_S^{\mathrm{gp}} \to R^1\pi_*\overline{M}_X \to R^1\pi_*\overline{M}_{X/S} = 0 \tag{4.2.2.1}$$

we can lift α to $\tilde{\alpha} \in H^1(X, \pi^*\overline{M}_S^{\mathrm{gp}}) = \mathrm{Hom}(H_1(\mathfrak{X}), \overline{M}_S^{\mathrm{gp}})$. The bounded monodromy condition for $\tilde{\alpha}$ can be checked by evaluating it on each of the finitely many generators of $H_1(\mathfrak{X})$, and for any one γ in $H_1(\mathfrak{X})$, we can see that $\tilde{\alpha}(\gamma)$ is bounded by $\ell(\gamma)$ in $\varinjlim \Gamma(S_i, \overline{M}_{S_i}^{\mathrm{gp}})$ if and only if it is bounded in $\Gamma(S_i, \overline{M}_{S_i}^{\mathrm{gp}})$ at some finite stage. This completes the proof. ☐

Corollary 4.2.3. *Suppose X is a proper, vertical logarithmic curve over S. Then the sheaf* $\mathrm{Log\,Pic}(X/S)$ *is locally of finite presentation over S.*

Proof. We can assume without loss of generality that X has connected fibers over S. Then $\mathbf{Log\,Pic}(X/S)$ is a gerbe over $\mathrm{Log\,Pic}(X/S)$ banded by $\mathbf{G}_m^{\log}$. Locally in S, this gerbe admits a section, making $\mathrm{Log\,Pic}(X/S)$ into a $\mathbf{G}_m^{\log}$-torsor over $\mathbf{Log\,Pic}(X/S)$. But $\mathbf{G}_m^{\log}$ is certainly locally of finite presentation and $\mathbf{Log\,Pic}(X/S)$ is locally of finite presentation over S by Proposition 4.2.2, so $\mathrm{Log\,Pic}(X/S)$ is locally of finite presentation over S, as required. ☐

4.3. **Line bundles on subdivisions.** The following statement is a corollary of Proposition 3.5.1. It says, effectively, that logarithmic line bundles can be represented by line bundles locally in the logarithmic étale topology.

Corollary 4.3.1. *Let X be a proper, vertical logarithmic curve over a logarithmic scheme S. A class $\alpha \in H^1(X, \overline{M}_X^{\mathrm{gp}})$ has bounded monodromy in the geometric fibers if and only if, étale-locally in S, we can find a logarithmic modification $\tilde{S} \to S$ and a model $\tilde{X}$ of X over $\tilde{S}$ such that $\alpha\big|_{\tilde{X}} = 0$.*

Proof. Replacing S by an étale cover, we can assume S is affine. We can then assume S is of finite type because the moduli space of logarithmic curves is locally of finite presentation and $\mathbf{Log\,Pic}(X/S)$ is locally of finite presentation (Proposition 4.2.2). Passing to a finer étale cover if necessary, we can arrange for S to be atomic (Proposition 2.2.2.4) and for the dual graph of X to be constant over the closed stratum. In particular, S is quasicompact.

Let S^{val} be the limit over all logarithmic modifications of S. This is a locally ringed space and its logarithmic structure is valuative. By Proposition 3.5.1, for each point s of S^{val}, we can find a subdivision $\mathfrak{Y}_s$ of $\mathfrak{X}_s$ to which the restriction of α is zero. If Y_s denotes the corresponding logarithmic modification of X_s then α restricts to 0 on Y_s.

The subdivision $\mathfrak{Y}_s$ only requires a finite number of elements of $\overline{M}_{S^{\mathrm{val}},s}$ that are not already in $\overline{M}_{S,s}$, so it is possible to recover $\mathfrak{Y}_s$ and Y_s as pulled back from a logarithmic modification Y_1 of X over a logarithmic modification S_1 of S. Moreover, there is an open neighborhood U_1 of s in S_1 where $\alpha\big|_{Y_1 \times_{S_1} U_1} = 0$.

Since S^{val} is quasicompact, the preimages of finitely many of these open neighborhoods U_i suffice to cover S^{val}. Let T be the fiber product of the finitely many logarithmic modifications S_i of S. Let $\mathfrak{Z}$ and Z be the common subdivision of the $\mathfrak{Y}_i\big|_T$ and the corresponding logarithmic modification of X over T, respectively. Then the U_i pull back to an open cover of T, from which it follows that $\alpha\big|_Z = 0$. ☐

Proposition 4.3.2. *Let X be a logarithmic curve over S. The following conditions are equivalent for a $\mathbf{G}_m^{\log}$-torsor P on X:*

1. *P has bounded monodromy.*
2. *In each valuative geometric fiber of S, there is a model Y of X where P is induced from a $\mathcal{O}_Y^*$-torsor.*
3. *Étale-locally in S there is a logarithmic modification $\tilde{S} \to S$ and a model $\tilde{X}$ of X over $\tilde{S}$ such that the restriction of P to $\tilde{X}$ is representable by a $\mathcal{O}_X^*$-torsor.*

Proof. From the exact sequence (4.3.2.1),

$$H^1(X, \mathcal{O}_X^*) \to H^1(X, M_X^{\mathrm{gp}}) \to H^1(X, \overline{M}_X^{\mathrm{gp}}) \tag{4.3.2.1}$$

to find a $Y \to X$ where P is representable by a $\mathcal{O}_Y^*$-torsor is equivalent to finding a cover where the class of P in $H^1(X, \overline{M}_X^{\mathrm{gp}})$ is trivial. With this observation, the equivalence of the first two conditions is Proposition 3.5.1 and the equivalence of the first and last conditions is Corollary 4.3.1. ☐

4.4. **Logarithmic étale descent.** By definition, $\mathbf{Log\,Pic}(X/S)$ is a stack in the étale topology. We show here that it is in fact a stack in the *logarithmic* étale topology. As the logarithmic étale topology is generated by étale covers, logarithmic modifications, and root stack constructions we still need to check descent along logarithmic modifications and root stacks.

The following theorem was proved for logarithmic modifications of elliptic curves by K. Kato [Kat, Section 2.2.4]:

Theorem 4.4.1. *Suppose that $\tau : Y \to X$ is a logarithmic modification or a root stack of a logarithmic scheme. Then the maps*

$$\tau^* : H^i(X, M_X^{\mathrm{gp}}) \to H^i(Y, M_Y^{\mathrm{gp}})$$

are isomorphisms for $i = 0, 1$.

Proof of Theorem 4.4.1 for logarithmic modifications. We will show that $M_X^{\mathrm{gp}} \to \mathrm{R}\tau_* M_Y^{\mathrm{gp}}$ is an isomorphism. The theorem will follow from the Leray spectral sequence. The claim is local in the étale topology on X, so we may assume that X has a global chart by an affine toric variety with cone σ and Y is the base change of a subdivision of σ with fan Σ.

If $\rho : Z \to Y$ is another logarithmic modification such that we can prove the theorem for $\rho\tau$ and ρ then we will have

$$\mathrm{R}^{\leq 1}\tau_* M_Y^{\mathrm{gp}} = \mathrm{R}^{\leq 1}\tau_* \mathrm{R}^{\leq 1}\rho_* M_Z^{\mathrm{gp}} = \mathrm{R}^{\leq 1}(\tau\rho)_* M_Z^{\mathrm{gp}} = M_X^{\mathrm{gp}}.$$

Every subdivision τ has a refinement by subdividing along hyperplanes such that $\tau\rho$ is also a subdivision along hyperplanes: given a subdivision of σ, one refines it by subdividing along all hyperplanes spanned by codimension 1 faces of σ. Therefore it is sufficient to prove the theorem for subdivisions along hyperplanes. By induction, we can then assume that Σ is the subdivision of σ along a single hyperplane. In other words, there is some $\alpha \in \Gamma(X, M_X^{\mathrm{gp}})$ such that Y is the union of the subfunctors where $\alpha \geq 0$ and $\alpha \leq 0$. Phrased still another way, Y is the base change of $\mathbf{P}^1$ along $\alpha : X \to \mathbf{G}_m^{\log}$ (see Lemma 2.2.7.3).

We use the commutative diagram of exact sequences (4.4.1.1) on X:

(4.4.1.1)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X^* \arrow[r] \arrow[d] & M_X^{\mathrm{gp}} \arrow[r] \arrow[d] & \overline{M}_X^{\mathrm{gp}} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \tau_*\mathcal{O}_Y^* \arrow[r] & \tau_* M_Y^{\mathrm{gp}} \arrow[r] & \tau_*\overline{M}_Y^{\mathrm{gp}} \arrow[r] & R^1\tau_*\mathcal{O}_Y^* \arrow[r] & R^1\tau_* M_Y^{\mathrm{gp}} \arrow[r] & R^1\tau_*\overline{M}_Y^{\mathrm{gp}}
\end{tikzcd}$$

The vertical arrows are clearly injections. Taking quotients along the vertical arrows and using the snake lemma, we get the exact sequence (4.4.1.2):

$$0 \to \tau_* M_Y^{\mathrm{gp}}/M_X^{\mathrm{gp}} \to \tau_*\overline{M}_Y^{\mathrm{gp}}/\overline{M}_X^{\mathrm{gp}} \to R^1\tau_*\mathcal{O}_Y^* \to R^1\tau_* M_Y^{\mathrm{gp}} \to R^1\tau_*\overline{M}_Y^{\mathrm{gp}} \to \cdots \tag{4.4.1.2}$$

We argue first that $R^1\tau_*\overline{M}_Y^{\mathrm{gp}} = 0$. By proper base change for étale cohomology, which implies that the base change map is injective for H^1 [Art73, Théorème 5.1 (ii)], it is sufficient to prove that

$$H^1(\tau^{-1}p, \overline{M}_Y^{\mathrm{gp}}) = 0$$

for all geometric points p of X. The fiber $Z = \tau^{-1}p$ is either a point (in which case the assertion is trivial) or it is isomorphic to $\mathbf{P}^1$, in which case we use the exact sequence (4.4.1.3):

$$0 \to \pi^{-1}\overline{M}_s^{\mathrm{gp}} \to \overline{M}_Z^{\mathrm{gp}} \to \overline{M}_{Z/s}^{\mathrm{gp}} \to 0 \tag{4.4.1.3}$$

We have $H^1(Z, \pi^{-1}\overline{M}_s^{\mathrm{gp}}) = 0$ since $\pi^{-1}\overline{M}_s^{\mathrm{gp}}$ is constant and Z is simply connected. We have $H^1(Z, \overline{M}_{Z/s}^{\mathrm{gp}}) = 0$ since $\overline{M}_{Z/s}$ is concentrated in dimension 0. Combined with proper base change, this gives $R^1\tau_*\overline{M}_Y^{\mathrm{gp}} = 0$.

Now we compute $R^1\tau_*\mathcal{O}_Y^*$. It vanishes except at those points p where $\tau^{-1}p$ is 1-dimensional. At such a point, we have $H^1(\tau^{-1}p, \mathcal{O}_Y^*) = \mathbf{Z}$. As $H^1(\tau^{-1}p, \mathcal{O}_Y) = H^2(\tau^{-1}p, \mathcal{O}_Y) = 0$, there is no obstruction to extending a class in $H^1(\tau^{-1}p, \mathcal{O}_Y^*)$ infinitesimally, and such infinitesimal extensions are unique. By Grothendieck's existence theorem, we can extend all the way to a formal neighborhood of p. It therefore follows that the stalk of $R^1\tau_*\mathcal{O}_Y^*$ at p is $H^1(\tau^{-1}p, \mathcal{O}_Y^*) = \mathbf{Z}$.

Next we compute $\tau_*\overline{M}_Y^{\mathrm{gp}}$. The stalk at a geometric point p can be identified with the piecewise linear $\overline{M}_s^{\mathrm{gp}}$-valued functions on the dual graph of $\tau^{-1}p$, which is the toric fan of $\mathbf{P}^1$. Such a function can be described by a value at the central point, along with integers representing the slopes along the two incident edges. For convenience, we give these two edges the same orientation, so a function is linear if the two slopes are the same. The stalk of $\tau_*\overline{M}_Y^{\mathrm{gp}}$ at p is therefore $\overline{M}_s^{\mathrm{gp}} \times \mathbf{Z}^2$ where s is the image of p in S.

We can likewise identify the stalk of $\overline{M}_X^{\mathrm{gp}}$ with the linear functions on the same graph. Under the identification of $\tau_*\overline{M}_Y^{\mathrm{gp}}$ with $\overline{M}_s^{\mathrm{gp}} \times \mathbf{Z}^2$, the subgroup $\overline{M}_X^{\mathrm{gp}} \subset \tau_*\overline{M}_Y^{\mathrm{gp}}$ goes over to $\overline{M}_s^{\mathrm{gp}} \times \Delta\mathbf{Z}$. The map

$$\overline{M}_s^{\mathrm{gp}} \times \mathbf{Z}^2 \to \mathbf{Z}$$

sends (λ, c_1, c_2) to $c_2 - c_1$. This induces a bijection

$$\mathbf{Z}^2/\Delta\mathbf{Z} \to \mathbf{Z}$$

which implies that $\tau_*(\overline{M}_Y^{\mathrm{gp}})/\overline{M}_X^{\mathrm{gp}} \to R^1\tau_*\mathcal{O}_Y^*$ is an isomorphism. We deduce that $\tau_*(M_Y^*)/M_X^* = R^1\tau_*M_Y^* = 0$, as required. ☐

Proof of Theorem 4.4.1 for root stacks. As in the proof for subdivisions, it is sufficient to demonstrate that $M_X^{\mathrm{gp}} \to \tau_*M_Y^{\mathrm{gp}}$ is an isomorphism and $R^1\tau_*M_Y^{\mathrm{gp}} = 0$. We again consider the diagram (4.4.1.1) and the exact sequence (4.4.1.2). By proper base change [ACV03, Proposition A.0.1], we are reduced to the case where X is a geometric point.

In that case, Y is the quotient of a finite, connected (possibly nonreduced) scheme Z over X by the action of a finite group G, the Cartier dual of $\overline{M}_Y^{\mathrm{gp}}/\overline{M}_X^{\mathrm{gp}}$, with order prime to the characteristic. Now, Z is finite over a geometric point, so it has trivial Picard group. The Picard group of Y is therefore $\mathrm{Hom}(G, \mathcal{O}_Z^*)$. Let I denote the quotient of $\mathcal{O}_Z^*$ by $\mathcal{O}_Y^*$. Then we have an exact sequence (4.4.1.4):

$$0 \to \mathrm{Hom}(G, \mathcal{O}_Y^*) \to \mathrm{Hom}(G, \mathcal{O}_Z^*) \to \mathrm{Hom}(G, I) \tag{4.4.1.4}$$

But G is torsion of order prime to the characteristic and I has no torsion prime to the characteristic. Thus the Picard group of Y is the Cartier dual of G, which is of course $\tau_*(\overline{M}_Y^{\mathrm{gp}})/\overline{M}_X^{\mathrm{gp}}$.

Now consider the exact sequence (4.4.1.2). We have just seen that the map $\tau_*(\overline{M}_Y^{\mathrm{gp}})/\overline{M}_X^{\mathrm{gp}} \to R^1\tau_*\mathcal{O}_Y^*$ is an isomorphism. The map $\mathcal{O}_X^* \to \tau_*\mathcal{O}_Y^*$ is easily seen to be an isomorphism, and $R^1\tau_*\overline{M}_Y^{\mathrm{gp}}$ vanishes because $H^1(Z, \overline{M}_Y^{\mathrm{gp}}) = 0$ (since Z is finite) and $\mathrm{Hom}(G, \overline{M}_Y^{\mathrm{gp}}) = 0$ (since G is finite and $\overline{M}_Y^{\mathrm{gp}}$ is torsion free). We conclude from the diagram (4.4.1.1) that $M_X \to \tau_*M_Y^{\mathrm{gp}}$ is an isomorphism and $R^1\tau_*M_Y^{\mathrm{gp}} = 0$, as required. ☐

Corollary 4.4.2. *Let X be a family of logarithmic curves over S. Then* $\mathbf{Log\,Pic}(X/S)$ *is a stack in the logarithmic étale topology over S.*

Proof. We show that logarithmic line bundles descend in the logarithmic étale topology and that boundedness of monodromy is a local property in the logarithmic étale topology. We address the second point first: by Proposition 4.3.2, boundedness of monodromy can be verified at the valuative geometric points of S, so we are free to assume S is a valuative geometric point. Any logarithmic étale cover of S can therefore be refined by an étale cover of a root stack. But by Lemma 2.1.3.2, boundedness is a local property with respect to root stacks.

Now we consider descent. Suppose that $R \to S$ is a logarithmically étale cover and that we have a descent datum over R for a logarithmic line bundle. Since $\mathbf{G}_m^{\log}$-torsors descend from étale covers, logarithmic modifications, and root stacks, we can replace S by any of these constructions, as necessary; we can also replace R by a finer cover. By [Nak17, Lemma 3.11], the map $R \to S$ can be factored locally as an étale map, followed by a logarithmic modification, followed by a root stack, followed by another étale map. We can assume S is quasicompact because it is so étale-locally, and we can then replace R by a finer cover by $\coprod R_i$, where each $R_i \to S$ factors as

$$R_i \to U_i \to V_i \to S$$

where $R_i \to U_i$ is étale, $U_i \to V_i$ is a logarithmic modification, $V_i \to S$ is a root stack. The fiber product of all the V_i is also a root stack of S, and pulling back to this root stack, we can assume all V_i are the same V. Similarly, we can assume that all U_i are the same U. Now, $R_i = R_i \times_S U$ since $U \to S$ is a monomorphism (in the category of fine, saturated logarithmic schemes). Therefore the R_i are an *étale* cover of U. We can now descend a descent datum from $R = \coprod R_i$ to U by étale descent, then to V by invariance along logarithmic modifications (Theorem 4.4.1 and Lemma 3.5.3), and finally to S by invariance along root stacks (Theorem 4.4.1 and Lemma 3.5.3). □

The following corollary complements Theorem 2.4.2.1. It also appeared in [MW17], but we give a proof here for the sake of a self-contained treatment.

Corollary 4.4.3. *With notation as in Theorem 2.4.2.1, let $j : X_\eta \to X$ be the inclusion of the generic fiber. Then $R^1 j_* M_{X_\eta}^{\mathrm{gp}} = 0$.*

Proof. We wish to show that any $M_{X_\eta}^{\mathrm{gp}}$-torsor can be trivializated étale-locally on X. As in the proof of Theorem 2.4.2.1, we let Y be an extension of X to a valuative base T with smooth general fiber. Since obstructions to deforming M_X^{gp}-torsors lie in $H^2(X, \mathcal{O}_X)$, there is no obstruction to extending an M_X^{gp}-torsor from X to Y. We can therefore replace X by Y and assume that the generic fiber of X is smooth.

An $M_{X_\eta}^{\mathrm{gp}}$-torsor is therefore a line bundle L_η on X_η. It can be represented by a divisor D_η on X_η. There is a semistable model $\tilde{X} \to X$ such that the closure D of D_η in $\tilde{X}$ lies in the smooth locus. Then $\mathcal{O}_{\tilde{X}}(D)$ extends L_η to a line bundle $\tilde{L}$ on $\tilde{X}$. In particular, L_η extends to an $M_{\tilde{X}}^{\mathrm{gp}}$-torsor on $\tilde{X}$. But $H^1(X, M_X^{\mathrm{gp}}) = H^1(\tilde{X}, M_{\tilde{X}}^{\mathrm{gp}})$ by Theorem 4.4.1, so L_η also extends to an M_X^{gp}-torsor L on X. This torsor can certainly be trivialized locally in X, so in particular L_η can be trivialized locally in X, and $R^1 j_* M_{X_\eta}^{\mathrm{gp}} = 0$. □

4.5. **Degree.** Let X be a proper, vertical logarithmic curve over S, whose underlying scheme is the spectrum of an algebraically closed field. We construct a dashed arrow making diagram (4.5.1) commute:

$$\begin{tikzcd} H^0(X, \overline{M}_X^{\mathrm{gp}}) \arrow[r] \arrow[dr] & H^1(X, \mathcal{O}_X^*) \arrow[r] \arrow[d] & \operatorname{Log}\operatorname{Pic}(X) \arrow[d, dashed, "\deg"] \\ & \mathbf{Z}^V \arrow[r, "\Sigma"] & \mathbf{Z} \end{tikzcd} \tag{4.5.1}$$

Here, V is the set of vertices of the dual graph of X and the solid vertical arrow is the multidegree. The map $\Sigma : \mathbf{Z}^V \to \mathbf{Z}$ is the sum. We regard a section of $\overline{M}_X^{\mathrm{gp}}$ as a piecewise linear, $\overline{M}_S^{\mathrm{gp}}$-valued function on the dual graph of X, with integer slopes along the edges. The diagonal map to $\mathbf{Z}^V$ sends such a function to tuple whose each component is the sum of the outgoing slopes from the corresponding vertex of the dual graph. The composed map to $\mathbf{Z}$ therefore takes the sum of the outgoing slopes from every vertex; since each edge gets counted twice with opposite orientations (X is vertical, so its dual graph is compact) the composition is zero. This gives the vertical arrow on the image of $H^1(X, \mathcal{O}_X^*)$. However, it is easily seen that if we replace S by a logarithmic modification, subdivide the dual graph of X, and replace X with the corresponding semistable model, the total degree homomorphism remains unchanged. By Proposition 3.5.1, every $P \in \operatorname{Log}\operatorname{Pic}(X)$ is represented in the image of $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^*)$ for some semistable model $\tilde{X}$ of X, and we therefore obtain a well-defined degree homomorphism on all of $\operatorname{Log}\operatorname{Pic}(X)$.

Proposition 4.5.2. *The degree of a $\mathbf{G}_m^{\log}$-torsor is locally constant in families.*

Proof. The total degree of a family of line bundles is constant in families. □

Definition 4.5.3. We write $\operatorname{Log}\operatorname{Pic}^d(X/S)$ for the open and closed substack of $\operatorname{Log}\operatorname{Pic}(X/S)$ parameterizing isomorphism classes of $\mathbf{G}_m^{\log}$-torsors with bounded monodromy and degree d.

4.6. **Quotient presentation.** We construct a quotient presentation of $\operatorname{Log}\operatorname{Pic}^0(X/S)$. Over the strata of S, this produces a logarithmic abelian variety with *constant degeneration*, in the terminology of Kajiwara, Kato, and Nakayama [KKN08c, KKN08b, KKN13, KKN15] (see Section 4.7). Our presentation is inspired by Kajiwara's [Kaj93].

Let X be a proper, vertical logarithmic curve over S, with connected geometric fibers. Write $\operatorname{Pic}^{[0]}(X/S)$ for the multidegree zero part of $\operatorname{Pic}^0(X/S)$.

Lemma 4.6.1. *Let X be a proper, vertical logarithmic curve over S with connected geometric fibers. Then the natural map $M_S^{\mathrm{gp}} \to \pi_* M_X^{\mathrm{gp}}$ is an isomorphism.*

Proof. This assertion is étale-local in S. We can therefore assume that S is atomic and that the dual graph of X is constant on the closed stratum. We denote it $\mathfrak{X}$. Now $H^0(X, \overline{M}_X^{\mathrm{gp}})$ is the group of piecewise linear function on $\mathfrak{X}$ having integer slopes along the edges and taking values in $\overline{M}_S^{\mathrm{gp}}$. Since sections of M_X^{gp} correspond generically on X to rational functions, the associated piecewise linear function on $\mathfrak{X}$ of such a section will be *linear*. That is, the sum of the outgoing slopes along the edges incident to any vertex of $\mathfrak{X}$ will be zero.

On the other hand $\mathfrak{X}$ is compact so every linear function on $\mathfrak{X}$ is constant by Lemma 3.4.4. Therefore the section of $\overline{M}_X^{\mathrm{gp}}$ induced from any section of M_X^{gp} lies in the image $\overline{M}_S^{\mathrm{gp}}$, which is to say that there is a diagonal arrow as shown in the commutative diagram of exact sequences (4.6.1.1):

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_S^* \arrow[r] \arrow[d] & M_S^{\mathrm{gp}} \arrow[r] \arrow[d] & \overline{M}_S^{\mathrm{gp}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi_*\mathcal{O}_X^* \arrow[r] & \pi_* M_X^{\mathrm{gp}} \arrow[r] \arrow[ur] & \pi_*\overline{M}_X^{\mathrm{gp}} \arrow[r] & R^1\pi_*\mathcal{O}_X^* \end{tikzcd} \tag{4.6.1.1}$$

As X is proper and reduced with connected fibers, the map $\mathcal{O}_S^* \to \pi_* \mathcal{O}_X^*$ is an isomorphism. We may therefore conclude by the 5-lemma, applied to the induced diagram (4.6.1.2):

(4.6.1.2)
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_S^* \arrow[r] \arrow[d, "\wr"] & M_S^{\mathrm{gp}} \arrow[r] \arrow[d] & \overline{M}_S^{\mathrm{gp}} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \pi_* \mathcal{O}_X^* \arrow[r] & \pi_* M_X^{\mathrm{gp}} \arrow[r] & \overline{M}_S^{\mathrm{gp}} \arrow[r] & 0 \end{tikzcd}$$

□

Proposition 4.6.2. *The map $R^1\pi_*\pi^* M_S^{\mathrm{gp}} \to R^1\pi_* M_X^{\mathrm{gp}}$ induces a surjection from the multi-degree 0 part onto the degree 0 part, with kernel $H_1(\mathfrak{X})$.*

Proof. We use the exact sequence (4.6.2.1)

$$0 \to \pi^* M_S^{\mathrm{gp}} \to M_X^{\mathrm{gp}} \to \overline{M}_{X/S}^{\mathrm{gp}} \to 0 \tag{4.6.2.1}$$

and its associated long exact sequence in the top row of (4.6.2.2):

(4.6.2.2)
$$\begin{tikzcd} 0 \arrow[r] & \pi_* \overline{M}_{X/S}^{\mathrm{gp}} \arrow[r] \arrow[d, "\wr"] & R^1\pi_*\pi^* M_S^{\mathrm{gp}} \arrow[r] \arrow[d] & R^1\pi_* M_X^{\mathrm{gp}} \arrow[r] \arrow[d, dashed] & 0 \\ & \mathbf{Z}^E \arrow[r] & \mathbf{Z}^V \arrow[r] & \mathbf{Z} \arrow[r] & 0 \end{tikzcd}$$

Here $\mathbf{Z}^V$ is the sheaf of abelian groups freely generated by the irreducible components of the fibers, and $\mathbf{Z}^E$ is the sheaf whose stalks are freely generated by the nodes. When a node is smoothed in X, the corresponding generator of the stalk of $\mathbf{Z}^E$ maps to zero under the generization map.

Note that the first map in the first row of (4.6.2.1) is injective because $M_S \to \pi_* M_X^{\mathrm{gp}}$ is an isomorphism by Lemma 4.6.1. A section of $R^1\pi_*\pi^* M_S^{\mathrm{gp}}$ induces isomorphism classes of line bundles on the components of X and therefore has a well-defined multidegree. This gives the vertical homomorphism in the middle term of diagram (4.6.2.2).

By an explicit calculation, the map $\pi_* \overline{M}_{X/S}^{\mathrm{gp}} \to R^1\pi_*\pi^* M_S^{\mathrm{gp}}$ commutes with the boundary map $\mathbf{Z}^E \to \mathbf{Z}^V$ computing the homology of the dual graph of X. Therefore we recover the degree homomorphism by passing to cokernels, as indicated by the dashed arrow in (4.6.2.2).

We write $R^1\pi_*(\pi^* M_S^{\mathrm{gp}})^{[0]}$ for the multidegree 0 part of $R^1\pi_*(\pi^* M_S^{\mathrm{gp}})$ and $R^1\pi_*(M_X^{\mathrm{gp}})^0$ for the degree zero part of $R^1\pi_* M_X^{\mathrm{gp}}$. As $\mathbf{Z}^E$ surjects on to the kernel of $\mathbf{Z}^V \to \mathbf{Z}$, the map (4.6.2.3)

$$R^1\pi_*(\pi^* M_S^{\mathrm{gp}})^{[0]} \to R^1\pi_*(M_X^{\mathrm{gp}})^0 \tag{4.6.2.3}$$

is surjective with kernel $H_1(\mathfrak{X})$. □

Corollary 4.6.3. *Let X be a proper, vertical logarithmic curve over S. Let $R^1\pi_*(\pi^* \mathbf{G}_m^{\log})$ denote the sheaf on logarithmic schemes over S whose value on a logarithmic scheme T over S is $R^1\pi_*\pi^* M_T^{\mathrm{gp}}$ where π abusively denotes the projection $X_T \to T$. There is an exact sequence*

$$0 \to H_1(\mathfrak{X}) \to R^1\pi_*(\pi^* \mathbf{G}_m^{\log})^{[0]\dagger} \to \operatorname{Log}\operatorname{Pic}^0(X/S) \to 0$$

where $R^1\pi_*(\pi^* \mathbf{G}_m^{\log})^{[0]\dagger}$ is the bounded monodromy, multidegree 0 subsheaf of $R^1\pi_*(\pi^* \mathbf{G}_m^{\log})$.

4.7. **Stratumwise description of the homology action.** We assume that X is a family of logarithmic curves over S with constant degeneracy. That is, the characteristic monoid of S is constant, as is the dual graph of X. Let X^ν be the normalization of the nodes of X. We have an exact sequence (4.7.1)

$$0 \to T \to \mathrm{Pic}^{[0]}(X/S) \to \mathrm{Pic}^{[0]}(X^\nu/S) \to 0 \tag{4.7.1}$$

where T is the torus $\mathrm{Hom}(H_1(\mathfrak{X}), \mathbf{G}_m)$ and $\mathfrak{X}$ is the dual graph of X.

We obtain a similar sequence with $\pi^* M_S^{\mathrm{gp}}$ in place of $\mathcal{O}_X^*$. The short exact sequence (4.7.2)

$$0 \to \mathcal{O}_{X^\nu}^* \to \nu^*\pi^* M_S^{\mathrm{gp}} \to \nu^*\pi^*\overline{M}_S^{\mathrm{gp}} \to 0 \tag{4.7.2}$$

yields the long exact sequence (4.7.3):

(4.7.3)
$$\pi_*\nu_*\nu^*\pi^* M_S^{\mathrm{gp}} \to \pi_*\nu_*\nu^*\pi^*\overline{M}_S^{\mathrm{gp}} \to R^1(\pi_*\nu_*)\mathcal{O}_{X^\nu}^* \to R^1(\pi_*\nu_*)\nu^*\pi^* M_S^{\mathrm{gp}} \to R^1(\pi_*\nu_*)\nu^*\pi^*\overline{M}_S^{\mathrm{gp}}$$

As $M_S^{\mathrm{gp}} \to \overline{M}_S^{\mathrm{gp}}$ is surjective, so is $\pi_*\nu_*\nu^*\pi^* M_S^{\mathrm{gp}} \to \pi_*\nu_*\nu^*\pi^*\overline{M}_S^{\mathrm{gp}}$. Furthermore, the components of X^ν are irreducible curves over S, so they have no first cohomology valued in $\overline{M}_S^{\mathrm{gp}}$ because it is torsion-free and constant on the fibers. The sequence therefore reduces to an isomorphism between $R^1(\pi_*\nu_*)\mathcal{O}_{X^\nu}^*$ an $R^1(\pi_*\nu_*)\nu^*\pi^* M_S^{\mathrm{gp}}$. That is, we have an isomorphism between $\mathrm{Pic}(X^\nu/S)$ and the functor $T \mapsto \Gamma(T, R^1\pi_*\nu_*\nu^*\pi^* M_T^{\mathrm{gp}})$ on logarithmic schemes over S

By pullback, we therefore obtain a morphism (4.7.4):

$$R^1\pi_*\pi^* M_S^{\mathrm{gp}} \to R^1\pi_*\nu_*\nu^*\pi^* M_S^{\mathrm{gp}} \simeq \mathrm{Pic}(X^\nu/S) \tag{4.7.4}$$

The kernel of this morphism consists of those M_S^{gp}-torsors on X that are trivial when restricted to X^ν. Such a torsor is specified by transition functions in M_S^{gp} along the nodes of X and the kernel may therefore be identified with $T^{\log} = \mathrm{Hom}(H_1(\mathfrak{X}), \mathbf{G}_m^{\log})$.

Passing to the multidegree 0 parts of $R^1\pi_*\pi^* M_S^{\mathrm{gp}}$ and $\mathrm{Pic}(X^\nu/S)$, we get an exact sequence (4.7.5):

$$0 \to \mathrm{Hom}(H_1(\mathfrak{X}), \mathbf{G}_m^{\log}) \to R^1\pi_*(\pi^* M_S^{\mathrm{gp}})^{[0]} \to \mathrm{Pic}^{[0]}(X^\nu/S) \to 0 \tag{4.7.5}$$

4.8. **Local description of the homology action.** We retain the assumptions of Section 4.7 and permit further étale localization in S.

Because we have assumed the logarithmic structure of S is constant, $\overline{M}_S^{\mathrm{gp}}$ is a constant sheaf of finitely generated free abelian groups. Working locally in S, we can assume that $M_S^{\mathrm{gp}} \to \overline{M}_S^{\mathrm{gp}}$ is split, and therefore that $M_S^{\mathrm{gp}} \simeq \mathcal{O}_S^* \times \overline{M}_S^{\mathrm{gp}}$. This implies $\pi^* M_S^{\mathrm{gp}} = \mathcal{O}_X^* \times \pi^*\overline{M}_S^{\mathrm{gp}}$, and therefore gives a splitting (4.8.1):

$$R^1\pi_*\pi^*\mathbf{G}_m^{\log} \simeq \mathrm{Pic}(X/S) \times \mathrm{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log}) \tag{4.8.1}$$

We have used the canonical identification $R^1\pi_*\pi^*\overline{\mathbf{G}}_m^{\log} \simeq \mathrm{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})$.

Our goal in this section is to explain the map $H_1(\mathfrak{X}) \to R^1\pi_*\pi^*(\mathbf{G}_m^{\log})^{[0]}$ from Corollary 4.6.3, which is induced from $\mathbf{Z}^E \to R^1\pi_*\pi^*\mathbf{G}_m^{\log}$, in terms of this splitting. Given $\alpha \in \Gamma(X, M_{X/S}^{\mathrm{gp}}) = \mathbf{Z}^E$, we write $\pi^* M_S(\alpha)$ for its image in $R^1\pi_*\pi^*\mathbf{G}_m^{\log}$.

We work out the pullback of $\pi^* M_S(\alpha)$ to the normalization X^ν of X along its nodes. We let $\mathfrak{X}^\nu$ be the union of the stars of $\mathfrak{X}$. In a sense that we do not make precise here, this is the tropicalization of X^ν when X^ν is given the logarithmic structure pulled back from X. Every section α of $\mathbf{Z}^E = \Gamma(X, \overline{M}_{X/S}^{\mathrm{gp}})$ can be lifted to a section $\tilde{\alpha}$ of $\overline{M}_{X^\nu}^{\mathrm{gp}}$, which can also be regarded as a piecewise linear function on $\mathfrak{X}^\nu$. Then $\nu^*\pi^* M_S^{\mathrm{gp}}(\alpha)$ is represented by the line bundle $\mathcal{O}_{X^\nu}(\tilde{\alpha})$. Note that the isomorphism class of $\mathcal{O}_{X^\nu}(\tilde{\alpha})$ depends only on α because $\tilde{\alpha}$ is uniquely determined up to the addition of a constant from $\overline{M}_S^{\mathrm{gp}}$ on each component, and the addition of a constant only changes $\mathcal{O}_{X^\nu}(\tilde{\alpha})$ by a line bundle pulled back from S.

Suppose that $X_0 \subset X^\nu$ is a component and $\mathfrak{X}_0$ is the corresponding component of $\mathfrak{X}^\nu$. Then (4.8.2) computes $\mathcal{O}_{X_0}(\tilde{\alpha})$:

$$\mathcal{O}_{X_0}(\tilde{\alpha}) = \mathcal{O}_{X_0}\big(\sum \alpha_e D_e\big) \tag{4.8.2}$$

The sum is taken over the edges e of $\mathfrak{X}_0$, with D_e denoting the node of X corresponding to e, and α_e denoting the slope of α along e when e is oriented away from the central vertex of $\mathfrak{X}_0$. In order to understand $\pi^* M_S(\alpha)$, we will need to see how the line bundles $\mathcal{O}_{X_i}(\tilde{\alpha})$ on the components of X^ν are glued to one another.

For each $\delta \in \overline{M}_S^{\mathrm{gp}}$, we write $m^\delta : \mathcal{O}_X \to \mathcal{O}_X(\delta)$ for map sending $\lambda \in \mathcal{O}_X$ to $\lambda m(\delta) \in \mathcal{O}_X(\delta)$. Suppose that D is a node of X joining components X_0 and X_1 and let e be the corresponding edge of $\mathfrak{X}$. Recall that we have (4.8.3),

$$\begin{aligned} \mathcal{O}_{X_0}(\tilde{\alpha})\big|_D &= \mathcal{O}_D(\tilde{\alpha}(0)) \otimes \mathcal{O}_{X_0}(\alpha_e D)\big|_D \\ \mathcal{O}_{X_1}(\tilde{\alpha})\big|_D &= \mathcal{O}_D(\tilde{\alpha}(1)) \otimes \mathcal{O}_{X_1}(-\alpha_e D)\big|_D \end{aligned} \tag{4.8.3}$$

where α_e is the slope of α along the edge e of $\mathfrak{X}$ corresponding to D, oriented from 0 to 1, and $\tilde{\alpha}(i) \in \overline{M}_S^{\mathrm{gp}}$ is the value of $\tilde{\alpha}$ on the vertex i of $\mathfrak{X}$. Using the trivializations m, we obtain an isomorphism:

$$m^{\tilde{\alpha}(1)-\tilde{\alpha}(0)} : \mathcal{O}_{X_0}(\alpha_e D)\big|_D \to \mathcal{O}_{X_1}(-\alpha_e D)\big|_D \tag{4.8.4}$$

If $\tilde{\alpha}$ is actually well-defined on $X_{01} = X_0 \cup_D X_1$ then $\tilde{\alpha}(1) - \tilde{\alpha}(0) = \alpha_e \delta_e$, where δ_e is the length of e. Then $m^{\tilde{\alpha}(1)-\tilde{\alpha}(0)} = m^{\alpha_e \delta_e}$. Note that this depends only on α and m, and not on $\tilde{\alpha}$. We glue $\mathcal{O}_{X_0}(\tilde{\alpha})$ to $\mathcal{O}_{X_1}(\tilde{\alpha})$ along D by $m^{-\alpha_e \delta_e}$ and repeat the same process for each edge of $\mathfrak{X}$ to produce a line bundle $L(\alpha, m)$ on X.

Proposition 4.8.5. *The isomorphism (4.8.1) sends $\pi^* M_S(\alpha)$ to $(L(\alpha, m), \partial(\alpha).\gamma)$.*

Proof. The second component of the formula is implied by Lemma 3.4.6. It can also be deduced from the argument below.

Let $\tilde{\mathfrak{X}}$ be the universal cover of $\mathfrak{X}$ and let $\rho : \tilde{X} \to X$ be the corresponding étale cover. The fundamental group of $\mathfrak{X}$ acts by deck transformations on $\tilde{X}$. Since $H_1(\tilde{\mathfrak{X}}) = 0$, we can find a lift of $\tilde{\alpha}$ to $\overline{M}_{\tilde{X}}$. Without loss of generality, we can assume that the function on X^ν constructed before the statement of the proposition is induced from this $\tilde{\alpha}$ by restriction along some embedding $X^\nu \subset \tilde{X}$.

By construction $\rho^* L(\alpha, m)$ induces $\rho^* \pi^* M_S(\alpha)$. We will prove that $\pi^* M_S(\alpha) = (L(\alpha, m), \partial(\alpha).\gamma)$ by comparing their transition data on the cover $\tilde{X}$.

If $\gamma \in \pi_1(\mathfrak{X})$ then γ acts by deck transformations on $\tilde{X}$ and $\gamma^* \mathcal{O}_{\tilde{X}}(\tilde{\alpha}) = \mathcal{O}_{\tilde{X}}(\tilde{\alpha}) \otimes \mathcal{O}_X(\partial(\alpha).\gamma)$, canonically. By definition, we have an inclusion $\mathcal{O}_X^*(\partial(\alpha).\gamma)$ inside $\pi^* M_S^{\mathrm{gp}}$ as the fiber over $\partial(\alpha).\gamma \in \pi^* \overline{M}_S^{\mathrm{gp}}$. This gives us a canonical identification $\gamma^* \rho^* \pi^* M_S(\tilde{\alpha}) = \rho^* \pi^* M_S(\tilde{\alpha})$ that serves as a descent datum for $\rho^* \pi^* M_S(\tilde{\alpha})$ from $\tilde{X}$ to $\pi^* M_S(\tilde{\alpha})$ on X.

In terms of the splitting m, the map from $\mathcal{O}_X^*(\partial(\alpha).\gamma)$ to $\pi^* M_S^{\mathrm{gp}}$ is given by (4.8.5.1):

$$(m^{-\partial(\alpha).\gamma}, \partial(\alpha).\gamma) : \mathcal{O}_X^*(\partial(\alpha).\gamma) \to \mathcal{O}_X^* \times \pi^* \overline{M}_S^{\mathrm{gp}} \tag{4.8.5.1}$$

The second component of this formula gives the homomorphism $H_1(\mathfrak{X}) \to \overline{M}_S^{\mathrm{gp}}$ that makes up the second component of (4.8.1).

The transition function for $L(\alpha, m)$ around the loop γ is given by (4.8.5.2):

$$\prod_e (m^{-\delta_e \alpha_e})^{\gamma_e} = m^{-\sum \alpha_e \gamma_e \delta_e} \tag{4.8.5.2}$$

By definition of the intersection pairing, $\sum \alpha_e \gamma_e \delta_e = \partial(\alpha).\gamma$, so (4.8.5.2) agrees with the first component of (4.8.5.1). □

4.9. **Tropicalizing the logarithmic Jacobian.** For any proper, vertical logarithmic curve X over S, we construct a morphism (4.9.1)

$$\operatorname{Log}\operatorname{Pic}^0(X/S) \to \operatorname{Tro}\operatorname{Jac}(X/S) \tag{4.9.1}$$

over S. For each logarithmic scheme T and object of $\operatorname{Log}\operatorname{Pic}^0(X/S)$, we must produce a section of $\operatorname{Tro}\operatorname{Jac}(X/S)$. By Corollary 4.2.3, it is sufficient to do this when T is of finite type. Under this assumption, the T-points of $\operatorname{Tro}\operatorname{Jac}(X/S)$ are generization-compatible objects of $\operatorname{Tro}\operatorname{Jac}(\mathfrak{X}_t)$, for each geometric point t of T. We therefore describe the morphism first under the assumption that X has constant dual graph over S and S has constant characteristic monoid (which covers the case of a geometric point) and then discuss generization.

If X has constant dual graph and S has constant characteristic monoid, we use the commutative diagram of exact sequences (4.9.2):

$$\begin{tikzcd}
0 \arrow[r] & H_1(\mathfrak{X}) \arrow[r] \arrow[dd, equal] & H^1(X, \pi^*\mathbf{G}_m^{\log})^{[0]\dagger} \arrow[r] \arrow[d] & \operatorname{Log}\operatorname{Pic}^0(X/S) \arrow[r] \arrow[dd, dashed] & 0 \\
& & H^1(X, \pi^*\overline{\mathbf{G}}_m^{\log})^{\dagger} \arrow[d, equal] & & \\
0 \arrow[r] & H_1(\mathfrak{X}) \arrow[r] & \operatorname{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})^{\dagger} \arrow[r] & \operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S) \arrow[r] & 0
\end{tikzcd} \tag{4.9.2}$$

The first row of the diagram comes from Corollary 4.6.3 and the bottom row is the definition of the tropical Jacobian from Section 3.6. The identification between $H^1(X, \pi^*\overline{\mathbf{G}}_m^{\log})$ and $\operatorname{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})$ comes from the fact that $\overline{M}_S^{\mathrm{gp}}$ is a torsion-free sheaf: since a smooth, proper curve has no nontrivial torsors under such a sheaf, any such torsor on a nodal curve can be trivialized on its normalization, and torsors under $\overline{M}_S^{\mathrm{gp}}$ on X are determined uniquely by monodromy around the loops of the dual graph. A unique dashed arrow exists by the universal property of the cokernel.

We show now that this morphism is compatible with generizations. Any specialization $s \rightsquigarrow t$ in $\operatorname{Log}\operatorname{Pic}^0(X/S)$ can be represented by a map $T \to \operatorname{Log}\operatorname{Pic}^0(X/S)$ where T is a strictly henselian valuation ring with some logarithmic structure, s is its generic point, and t is its closed point. This map gives a logarithmic curve $X_T = X \times_S T$ over T and an $M_{X_T}^{\mathrm{gp}}$-torsor P on T with bounded monodromy and degree 0. Since $\operatorname{Log}\operatorname{Pic}^0(X/S)$ is the quotient of $H^1(X, \pi^*\mathbf{G}_m^{\log})$ by a discrete group, we can lift P to a $\pi^* M_T^{\mathrm{gp}}$-torsor, Q, on X_T.

We now have a commutative diagram (4.9.3):

(4.9.3)
$$\begin{tikzcd}
H^1(X_t, \pi^* M_t^{\mathrm{gp}})^\dagger \arrow[d] & H^1(X_T, \pi^* M_T^{\mathrm{gp}})^\dagger \arrow[l] \arrow[r, dotted] \arrow[d] \arrow[ddl, dotted] & H^1(X_s, \pi^* M_s^{\mathrm{gp}})^\dagger \arrow[d, dotted] \\
H^1(X_t, \pi^* \overline{M}_t^{\mathrm{gp}})^\dagger \arrow[d, equal] & H^1(X_T, \pi^* \overline{M}_T^{\mathrm{gp}})^\dagger \arrow[l] \arrow[r] & H^1(X_s, \pi^* \overline{M}_s^{\mathrm{gp}})^\dagger \arrow[d, equal] \\
\mathrm{Hom}(H_1(\mathfrak{X}_t), \overline{M}_t^{\mathrm{gp}})^\dagger \arrow[rr, dotted] & & \mathrm{Hom}(H_1(\mathfrak{X}_s), \overline{M}_s^{\mathrm{gp}})^\dagger
\end{tikzcd}$$

The commutativity of the trapezoid rendered in dotted arrows is precisely the compatibility of our map with generization.

Theorem 4.9.4. *Let X be a proper, vertical logarithmic curve over S. There is an exact sequence:*

(4.9.4.1)
$$0 \to \mathrm{Pic}^{[0]}(X/S) \to \mathrm{Log\,Pic}^0(X/S) \to \mathrm{Tro\,Jac}(X/S) \to 0$$

Proof. Applying the snake lemma to (4.9.2), and identifying $H^1(X, \pi^* \overline{\mathbf{G}}_m^{\log}) = \mathrm{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})$, we see that the exactness of (4.9.4.1) is equivalent to that of (4.9.4.2):

(4.9.4.2)
$$0 \to \mathrm{Pic}^{[0]}(X/S) \to R^1\pi_*(\pi^* \mathbf{G}_m^{\log})^{[0]^\dagger} \to R^1\pi_*(\pi^* \overline{\mathbf{G}}_m^{\log})^\dagger \to 0$$

We note that the bounded monodromy subgroup of $R^1\pi_*(\pi^*\mathbf{G}_m^{\log})^{[0]}$ is simply the preimage of that in $\mathrm{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})$, and that the multidegree 0 subgroup of $\mathrm{Pic}(X/S)$ is the preimage of the multidegree 0 subgroup of $R^1\pi_*(\pi^*\mathbf{G}_m^{\log})$. Therefore it will be sufficient to demonstrate the exactness of (4.9.4.3):

(4.9.4.3)
$$0 \to \mathrm{Pic}(X/S) \to R^1\pi_*(\pi^*\mathbf{G}_m^{\log}) \to \mathrm{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log}) \to 0$$

This amounts to showing that, for each logarithmic scheme T over S, the sequence (4.9.4.4) is exact, where $Y = X \times_S T$:

(4.9.4.4)
$$0 \to R^1\pi_*\mathcal{O}_Y^* \to R^1\pi_*\pi^* M_T^{\mathrm{gp}} \to R^1\pi_*\pi^*\overline{M}_T^{\mathrm{gp}} \to 0$$

The exact sequence (4.9.4.4) arises from the long exact sequence (4.9.4.6) associated with the short exact sequence (4.9.4.5):

(4.9.4.5)
$$0 \to \mathcal{O}_Y^* \to \pi^* M_T^{\mathrm{gp}} \to \pi^*\overline{M}_T^{\mathrm{gp}} \to 0$$

(4.9.4.6)
$$\pi_*\pi^* M_T^{\mathrm{gp}} \to \pi_*\pi^*\overline{M}_T^{\mathrm{gp}} \to R^1\pi_*\mathcal{O}_Y^* \to R^1\pi_*\pi^* M_T^{\mathrm{gp}} \to R^1\pi_*\pi^*\overline{M}_T^{\mathrm{gp}} \to R^2\pi_*\mathcal{O}_Y^*$$

We have $R^2\pi_*\mathcal{O}_Y^* = 0$ by Tsen's theorem. As $\pi^*\overline{M}_T^{\mathrm{gp}}$ is a constant sheaf on the fibers and $M_T^{\mathrm{gp}} \to \overline{M}_T^{\mathrm{gp}}$ is surjective, the map $\pi_*\pi^* M_T^{\mathrm{gp}} \to \pi_*\pi^*\overline{M}_T^{\mathrm{gp}}$ is surjective as well. This gives the exactness of (4.9.4.4) and completes the proof. □

Corollary 4.9.5. *Let X be a proper, vertical logarithmic curve over S. For each degree d, the sheaf* $\mathrm{Log\,Pic}^d(X/S)$ *and the stack* $\mathbf{Log\,Pic}^d(X/S)$ *are bounded.*

Proof. As $\mathrm{Log\,Pic}^d(X/S)$ is a torsor under $\mathrm{Log\,Pic}^0(X/S)$, it is sufficient to prove the corollary for $d = 0$. By the exact sequence (4.9.4.1), $\mathrm{Log\,Pic}^0(X/S)$ is a $\mathrm{Pic}^{[0]}(X/S)$-torsor over $\mathrm{Tro\,Jac}(X/S)$. As both $\mathrm{Pic}^{[0]}(X/S)$ and $\mathrm{Tro\,Jac}(X/S)$ are bounded — in the latter case by Theorem 3.10.2 — it follows that $\mathrm{Log\,Pic}^0(X/S)$ is also bounded.

Finally, we note that $\mathbf{Log\,Pic}^d(X/S)$ is isomorphic, locally in S, to $\operatorname{Log\,Pic}^d(X/S) \times \mathrm{B}\mathbf{G}_m^{\log}$, so the conclusion follows from the boundedness of $\mathrm{B}\mathbf{G}_m^{\log}$. □

4.10. **The valuative criterion for properness.**

Theorem 4.10.1. *Let X be a proper, vertical logarithmic curve over S. Then $\mathbf{Log\,Pic}(X/S) \to S$ satisfies the valuative criterion for properness (Theorem 2.2.5.2) over S.*

Proof. Let R be a valuation ring with a valuative logarithmic structure and with field of fractions K. We consider a lifting problem (4.10.1.1) and show it has a unique solution:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathbf{Log\,Pic}(X/S) \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \tag{4.10.1.1}$$

These data give us a logarithmic curve X_R over R and a $M_{X_K}^{\mathrm{gp}}$-torsor P on X_K with bounded monodromy. Let $j : X_K \to X_R$ denote the inclusion. By Theorem 2.4.2.1 and Corollary 4.4.3, we have $R^1 j_* \mathcal{O}_{X_K}^* = 0$ and $M_{X_R}^{\mathrm{gp}} \to j_* M_{X_K}^{\mathrm{gp}}$ is an isomorphism. These imply that the morphism of group stacks $\mathrm{B}M_{X_R}^{\mathrm{gp}} \to j_* \mathrm{B}M_{X_K}^{\mathrm{gp}}$ induces isomorphisms on sheaves of isomorphism classes and sheaves of automorphisms, hence is an equivalence. Pushing forward to S gives (4.10.1.2):

$$\pi_* \mathrm{B}M_{X_R}^{\mathrm{gp}} = j_* \pi_* \mathrm{B}M_{X_K}^{\mathrm{gp}} \tag{4.10.1.2}$$

But a section of $j_* \pi_* \mathrm{B}M_{X_K}^{\mathrm{gp}}$ is a commutative square (4.10.1.1), and a section of $\pi_* \mathrm{B}M_{X_R}^{\mathrm{gp}}$ is a diagonal arrow lifting it. This completes the proof. □

Corollary 4.10.2. *The projection* $\operatorname{Log\,Pic}(X/S) \to S$ *satisfies the valuative criterion for properness.*

Proof. Locally in S the projection $\mathbf{Log\,Pic}(X/S) \to \operatorname{Log\,Pic}(X/S)$ has a section making $\operatorname{Log\,Pic}(X/S)$ into a $\mathbf{G}_m^{\log}$-torsor over S. But $\mathbf{G}_m^{\log}$ satisfies the valuative criterion for properness, so $\operatorname{Log\,Pic}(X/S)$ does as well. □

Once we have demonstrated the algebraicity of $\operatorname{Log\,Pic}^d$, we will be able to conclude that it is proper in Corollary 4.12.5.

4.11. **Existence of a smooth cover.**

Definition 4.11.1. We call a presheaf X on logarithmic schemes a *logarithmic space* if there is a logarithmic scheme U and a morphism $U \to X$ that is surjective on valuative geometric points and representable by logarithmically smooth logarithmic schemes.

Theorem 4.11.2. *Let X be a proper logarithmic curve over S. Then there is a logarithmic scheme and a logarithmically smooth morphism to $\mathbf{Log\,Pic}(X)$ that is representable by logarithmic spaces.*

Proof. We consider a map $T \to S$ that is a composition of étale maps and logarithmic modifications. Let Y be a logarithmic model of $X \times_S T$ over T. Then $\mathbf{Pic}(Y/T)$ is representable by an algebraic stack over T. When equipped with the logarithmic structure pulled back from T, we have a morphism to $\mathbf{Log\,Pic}(X/S)$:

$$\mathbf{Pic}(Y/T) \to \mathbf{Log\,Pic}(Y/T) \to \mathbf{Log\,Pic}(X_T/T) \to \mathbf{Log\,Pic}(X/S) \tag{4.11.2.1}$$

We will argue that these maps are a logarithmically étale cover of $\operatorname{Log}\operatorname{Pic}(X/S)$ using the following two lemmas.

Lemma 4.11.3. *For any logarithmic curve Y over T, the map $\mathbf{Pic}(Y/T) \to \mathbf{Log\,Pic}(Y/T)$ is representable by logarithmic schemes and is logarithmically étale.*

Lemma 4.11.4. *Every valuative geometric point of $\mathbf{Log\,Pic}(X/S)$ lifts to some $\mathbf{Pic}(Y/T)$ for some logarithmic model Y of X over some étale-local logarithmic modification T of S, as described above.*

Granting these lemmas, we complete the proof of Theorem 4.11.2. We show first of all that $\mathbf{Pic}(Y/T) \to \mathbf{Log\,Pic}(X/S)$ is representable by logarithmic schemes and is logarithmically étale. The first arrow in the sequence (4.11.2.1) has these properties by Lemma 4.11.3, the second is an isomorphism by Theorem 4.4.1 and Lemma 3.5.3, and the last arrow is the base change of the logarithmically étale morphism $T \to S$, by definition. Their composition is therefore representable by logarithmic schemes and is logarithmically étale.

The second lemma shows that the family of maps $\mathbf{Pic}(Y/T) \to \mathbf{Log\,Pic}(X/S)$ is universally surjective, and therefore completes the proof. □

Proof of Lemma 4.11.3. Let K be the kernel of $\mathbf{Pic}(Y/T) \to \mathbf{Log\,Pic}(Y/T)$. As the base change of $\mathbf{Pic}(Y/T) \to \mathbf{Log\,Pic}(Y/T)$ to any logarithmic T-scheme is a K-torsor, it will suffice to show that K is an algebraic space that is étale over T.

For any logarithmic scheme U over T, we have an exact sequence of commutative group stacks:

$$0 \to \overline{M}_Y^{\mathrm{gp}} \to \mathrm{B}\mathcal{O}_Y^* \to \mathrm{B}M_Y^{\mathrm{gp}} \to 0$$

This sequence corresponds to an exact sequence on the big étale site of Y:

$$0 \to \overline{\mathbf{G}}_m^{\log} \to \mathrm{B}\mathbf{G}_m \to \mathrm{B}\mathbf{G}_m^{\log} \to 0$$

Using the left exactness of pushforward to T, we get the exact sequence (4.11.4.1),

$$0 \to \pi_* \overline{\mathbf{G}}_m^{\log} \to \mathbf{Pic}(Y/T) \to \mathbf{Log\,Pic}(Y/T) \tag{4.11.4.1}$$

which implies that $K = \pi_* \overline{\mathbf{G}}_m^{\log}$. We have therefore to show that $\pi_* \overline{\mathbf{G}}_m^{\log}$ has a logarithmically smooth cover by a logarithmic scheme that is logarithmically étale over T.

We check first that $\pi_* \overline{\mathbf{G}}_m^{\log}$ satisfies the infinitesimal criterion for being logarithmically étale. Indeed, if $U \subset U'$ is a strict infinitesimal extension of logarithmic schemes over T then Y_U and $Y_{U'}$ have isomorphic étale sites and $\overline{M}_{Y_U} = \overline{M}_{Y_{U'}}$ when the étale sites are identified. Therefore we have the requirement of the infinitesimal criterion:

$$\pi_* \overline{\mathbf{G}}_m^{\log}(U) = \Gamma(Y, \overline{M}_{Y_U}) = \Gamma(Y, \overline{M}_{Y_{U'}}) = \pi_* \overline{\mathbf{G}}_m^{\log}(U') \tag{4.11.4.2}$$

Now we address the existence of a logarithmically smooth cover. There is an exact sequence (4.11.4.3) of sheaves on the small étale site of Y,

$$0 \to \pi^* \overline{M}_T^{\mathrm{gp}} \to \overline{M}_Y^{\mathrm{gp}} \to Q \to 0 \tag{4.11.4.3}$$

where the formations of $Q = \overline{M}_{Y/T}^{\mathrm{gp}}$ commutes with pullback along an arbitrary logarithmic base change $T' \to T$. Pushing forward to T, we obtain a map $\pi_* \overline{\mathbf{G}}_m^{\log} \to \pi_* Q$. As the formation of $\pi_* Q$ commutes with arbitrary base change, it is representable by a strict étale morphism from an algebraic space to T. We can now work locally in $\pi_* Q$ and show that the preimage of $\pi_* \overline{\mathbf{G}}_m^{\log}$ has a logarithmically étale cover by a logarithmic scheme.

Recall that sections of $\overline{\mathbf{G}}_m^{\log}$ over Y correspond to piecewise linear functions on the fibers of the tropicalization, $\mathfrak{Y}$, that are linear along the edges with integer slopes. Working locally in $\pi_* Q$ amounts to fixing these slopes. We write $\pi_*(\overline{\mathbf{G}}_m^{\log})_\alpha$ to be the subsheaf of $\pi_*\overline{\mathbf{G}}_m^{\log}$ with fixed slopes α.

It is sufficient to work étale-locally in T, so we assume that T is an atomic neighborhood of a geometric point t. We may also assume that the dual graph $\mathfrak{Y}$ of Y on the closed stratum of T is constant. Let V be the set of vertices of $\mathfrak{Y}_t$ and let E be its set of edges with orientations chosen arbitrarily. Note that a vertex $v \in V$ induces a vertex of $\mathfrak{Y}_u$ for every geometric point u of every logarithmic scheme U over T.

A piecewise linear function f on $\mathfrak{Y}$ has a value $f(v)$ at each vertex $v \in V$. This gives a morphism:

$$\pi_*\overline{\mathbf{G}}_m^{\log} \to \operatorname{Hom}(V, \overline{\mathbf{G}}_m^{\log})$$

Then $\pi_*\overline{\mathbf{G}}_m^{\log}$ is determined by the condition that $f(v) - f(w) = \alpha_e \ell(e)$ whenever e is an edge directed from v to w in $\mathfrak{Y}_t$ and ℓ is its length. In particular, we can recover $\pi_*(\overline{\mathbf{G}}_m^{\log})_\alpha$ as a fiber product $T \times_{\operatorname{Hom}(E, \overline{\mathbf{G}}_m^{\log})} \operatorname{Hom}(V, \overline{\mathbf{G}}_m^{\log})$ where the map $T \to \operatorname{Hom}(E, \overline{\mathbf{G}}_m^{\log})$ is $e \mapsto \alpha_e \ell(e)$ and the map $\operatorname{Hom}(V, \overline{\mathbf{G}}_m^{\log}) \to \operatorname{Hom}(E, \overline{\mathbf{G}}_m^{\log})$ sends f to $e \mapsto f(v) - f(w)$.

We now conclude by noting that $\operatorname{Hom}(V, \overline{\mathbf{G}}_m^{\log})$ and $\operatorname{Hom}(E, \overline{\mathbf{G}}_m^{\log})$ are both isomorphic to products of copies of $\overline{\mathbf{G}}_m^{\log}$, which has a logarithmically smooth cover by a logarithmic scheme by Corollary 2.2.7.4. □

Proof of Lemma 4.11.4. Let P be valuative geometric point and let L be a P-point of $\mathbf{Log\,Pic}(X/S)$. Then by Proposition 3.5.1, there is a logarithmic modification Z of X_P over which L lies in the image of $\mathbf{Pic}(Z/P)$. But the construction of Z only involves finitely many values from the characteristic monoid $\overline{M}_P$. Therefore there is a submonoid of $\overline{M}_P^{\mathrm{gp}}$ containing the pullback of $\overline{M}_S$ and over which L can be defined. This modification can be extended to a logarithmic modification Y of $X \times_S T$ in some étale neighborhood T of P in S. We conclude that L lies in the image of $\operatorname{Pic}(Y/T)$, as required. □

Corollary 4.11.5. *There is a cover of* $\operatorname{Log\,Pic}(X/S)$ *by a logarithmic scheme that is representable by logarithmic spaces and is logarithmically smooth.*

Proof. Locally in S, we can identify $\mathbf{Log\,Pic}(X/S) = \operatorname{Log\,Pic}(X/S) \times \mathrm{B}\mathbf{G}_m^{\log}$ by identifying $\operatorname{Log\,Pic}(X/S)$ with the sheaf of logarithmic line bundles on X trivialized over a section. A section $\operatorname{Log\,Pic}(X/S) \to \mathbf{Log\,Pic}(X/S)$ makes $\operatorname{Log\,Pic}(X/S)$ into a $\mathbf{G}_m^{\log}$-bundle over $\mathbf{Log\,Pic}(X/S)$. If $U \to \mathbf{Log\,Pic}(X/S)$ is a logarithmically smooth cover by a logarithmic scheme, then its pullback is a logarithmically smooth cover $W \to \operatorname{Log\,Pic}(X/S)$, and W is a $\mathbf{G}_m^{\log}$-torsor over the logarithmic schcme U, hence a logarithmic space. □

Corollary 4.11.6. *The diagonals of* $\operatorname{Log\,Pic}(X/S)$ *and* $\mathbf{Log\,Pic}(X/S)$ *are representable by logarithmic spaces.*

Proof. Let Z be $\operatorname{Log\,Pic}(X/S)$ or $\mathbf{Log\,Pic}(X/S)$. We have a logarithmically smooth cover $U \to Z$ that is representable by logarithmic spaces. We wish to show that $W = V \times_{Z \times Z} Z$ is representable by logarithmic spaces whenever V is a logarithmic scheme. But

$$W \underset{Z \times Z}{\times} (U \times U) = (V \underset{Z \times Z}{\times} (U \times U)) \underset{U \times U}{\times} (U \underset{Z}{\times} U)$$

is the fiber product of the logarithmic space $U \times_Z U$ with the logarithmic space $V \times_{Z \times Z} (U \times U)$ over the logarithmic scheme $U \times U$, hence is a logarithmic space. □

4.12. **Representability of the diagonal.** Our algebraicity result is slightly stronger for Log Pic.

Theorem 4.12.1. *The diagonal of* $\mathrm{Log\,Pic}(X/S)$ *over* S *is representable by finite morphisms of logarithmic schemes.*

In other words, we are to show that if X is a proper, vertical logarithmic curve over S with two logarithmic line bundles L and L' then there is a universal logarithmic scheme T over S such that $L_T \simeq L'_T$ and, moreover, the underlying scheme of T is finite over that of S. This assertion only depends on the difference between L and L' in the group structure of Log Pic, so we can assume L' is trivial. The assertion is also local in the strict étale topology on S, so we freely replace S by an étale cover. By Corollary 4.10.2, the diagonal of $\mathrm{Log\,Pic}^d(X/S)$ satisfies the valuative criterion for properness, so it will suffice to prove that the diagonal is schematic, quasicompact, and locally quasifinite. In fact, morphisms of algebraic spaces that are separated and locally quasifinite are schematic [Sta18, Tag 03XX], so we only need to show the diagonal is representable by algebraic spaces, locally quasifinite, and quasicompact.

Lemma 4.12.2. *The relative diagonal of* $\mathrm{Log\,Pic}(X/S)$ *over* S *is quasicompact.*

Proof. It is sufficient to demonstrate that $\mathrm{Log\,Pic}^0(X/S)$ has quasicompact diagonal over S. This assertion is local in the constructible topology on S, so we assume that the dual graph of X is constant over S and that $\overline{M}_S$ is a constant sheaf on S. In this situation, Corollary 4.6.3 gives an étale cover of $\mathrm{Log\,Pic}^0(X/S)$ by $V = R^1\pi_*(\pi^*\mathbf{G}_m^{\log})^{[0]\dagger}$. By étale descent, it is sufficient to show that $V \times_{\mathrm{Log\,Pic}^0(X/S)} V \to V \times V$ is quasicompact.

Applying the change of coordinates $(v, w) \mapsto (v, w - v)$ to $V \times V$, we can recognize this map as the base change to V of the inclusion $H_1(\mathfrak{X}) = \ker(V \to \mathrm{Log\,Pic}^0(X/S)) \to V$. It therefore suffices to demonstrate that $H_1(\mathfrak{X}) \to V$ is quasicompact.

By the compatibility commutative square on the left side of (4.9.2), it suffices to demonstrate that $H_1(\mathfrak{X}) \to \mathrm{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})^\dagger$ is quasicompact. But this is precisely Corollary 3.11.6. □

Lemma 4.12.3. *Let* S *be a logarithmic scheme, let* $\mathfrak{X}$ *be a compact logarithmic curve over* S*. Then the zero section of* $\mathrm{Tro\,Jac}(\mathfrak{X}/S)$ *is representable by affine logarithmic schemes of finite type.*

Proof. Suppose we are given a section $S \to \mathrm{Tro\,Jac}(\mathfrak{X}/S)$. Let Z be the pullback of the zero section of $\mathrm{Tro\,Jac}(\mathfrak{X}/S)$ to S. We wish to show Z is representable by a finite type, affine logarithmic scheme over S. This is an étale-local assertion on S, so we can work locally in S and find a lift $S \to \mathrm{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})$. We can realize Z as the pullback of $\partial : H_1(\mathfrak{X}) \to \mathrm{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})$ to S. We therefore have (4.12.3.1):

$$Z = S \underset{\mathrm{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})}{\times} H_1(\mathfrak{X}) = \coprod_{\alpha \in H_1(\mathfrak{X})} S \underset{\mathrm{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})}{\times} \{\partial(\alpha)\} \tag{4.12.3.1}$$

But ∂ is quasicompact by Corollary 3.11.6, so we only need to consider finitely many of the $\alpha \in H_1(\mathfrak{X})$. We can therefore assume there is a single α. We write Z_α for the component of Z that corresponds.

Locally in S, we can choose a surjection from a finitely generated free abelian group A onto $H_1(\mathfrak{X})$. This induces an embedding $\mathrm{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log}) \to \mathrm{Hom}(A, \overline{\mathbf{G}}_m^{\log})$, which is a product of copies of $\overline{\mathbf{G}}_m^{\log}$. Applying Proposition 2.2.7.5 on each copy, we get the result. □

Corollary 4.12.4. *Let S be a logarithmic scheme and let X be a proper, vertical logarithmic curve over S. Then the zero section of* $\operatorname{Log}\operatorname{Pic}(X/S)$ *is representable by affine logarithmic schemes of finite type.*

Proof. We use the exact sequence from Theorem 4.9.4. By Lemma 4.12.3, the map from $\operatorname{Pic}^{[0]}(X/S)$ to $\operatorname{Log}\operatorname{Pic}^0(X/S)$ is representable by affine logarithmic schemes of finite type. But the zero section of $\operatorname{Pic}^{[0]}(X/S)$ is a closed embedding because $\operatorname{Pic}^{[0]}(X/S)$ is separated and schematic over S; in particular, it is affine and of finite type. We deduce that the zero section of $\operatorname{Log}\operatorname{Pic}(X/S)$ is affine and of finite type. $\square$

Proof of Theorem 4.12.1. The diagonal of $\operatorname{Log}\operatorname{Pic}(X/S)$ is the base change of the embedding of the zero section, so it is sufficient to demonstrate that the embedding of the zero section is finite. We have seen that it is affine and of finite type in Corollary 4.12.4. On the other hand, it also satisfies the valuative criterion for properness by Corollary 4.10.2. We conclude that it is finite, as requried. $\square$

Corollary 4.12.5. *For each integer d, the sheaf* $\operatorname{Log}\operatorname{Pic}^d(X/S)$ *and the stack* $\mathbf{Log}\,\mathbf{Pic}^d(X/S)$ *are proper over S.*

Proof. We have shown that $\operatorname{Log}\operatorname{Pic}^d(X/S)$ has finite diagonal by Theorem 4.12.1, is bounded by Corollary 4.9.5, and satisfies the valuative criterion by Theorem 4.10.1. The properness of $\mathbf{Log}\,\mathbf{Pic}^d(X/S)$ follows because it is a gerbe banded by the proper group $\mathbf{G}_m^{\log}$ over $\operatorname{Log}\operatorname{Pic}^d(X/S)$. $\square$

4.13. **Smoothness.**

Theorem 4.13.1. *Let X be a logarithmic curve over S. Then* $\operatorname{Log}\operatorname{Pic}(X/S)$ *is logarithmically smooth.*

There are two parts to smoothness: the infinitesimal criterion and local finite presentation. Local finite presentation was addressed in Proposition 4.2.2.

Lemma 4.13.2. $\mathbf{Log}\,\mathbf{Pic}(X/S)$ *satisfies the infinitesimal criterion for smoothness over S. Its logarithmic tangent stack is $\pi_* \mathrm{B}\mathbf{G}_a$, meaning isomorphism classes of deformations are a torsor under $H^1(X, \mathcal{O}_X)$ and automorphisms are in bijection with $H^0(X, \mathcal{O}_X)$.*

Proof. Consider a lifting problem (4.13.2.1)

(4.13.2.1)
$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathbf{Log}\,\mathbf{Pic}(X/S) \arrow[d] \\ S' \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

in which S' is a strict infinitesimal square-zero extension of S. The lower horizontal arrow gives a logarithmic curve X' over S' with fiber X over S, and the upper horizontal arrow gives a logarithmic line bundle L on X. We wish to extend this to S. It is sufficient to assume that S' is a square-zero extension with ideal J.

Let $\overline{L}$ be the $\overline{M}_X^{\mathrm{gp}}$-torsor induced from L. As this is a torsor under an étale sheaf, and the étale sites of X and X' are identical, $\overline{L}$ extends uniquely to $\overline{L}'$. We therefore assume $\overline{L}'$ is fixed. We note that the bounded monodromy condition for a putative L' extending L depends only on $\overline{L}'$, and is equivalent to that for $\overline{L}$, hence is automatically satisfied.

We wish to show that $\overline{L}'$ can be lifted to an $\overline{M}_{X'}^{\mathrm{gp}}$-torsor. Locally in X there is no obstruction to extending L to L'. If we take any two local extensions of L, their difference $L' \otimes L''^{\vee}$ is a $M^{\mathrm{gp}}_{X'}$-torsor whose restriction to X is trivialized, as is its induced $\overline{M}^{\mathrm{gp}}_{X'}$-torsor. Therefore $L' \otimes L''^{\vee}$ is induced from a uniquely determined $\mathcal{O}^*_{X'}$-torsor exending the trivial one from X.

It follows that extensions of L form a gerbe on X banded by $\mathcal{O}_X \otimes J$. Obstructions to producing a lift — that is, a section of this gerbe — lie in $H^2(X, \mathcal{O}_X \otimes J)$, which vanishes locally in S because X is a curve over S. By the cohomological classification of banded gerbes, deformations form a torsor under $H^1(X, \mathcal{O}_X \otimes J)$ and automorphisms are in bijection with $H^0(X, \mathcal{O}_X \otimes J)$.

To get the logarithmic tangent space, we take a trivial extension S' of S by $J = \mathcal{O}_S$. □

4.14. **Tropicalizing the logarithmic Picard group.** Let X be a proper, vertical logarithmic curve over S and let $\mathfrak{X}$ denote the tropicalization of X. We construct a tropicalization map (4.14.1):

$$\mathbf{Log\,Pic}(X/S) \to \mathbf{Tro\,Pic}(\mathfrak{X}/S) \tag{4.14.1}$$

Since $\mathbf{Tro\,Pic}(\mathfrak{X}/S)$ is locally constant on the logarithmic strata of S, our strategy will be to construct (4.14.1) stratumwise and then show its compatibility with generization.

Assume first that S has constant characteristic monoid and that the dual graph of X is constant over S. Under these assumptions, we have an anticontinuous tropicalization map $t : X \to \mathfrak{X}$.

Suppose that Q is a M^{gp}_X-torsor on X. Let $\mathfrak{U} \to \mathfrak{X}$ be a local isomorphism and let $U = t^{-1}\mathfrak{U}$. Let $\mathrm{NS}(U)$ denote the Néron–Severi group of U. Then NS is a functor on finite X-schemes and we observe that the sheaf $\mathfrak{V}$ on $\mathfrak{X}$ (whose sections are members of the free abelian group generated by the vertices) is isomorphic to $t_* \mathrm{NS}$. Combined with Lemma 2.4.3.4 and the exact sequence in the middle column of (3.4.1), this proves Proposition 4.14.2:

Proposition 4.14.2. *Let X be a logarithmic curve over S, where S has constant characteristic monoid and X has constant dual graph. Let $\mathfrak{X}$ be the tropicalization of X. Then the sheaf of linear functions $\mathfrak{L}$ on $\mathfrak{X}$ is quasi-isomorphic to $t_*[\overline{M}^{\mathrm{gp}}_X \to \mathrm{NS}]$.*

Suppose that L is a logarithmic line bundle on X, continuing to assume that the characteristic monoid of S and the dual graph of X are constant over S. Then L induces an $\overline{M}^{\mathrm{gp}}_X$-torsor $\overline{L}$ on X with a trivialization of its induced $\mathrm{B}\mathcal{O}^*_X$-torsor, which we denote $\mathrm{B}\mathcal{O}^*_X(\overline{L})$. This trivialization implies that $\pi_*\mathrm{B}\mathcal{O}^*_X(\overline{L})$ is a trivialized torsor under $\pi_*\mathrm{B}\mathcal{O}^*_X = \mathbf{Pic}(X/S)$. Passing to the Néron–Severi group, we acquire a trivialized $\mathrm{NS}(X)$-torsor.

Remark 4.14.3. In what may be more concrete terms, the trivialization of $\mathrm{B}\mathcal{O}^*_X(\overline{L})$-torsor amounts to the specification of a $\mathcal{O}^*_X$-torsor $L(\gamma)$ for each local section γ of $\overline{L}$, along with isomorphisms $L(\alpha + \gamma) \simeq \mathcal{O}_X(\alpha) \otimes L(\gamma)$ for each local section α of $\overline{M}^{\mathrm{gp}}_X$, subject to a compatibility condition concerning the two isomorphisms $L(\alpha + \beta + \gamma) \simeq \mathcal{O}_X(\alpha) \otimes \mathcal{O}_X(\beta) \otimes L(\gamma)$, that we leave to the reader to make explicit. In this case, the trivialization arises by taking $L(\gamma)$ to be the fiber of L over $\gamma \in \overline{L}$.

If $\gamma \in \Gamma(U, \overline{L})$, then let $[L(\gamma)]$ denote the class of $L(\gamma)$ in $\mathrm{NS}(U)$. Then we have $[L(\alpha+\gamma)] = [\mathcal{O}^*_X(\alpha)] + [L(\gamma)]$.

The logarithmic line bundle L therefore induces a $\mathfrak{P}$-torsor on $\mathfrak{X}$ with a trivialization of its induced $\mathfrak{V}$-torsor. By the quasi-isomorphism $\mathfrak{L} \simeq [\mathfrak{P} \to \mathfrak{V}]$, we obtain a $\mathfrak{L}$-torsor on $\mathfrak{X}$. This gives the map (4.14.1) in the case of a constant characteristic monoid and constant dual graph.

In order to extend this construction to one valid over a general base, we will need to prove its compatibility with the generization maps for $\mathbf{Tro\,Pic}(\mathfrak{X}/S)$, given by Proposition 3.8.2.

Proposition 4.14.4. *Let X be a proper logarithmic curve over S and let s be a geometric point of S. Then $\pi_*(\mathrm{B}\overline{M}_X^{\mathrm{gp}})_s \to \Gamma(X_s, \mathrm{B}\overline{M}_{X_s}^{\mathrm{gp}})$ is injective and restricts to an isomorphism on the bounded monodromy subgroups.*

Proof. Injectivity follows from proper base change for étale cohomology [Art73, Théorème 5.1 (i) and (ii)], so the point is to prove surjectivity on the bounded monodromy subgroup. The assertion is étale-local in S, so we may assume that S is an atomic neighborhood of s and that the dual graph of X is constant on the closed stratum of S.

Suppose that $\overline{L}_s$ is an $\overline{M}_X^{\mathrm{gp}}$-torsor on X_s with bounded monodromy. We extend $\overline{L}_s$ to an $\overline{M}_X^{\mathrm{gp}}$-torsor on X inductively over the strata of S. By induction, we can assume that $\overline{L}_Z$ has already been constructed on a closed union of strata Z containing s and that the complement of Z in S is an open subset U on which $\overline{M}_S$ is constant. Let j denote the inclusion of U in S.

The homomorphism $\overline{M}_S^{\mathrm{gp}} \to j_* \overline{M}_U^{\mathrm{gp}}$ induces a homomorphism $\overline{M}_X^{\mathrm{gp}} \to \overline{N}_X^{\mathrm{gp}}$ by pushout. Let $\overline{K}_Z$ be the $\overline{N}_X^{\mathrm{gp}}$-torsor on X_Z induced along this homomorphism.

Let $\mathfrak{X}_U$ denote the dual graph of a geometric fiber of X over U and let $\mathfrak{V}_U$ be its universal cover. Pulling back along the projection $\mathfrak{X}_\xi \to \mathfrak{X}_U$ we obtain an étale cover $\mathfrak{V}_\xi$ of $\mathfrak{X}_\xi$, which corresponds to an étale cover of X_ξ. By construction, this cover extends to an étale cover $\rho : V \to X$ of all of X.

We also use ρ to denote the restriction of ρ to the preimage of Z. The pullback $\rho^* \overline{K}_Z$ is trivial. Indeed, it suffices to trivialize $\rho^* \overline{K}_\xi$, and $\overline{K}_\xi$ has trivial monodromy around all loops $\mathfrak{V}_\xi$, by its construction and the assumption of bounded monodromy in $\overline{L}$. Then $\rho^* \overline{K}_Z$ extends trivially to an $\overline{N}_X^{\mathrm{gp}}$-torsor $\overline{K}'$ on V and the action of deck transformations extends as well. By descent, we obtain an $\overline{N}_X$-torsor $\overline{K}$ on X extending $\overline{K}_Z$.

We may now define $\overline{L} = \overline{K} \times_{i_* \overline{K}_Z} i_* \overline{L}_Z$ where i is the inclusion of X_Z in X. This is a torsor under $\overline{N} \times_{i_* \overline{N}_{X_Z}} i_* \overline{M}_{X_Z}$, which is isomorphic to $\overline{M}_X$ by the canonical map. $\square$

Suppose now that S is a strictly henselian valuation ring with special point ξ and generic point η. We have a commutative diagram with exact columns:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\Gamma(X_\xi, \mathrm{B}M_{X_\xi}^{\mathrm{gp}}) \arrow[d] & \Gamma(X, \mathrm{B}M_X^{\mathrm{gp}}) \arrow[l] \arrow[r] \arrow[d] & \Gamma(X_\eta, \mathrm{B}M_{X_\eta}^{\mathrm{gp}}) \arrow[d] \\
\Gamma(X_\xi, \mathrm{B}\overline{M}_{X_\xi}^{\mathrm{gp}}) \arrow[d] & \Gamma(X, \mathrm{B}M_X^{\mathrm{gp}}) \arrow[l] \arrow[r] \arrow[d] & \Gamma(X_\eta, \mathrm{B}\overline{M}_{X_\eta}^{\mathrm{gp}}) \arrow[d] \\
\Gamma(X_\xi, \mathrm{B}^2\mathcal{O}_{X_\xi}^*) & \Gamma(X, \mathrm{B}^2\mathcal{O}_{X_\xi}^*) \arrow[l] \arrow[r] & \Gamma(X_\eta, \mathrm{B}^2\mathcal{O}_{X_\xi}^*)
\end{tikzcd}$$

Upon passage to the bounded monodromy subgroups and composing with the projection from the Picard group to the Néron–Severi group, we obtain (4.14.5):

$$\begin{tikzcd}
\mathbf{Log\,Pic}(X/S)(\xi) \arrow[d] & \mathbf{Log\,Pic}(X/S)(S) \arrow[l] \arrow[r] \arrow[d] & \mathbf{Log\,Pic}(X/S)(\eta) \arrow[d] \\
\Gamma(X_\xi, \mathrm{B}\overline{M}^{\mathrm{gp}}_{X_\xi})^\dagger \arrow[d] & \Gamma(X, \mathrm{B}\overline{M}^{\mathrm{gp}}_X)^\dagger \arrow[l, "\sim"'] \arrow[r] \arrow[d] & \Gamma(X_\eta, \mathrm{B}\overline{M}_{X_\eta})^\dagger \arrow[d] \\
\mathrm{B}\,\mathrm{NS}(X_\xi) & (\mathrm{B}\,\mathrm{NS}(X/S))(S) \arrow[l, "\sim"'] \arrow[r] & \mathrm{B}\,\mathrm{NS}(X_\eta)
\end{tikzcd} \tag{4.14.5}$$

The isomorphism in the second row is Proposition 4.14.4 and we get the isomorphism $(\mathrm{B}\,\mathrm{NS}(X/S))(S) \simeq \mathrm{B}\,\mathrm{NS}(X_\xi)$ from the knowledge that $\mathrm{NS}(X/S)$ is an étale sheaf over S.

The vertical compositions in diagram (4.14.5) are canonically trivialized, as was discussed earlier. Proposition 4.14.2 implies that $\mathbf{Tro\,Pic}(\mathfrak{X}_\xi)$ is the kernel of $\Gamma(\mathfrak{X}_\xi, \mathrm{B}\overline{M}^{\mathrm{gp}}_{X_\xi})^\dagger \to \mathrm{B}\,\mathrm{NS}(X_\xi)$ (and similarly over η) so we obtain a commutative diagram (4.14.6) :

$$\begin{tikzcd}
\mathbf{Log\,Pic}(X/S)(\xi) \arrow[d] & \mathbf{Log\,Pic}(X/S)(S) \arrow[l] \arrow[r] \arrow[dl] & \mathbf{Log\,Pic}(X/S)(\eta) \arrow[d] \\
\mathbf{Tro\,Pic}(\mathfrak{X}_\xi) \arrow[rr] & & \mathbf{Tro\,Pic}(X_\eta)
\end{tikzcd} \tag{4.14.6}$$

We leave it to the reader to verify that the construction in the proof of Proposition 4.14.4 is the same as the one used in the proof of Proposition 3.8.2 so that the map $\mathbf{Tro\,Pic}(\mathfrak{X}_\xi) \to \mathbf{Tro\,Pic}(X_\eta)$ displayed above is indeed the same as the one guaranteed by Proposition 3.8.2. The commutativity of the inner trapezoid gives the compatibility of the tropicalization map with generization.

Theorem 4.14.7. *Let X be a proper, vertical logarithmic curve over S and let $\mathfrak{X}$ be its tropicalization. Then there are exact sequences (in the étale topology):*

$$0 \to \mathbf{Pic}^{[0]}(X/S) \to \mathbf{Log\,Pic}(X/S) \to \mathbf{Tro\,Pic}(\mathfrak{X}/S) \to 0$$
$$0 \to \mathrm{Pic}^{[0]}(X/S) \to \mathrm{Log\,Pic}(X/S) \to \mathrm{Tro\,Pic}(\mathfrak{X}/S) \to 0$$

Proof. The second exact sequence is obtained from the first by dividing, term by term, by the exact sequence (4.14.7.1):

$$0 \to \mathrm{B}\mathbf{G}_m \to \mathrm{B}\mathbf{G}_m^{\log} \to \mathrm{B}\overline{\mathbf{G}}_m^{\log} \to 0 \tag{4.14.7.1}$$

We have exact sequences (4.14.7.2):

$$\begin{aligned} 0 \to \mathcal{O}_X^* \to M_X^{\mathrm{gp}} \to \overline{M}_X^{\mathrm{gp}} \to 0 \qquad &(\text{on } X) \\ 0 \to \mathfrak{L} \to \mathfrak{P} \to \mathfrak{V} \to 0 \qquad &(\text{on } \mathfrak{X}) \end{aligned} \tag{4.14.7.2}$$

Rotating these sequences, pushing forward to S, and restricting to bounded monodromy, we get a commutative diagram of exact sequences (with $\rho_*\mathrm{B}\mathfrak{P}$ denoting the stack on S of

$\mathfrak{P}$-torsors on $\mathfrak{X}$):

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Pic}(X/S) \arrow[r] \arrow[d] & \mathbf{Log\,Pic}(X/S) \arrow[r] \arrow[d] & \pi_* \mathrm{B}\overline{M}_X^{\mathrm{gp}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{NS}(X/S) \arrow[r] & \mathbf{Tro\,Pic}(\mathfrak{X}/S) \arrow[r] & \rho_* \mathrm{B}\mathfrak{P} \arrow[r] & 0
\end{tikzcd}$$

The kernel of $\mathbf{Log\,Pic}(X/S) \to \mathbf{Tro\,Pic}(X/S)$ therefore coincides with the kernel of the map $\mathbf{Pic}(X/S) \to \mathrm{NS}(X/S)$, which is $\mathbf{Pic}^{[0]}(X/S)$. Likewise, $\mathbf{Pic}(X/S)$ surjects onto $\mathrm{NS}(X/S)$ is surjective so $\mathbf{Log\,Pic}(X/S) \to \mathbf{Tro\,Pic}(X/S)$ is surjective as well. □

4.15. **Logarithmic abelian variety structure.** In this section, we explain how the logarithmic Jacobian carries the structure of a log abelian variety, in the sense of [KKN15]. For the convenience of the reader, we recall briefly the necessary definitions. For details and proofs, we refer the reader to [KKN15]. We try to keep the notations of [KKN15] as much as possible, but some changes will be necessary in order to avoid conflicts with notation already introduced here. We fix a base log scheme S, and form the site fs/S, whose objects are fine and saturated log schemes over S and whose coverings are strict étale maps.

Let G be a semiabelian group scheme, that is, an extension

$$1 \to T \to G \to A \to 1 \tag{4.15.1}$$

of an abelian variety A by a torus $T = \operatorname{Spec} \mathbf{Z}[H]$. Here H is a sheaf of lattices over fs/S. Just as $\mathbf{G}_m^{\log}$ extends $\mathbf{G}_m$, there is a sheaf $T^{\log} = \mathbf{G}_m^{\log} \otimes_{\mathbf{G}_m} T$ extending T that can be defined on fs/S by the folowing formula:

$$T^{\log}(S') = \operatorname{Hom}(H, M_{S'}^{\mathrm{gp}}) \tag{4.15.2}$$

Equivalently, $T^{\log} = \underline{\operatorname{Hom}}(H, \mathbf{G}_m^{\log})$, where we regard H as a sheaf on fs/S, and $\underline{\operatorname{Hom}}$ denotes the sheaf of homomorphisms. There is an evident inclusion $T \to T^{\log}$, and pushing out $T \to G$ along this map we obtain an exact sequence

$$1 \to T^{\log} \to G^{\log} \to A \to 1 \tag{4.15.3}$$

where $G^{\log} = T^{\log} \oplus_T G$.

Definition 4.15.4 ([KKN08b, Definition 2.2])**.** A *log 1-motif* is a map $K \to G^{\log}$, where G is a semiabelian group scheme and K is locally free sheaf of abelian groups on fs/S.

The map $K \to G^{\log}$ naturally defines a subsheaf $G^{\log}_{(K)} \subset G^{\log}$ as follows. The composed map from K to the quotient $G^{\log}/G \cong T^{\log}/T = \overline{T}^{\log}$ determines a pairing $\langle\, , \rangle : H \times K \to \overline{\mathbf{G}}_m^{\log}$, and a subsheaf $\overline{T}^{\log}_{(K)}$, determined by the formula

$$\overline{T}^{\log}_{(K)}(S') = \left\{ \phi \in \overline{T}^{\log}(S') \;\middle|\; \begin{array}{l} \forall \text{ geometric points } s \in S', x \in H_s, \\ \exists y, y' \in K \text{ s.t } \langle x, y\rangle \le \phi(x) \le \langle x, y'\rangle \end{array} \right\} \tag{4.15.5}$$

We thus obtain $G^{\log}_{(K)}$ by simply pulling back $\overline{T}^{\log}_{(K)}$ under the map $G \to \overline{T}^{\log}$. A log 1-motif defines an *abelian variety with constant degeneration*, by assigning to $Y \to G^{\log}$ the quotient sheaf $G^{\log}_{(K)}/K$.

Definition 4.15.6 ([KKN08a, Definition 4.1])**.** A *log abelian variety* is a sheaf $\mathcal{A}$ on fs/S such that all of the following properties hold:

(1) For each geometric point $s \in S$, the pullback of $\mathcal{A}$ to fs/s is a *log abelian variety* with constant degeneration.
(2) Étale locally on S, there is an exact sequence (4.15.6.1) for some bilinear form $H \times K \to \Gamma(S, \overline{M}_S^{\mathrm{gp}})$ and $\overline{T}^{\log} = \underline{\mathrm{Hom}}(H, \overline{\mathbf{G}}_m^{\log})$:

$$0 \to G \to \mathcal{A} \to \overline{T}^{\log}_{(K)}/K \to 0 \qquad (4.15.6.1)$$

(3) Let $\overline{K}$ denote the image of K in $\underline{\mathrm{Hom}}(H, \overline{\mathbf{G}}_m^{\log})$ and $\overline{H}$ the image of H in $\underline{\mathrm{Hom}}(K, \overline{\mathbf{G}}_m^{\log})$. For each geometric point $s \in S$, there exists a map $\phi : \overline{K}_s \to \overline{H}_s$ with finite cokernel such that $\langle \phi(y), z\rangle = \langle y, \phi(z)\rangle$ for all $y, z \in \overline{K}_s$, and $\langle \phi(y), y\rangle \in \overline{M}_{S,s}$.
(4) The diagonal $\mathcal{A} \to \mathcal{A} \times \mathcal{A}$ is representable by finite morphisms.

We are now ready to indicate how the logarithmic Jacobian fits into this context.

Theorem 4.15.7. *Let X be a proper, vertical logarithmic curve over S. Then* $\mathrm{Log\,Pic}^0(X/S)$ *is a logarithmic abelian variety in the sense of Kajiwara, Kato, and Nakayama* [KKN08a].

Proof. We verify the conditions of Definition 4.15.6.

Given a family of logarithmic curves $X \to S$, with dual graph $\mathfrak{X}$, we obtain a sheaf of lattices $H_1(\mathfrak{X})$. We set $H = K = H_1(\mathfrak{X})$ for the lattices appearing in the definition above, and take $\langle\,,\,\rangle$ to be the intersection pairing. We let $G = \mathrm{Pic}^{[0]}(\mathrm{X})$ denote the multidegree 0 part of $\mathrm{Pic}(X/S)$.

The third condition in the definition is immediate in our context: The two lattices X and Y are $H_1(\mathfrak{X})$, and we may take $\phi = \mathrm{id}$. For any $y \in H_1(\mathfrak{X}_s)$, the pairing $\langle y, y\rangle$ is a sum of elements of $\overline{M}_{S,s}$ by Definition 3.3.1, and therefore is in $\overline{M}_{S,s}$.

The last condition is exactly Theorem 4.12.1.

The first and second condition follow from Corollary 4.6.3 and the exact sequence of 4.9.4 respectively, once we observe:

Lemma 4.15.8. *For $K = H_1(\mathfrak{X})$, the subsheaf $\overline{T}^{\log}_{(K)}$ coincides with the subsheaf of elements with bounded monodromy $(\overline{T}^{\log})^\dagger$ in $\overline{T}^{\log}$.*

Proof. Since both the bounded monodromy condition and the condition defining $\overline{T}^{\log}_{(Y)}$ are defined pointwise, we may check that the two groups are the same on a logarithmic scheme s whose underlying scheme is the spectrum of an algebraically closed field. If $\phi : H_1(\mathfrak{X}) \to \overline{M}^{\mathrm{gp}}$ has bounded monodromy then, by definition, there are integers m and n such that $m\langle x, x\rangle \le \phi(x) \le n\langle x, x\rangle$. Thus $\phi \in \overline{T}^{\log}_{(Y)}$ as it verifies the definition with $y = mx, y' = nx$.

For the converse, suppose that $\phi : H_1(\mathfrak{X}) \to \overline{M}_s^{\mathrm{gp}}$ and, for every $x \in H_1(\mathfrak{X})$, there are $y, y' \in H_1(\mathfrak{X})$ such that $\langle x, y\rangle \le \phi(x) \le \langle x, y'\rangle$. For any $y \in H_1(\mathfrak{X})$, we have $\langle x, y\rangle \le n\langle x, x\rangle$ for some positive integer n. Indeed, we may take n to be the maximum of the coefficients of y as a linear combination of edges of $\mathfrak{X}$. We likewise have $\langle x, y'\rangle \le m\langle x, x\rangle$ for some positive integer m, and therefore $-m\langle x, x\rangle \le \phi(x) \le n\langle x, x\rangle$, as required. ∎

∎

4.16. **Prorepresentability.** The logarithmic Picard group and logarithmic Jacobian cannot be represented by schemes, or even by algbraic stacks, with logarithmic structures. This follows from the nonrepresentability of the logarithmic multiplicative group, which was proved in Proposition 2.2.7.2. We have already seen in Section 4.11 that it is nearly representable in the sense that it has a logarithmically smooth cover by a logarithmic scheme. In this section we will consider another near-representability property.

Let $\mathfrak{X}$ be the tropicalization of a logarithmic curve X over S. Theorem 4.9.4 shows that $\operatorname{Log}\operatorname{Pic}^0(X/S)$ is a torsor under the algebraic group $\operatorname{Pic}^{[0]}(X/S)$ over $\operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S)$ and Theorem 4.14.7 shows that $\operatorname{Log}\operatorname{Pic}(X/S)$ is a $\operatorname{Pic}^{[0]}(X/S)$-torsor over $\operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S)$. Therefore the nonrepresentability of $\operatorname{Log}\operatorname{Pic}(X/S)$ can be attributed to the nonrepresentability of $\operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S)$. However, we saw in Section 3.9 that $\operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S)$ is prorepresentable. We might therefore reasonably expect $\operatorname{Tro}\operatorname{Pic}(X/S)$ to be similarly prorepresentable.

We saw in Lemma 3.9.1 that $\operatorname{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})^\dagger$ is, locally in S, pro-representable a collection of submonoids of $\overline{M}_S^{\mathrm{gp}} + H_1(\mathfrak{X})$. Each of these submonoids represents a functor on logarithmic schemes that can be represented by an algebraic stack with a logarithmic structure (see [CCUW17, Section 6] for further details). Therefore we can think of $\operatorname{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})$ as ind-representable on logarithmic schemes by algebraic stacks with logarithmic structure. Since $\operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S)$ is a quotient of $\operatorname{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})^\dagger$ by $H_1(\mathfrak{X})$, we conclude that $\operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S)$ is, locally in S, the quotient of an ind-algebraic stack with logarithmic structure by $H_1(\mathfrak{X})$. The same applies to $\operatorname{Tro}\operatorname{Pic}^d(\mathfrak{X}/S)$ for all d, since it is a torsor under $\operatorname{Tro}\operatorname{Pic}^0(\mathfrak{X}/S) = \operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S)$.

Proposition 4.16.1. *Therefore* $\operatorname{Log}\operatorname{Pic}(X/S)$ *is, locally in* S*, the quotient of an ind-algebraic stack with a logarithmic structure by the action of* $H_1(\mathfrak{X})$*.*

Proof. $\operatorname{Log}\operatorname{Pic}(X/S)$ is a torsor over $\operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S)$ under the algebraic group $\operatorname{Pic}^{[0]}(X/S)$. □

For a moduli problem F on logarithmic schemes, one defines a minimal logarithmic structure on an S-point of F in such a way that when F is representable, minimality corresponds to strictness of the morphism $S \to F$. We introduce a similar notion that corresponds to strictness *at the level of associated groups.*

Definition 4.16.2. Let S be a logarithmic scheme and let F be a covariant functor valued in sets on logarithmic structures over M_S such that $F(M_S)$ has one element. We say that a logarithmic structure N over M_S and an object $\xi \in F(N)$ is *pseudominimal* if, for every $\eta \in F(P)$, there is a unique morphism $u : N^{\mathrm{gp}} \to P^{\mathrm{gp}}$ and $\xi' \in F(u^{-1}P \cap N)$ that is sent to ξ under $u^{-1}P \cap N \to N$ and is sent to η under $u^{-1}P \cap N \to P$.

If F is a presheaf on logarithmic schemes then we say $\xi \in F(T)$ is *pseudominimal* if ξ is pseudominimal when F is regarded as a functor on logarithmic structures over M_T.

Proposition 4.16.3. *An* T*-point of* $\operatorname{Log}\operatorname{Pic}^0(X/S)$ *over* $f : T \to S$ *is pseudominimal if and only if the canonical map* $f^* M_S^{\mathrm{gp}} + H_1(\mathfrak{X}) \to \overline{M}_T^{\mathrm{gp}}$ *is a bijection.*

Proof. Since $\operatorname{Log}\operatorname{Pic}^0(X/S) \to \operatorname{Tro}\operatorname{Jac}(X/S)$ is strict, a T-point of $\operatorname{Log}\operatorname{Pic}^0(X/S)$ is pseudominimal if and only if the induced T-point of $\operatorname{Tro}\operatorname{Jac}(X/S)$ is pseudominimal. The proposition therefore follows from Lemma 3.9.1. □

4.17. **Schematic models.** We show that the combinatorics of the tropical Picard group can be used to construct toroidal compactifications of $\operatorname{Log}\operatorname{Pic}^d(X/S)$. This section is inspired directly by Kajiwara, Kato, and Nakayama [Kaj93, KKN15] and is, for the most part, only a tropical reinterpretation of their results.

Suppose that X is a logarithmic curve over a logarithmic scheme S with tropicalization $\mathfrak{X}$. For simplicity, we assume that S is atomic, or at least that it has a morphism to h_σ for some rational polyhedral cone σ, dual to $\overline{M}$, and that $\mathfrak{X}$ is pulled back from a tropical curve $\mathfrak{Y}$ over h_σ. Then $\operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S)$ is pulled back from $\operatorname{Tro}\operatorname{Pic}(\mathfrak{Y})$. A subdivision $\mathfrak{Z}$ of $\operatorname{Tro}\operatorname{Pic}(\mathfrak{Y})$ induces a subdivision of $\operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S)$ and the a subdivision $\operatorname{Log}\operatorname{Pic}(X/S)_{\mathfrak{Z}}$ of $\operatorname{Log}\operatorname{Pic}(X/S)$ by pullback. Since subdivisions are proper and $\operatorname{Log}\operatorname{Pic}(X/S)$ is proper, the subdivision, $\operatorname{Log}\operatorname{Pic}(X/S)_{\mathfrak{Z}}$, is proper as well.

Suppose now that $\mathfrak{Z}$ is actually representable by a cone space in the sense of [CCUW17]. Then $\mathfrak{Z} \times_\sigma S$ is representable by an algebraic stack over S with a logarithmic structure. By Theorem 4.14.7, $\operatorname{Log}\operatorname{Pic}(X/S)$ is a torsor over $\operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S)$ under the group scheme $\operatorname{Pic}^{[0]}(X/S)$. Therefore $\operatorname{Log}\operatorname{Pic}(X/S)_{\mathfrak{Z}}$ is also a torsor over $\mathfrak{Z}_S$ under the same group scheme. This implies that $\operatorname{Log}\operatorname{Pic}(X/S)_{\mathfrak{Z}}$ is representable by an algebraic stack with a logarithmic structure.

Lemma 4.17.1. *Let L be a logarithmic line bundle on a proper, vertical logarithmic curve X over S. Assume that the logarithmic structure of S is pseudominimal. Then the automorphism group of L, fixing X and S and the minimal logarithmic structure of X, is $\Gamma(S, M_S^{\mathrm{gp}})$.*

Proof. Consider the sheaf A on X whose sections over an étale $U \to X$ consist of an automorphism ϕ of $\pi^* M_S\big|_U$ and an isomorphism between $L\big|_U$ and $\phi^* L\big|_U$. Since logaritihmic line bundles are locally trivial, an isomorphism between $L\big|_U$ and $\phi^* L\big|_U$ always exists locally in X and there is therefore an exact sequence:

$$0 \to M_X^{\mathrm{gp}} \to A \to \operatorname{Hom}(H_1(\mathfrak{X}), \mathbf{G}_m)_X \to 0$$

Pushing forward to S, we get (4.17.1.1):

$$0 \to \pi_* M_X^{\mathrm{gp}} \to \pi_* A \to \operatorname{Hom}(H_1(\mathfrak{X}), \mathcal{O}_S^*) \to R^1 \pi_* M_X^{\mathrm{gp}} \tag{4.17.1.1}$$

The map $\operatorname{Hom}(H_1(\mathfrak{X}), \mathcal{O}_S^*) \to R^1 \pi_* M_X^{\mathrm{gp}}$ sends a homomorphism ϕ to the multidegree 0 line bundle on X obtained by gluing using ϕ around the loops of $\mathfrak{X}$. It is, in other words, the inclusion of the torus part of $\operatorname{Pic}^{[0]}(X)$ in $\operatorname{Log}\operatorname{Pic}(X)$, and in particular is injective. It follows that $\pi_* M_X^{\mathrm{gp}} \to \pi_* A$ is bijective. By Lemma 4.6.1, $\pi_* M_X^{\mathrm{gp}} = M_S^{\mathrm{gp}}$ and the lemma is proved. $\square$

Corollary 4.17.2. *Let $\operatorname{Log}\operatorname{Pic}(X/S)_{\mathfrak{Z}}$ be a subdivision of $\operatorname{Log}\operatorname{Pic}(X/S)$ that is representable by an algebraic stack with a logarithmic structure. Then $\operatorname{Log}\operatorname{Pic}(X/S)_{\mathfrak{Z}}$ is representable by an algebraic* space *with a logarithmic structure.*

Proof. Since objects of $\operatorname{Log}\operatorname{Pic}(X/S)_{\mathfrak{Z}}$ are pseudominimal, Lemma 4.17.1 shows that objects of $\operatorname{Log}\operatorname{Pic}(X/S)_{\mathfrak{Z}}$ have no nontrivial automorphisms. Therefore $\operatorname{Log}\operatorname{Pic}(X/S)_{\mathfrak{Z}}$ is a sheaf, and hence an algebraic space. $\square$

4.18. **Unintegrable torsors.** We will show that a $\mathbf{G}_m^{\log}$-torsor on a logarithmic curve that deforms to all infinitesimal orders does not necessarily integrate to a $\mathbf{G}_m^{\log}$-torsor over a complete noetherian local ring. Such objects are excluded from the logarithmic Picard group by the bounded monodromy condition of Definition 3.5.4, and this section is meant to explain the reason behind that condition.

In this section, we can take cohomology either in the Zariski topology or the étale topology.

Let P be a $\mathbf{G}_m^{\log}$-torsor on a logarithmic scheme X. By the projection $\mathbf{G}_m^{\log} \to \overline{\mathbf{G}}_m^{\log}$, this induces a $\overline{\mathbf{G}}_m^{\log}$-torsor $\overline{P}$ over X. We note that there is an exact sequence:

$$H^1(X, M_X^{\mathrm{gp}}) \to H^1(X, \overline{M}_X^{\mathrm{gp}}) \to H^2(X, \mathcal{O}_X^*)$$

As $H^2(X, \mathcal{O}_X^*)$ vanishes for a curve over an algebraically closed field (or, more generally, over an artinian local ring with algebraically closed residue field), every $\overline{\mathbf{G}}_m^{\log}$-torsor on such a curve lifts to a $\mathbf{G}_m^{\log}$-torsor. To prove the existence of an unintegrable $\mathbf{G}_m^{\log}$-torsor, it will therefore suffice to give an example of an unintegrable $\overline{\mathbf{G}}_m^{\log}$-torsor on a family of logarithmic curves over a complete noetherian local ring with algebraically closed residue field.

Let $S = \operatorname{Spec} \mathbf{C}[[t]]$ and let X be a family of curves with smooth total space such that the general fiber is smooth and connected, but the special fiber has two irreducible components, joined to each other at two ordinary double points, but is otherwise smooth. This is essentially the simplest example where étale cohomology with non-torsion coefficients does not commute with base change [Art73, §2]. In this example, cohomology in the Zariski topology also fails to commute with base change.

Let M_S be the divisorial logarithmic structure on S and let M'_S be the trivial extension of M_S by $\mathbf{N}$ (the generator of $\mathbf{N}$ corresponding to the trivial line bundle and zero section on S). It is convenient to write $S' = (S, M'_S)$, so that $M'_S = M_{S'}$. If M_X is the divisorial logarithmic structure on X then let $X' = (X, M'_X)$ be the pullback of $(X, M_X) \to (S, M_S)$ along $S' \to (S, M_S)$. We construct a $\overline{M}_{X'}^{\mathrm{gp}}$-torsor on the special fiber X'_0 that lifts to all finite orders (this is automatic, by infinitesimal invariance of the étale site) but not to X'.

We compute $H^1(X, \overline{M}_{X'}^{\mathrm{gp}})$ by means of the following exact sequence:

$$H^0(X, \overline{M}_{X'/S'}^{\mathrm{gp}}) \to H^1(X, \pi^{-1}\overline{M}_{S'}^{\mathrm{gp}}) \to H^1(X, \overline{M}_{X'}^{\mathrm{gp}}) \to H^1(X, \overline{M}_{X'/S'}^{\mathrm{gp}})$$

As $\overline{M}_{X'/S'}^{\mathrm{gp}}$ is concentrated in dimension 0 on X, the last term in the sequence vanishes. The group $H^1(X, \pi^{-1}\overline{M}_{S'}^{\mathrm{gp}})$ vanishes because X is normal (see [Art73, §2]). Hence $H^1(X, \overline{M}_{X'}^{\mathrm{gp}}) = 0$.

On the other hand, in the exact sequence

$$H^0(X_0, \overline{M}_{X'/S'}^{\mathrm{gp}}) \xrightarrow{\partial} H^1(X_0, \pi^{-1}\overline{M}_{S'}^{\mathrm{gp}}) \to H^1(X_0, \overline{M}_{X'}^{\mathrm{gp}}) \to H^1(X_0, \overline{M}_{X'/S'}^{\mathrm{gp}})$$

we still have $H^1(X_0, \overline{M}_{X'/S'}^{\mathrm{gp}}) = 0$, for the same reason, but

$$H^1(X_0, \pi^{-1}\overline{M}_{S'}^{\mathrm{gp}}) = H^1(X_0, \mathbf{Z}^2) \simeq \mathbf{Z}^2$$

since the fundamental group of X_0 is $\mathbf{Z}$ in the Zariski topology. (In the étale topology, it is the non-torsion part of the fundamental group that is $\mathbf{Z}$.)

The sheaf $\overline{M}_{X'/S'}^{\mathrm{gp}}$ is a skyscraper $\mathbf{Z}$, concentrated at the nodes of X_0. Therefore $H^0(X_0, \overline{M}_{X'/S'}^{\mathrm{gp}}) = \mathbf{Z}^2$. The map ∂ is the intersection pairing and one can verify directly that its rank is 1. Alternatively, one may observe that it is induced by pushout from the intersection pairing on X, which certainly has rank at most 1 because $H^1(X_0, \pi^{-1}\overline{M}_S) \simeq \mathbf{Z}$. In any case, there is a nonzero element in $H^1(X_0, \overline{M}_{X'}^{\mathrm{gp}})$ (and one can verify that this group is free of rank 1).

This gives a formal collection of elements of $H^1(X_n, M_{X'}^{\mathrm{gp}})$, where X_n is the reduction of X modulo t^{n+1}, for every $n \geq 0$, whose image in $H^1(X_n, \overline{M}_{X'}^{\mathrm{gp}})$ is nonzero. However, $H^1(X, \overline{M}_{X'}^{\mathrm{gp}}) = 0$, so this formal collection cannot be integrated.

Proposition 4.18.1. *Let X' and S' be as above and let Z be either the category fibered in groupoids on $\mathbf{LogSch}/S'$ whose value is the groupoid of $\mathbf{G}_m^{\log}$-torsors on X'_T, or the the sheaf of isomorphism classes of such. Then Z has no logarithmically smooth cover by a logarithmic scheme.*

Proof. Suppose that U is a logarithmic scheme and $U \to Z$ is a logarithmically smooth cover. Then the formal family of points $S'_n \to Z$ constructed above lifts to $S'_n \to U$. Since U is a logarithmic scheme, this family can be integrated to a map $S' \to U$, and therefore the maps $S'_n \to Z$ can be integrated to $S' \to Z$. We have just seen no such integration exists. ∎

5. Examples

We calculate some examples of LogPic(X/S), over a base S whose underlying scheme is the spectrum of an algebraically closed field k. We use the quotient presentation of Corollary 4.6.3, which requires an explicit understanding of $H^1(X^\nu, \mathbf{G}_m^{\log})$ and the map $H_1(\mathfrak{X}) \to H^1(X^\nu, \mathbf{G}_m^{\log})$.

5.1. **The Tate curve.** Let $Y \to \operatorname{Spec} k[[t]]$ be a family of curves whose generic fiber Y_η is a smooth curve of genus 1 and whose special fiber X consists of n rational curves arranged in a circle. We give $\operatorname{Spec} k[[t]]$ its divisorial logarithmic structure and we take S to be the closed point of $\operatorname{Spec} k[[t]]$, with the logarithmic structure induced by restriction.

Let $\mathfrak{X}$ be the tropicalization of X. This is a graph with n vertices in a circle, and we have $H_0(\mathfrak{X}) = \mathbf{Z}$ and $H_1(\mathfrak{X}) = \mathbf{Z}$. The intersection pairing $\mathbf{Z} \times \mathbf{Z} \to \overline{M}_S^{\mathrm{gp}}$ sends (a, b) to $ab\delta$ where δ is the sum of the lengths of the edges of $\mathfrak{X}$. Corollary 3.4.7 then gives exact sequences (5.1.1) and (5.1.2):

$$0 \to \mathbf{Z} \xrightarrow{\delta} \overline{\mathbf{G}}_m^{\log} \to \operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S) \to 0 \tag{5.1.1}$$

$$0 \to \operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S) \to \operatorname{Tro}\operatorname{Pic}(\mathfrak{X}/S) \to \mathbf{Z} \to 0 \tag{5.1.2}$$

That is $\operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S) = \overline{\mathbf{G}}_m^{\log}/\mathbf{Z}\delta$. In particular, if $\overline{M}_T = \mathbf{R}_{\geq 0}$ then the T-points of $\operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S)$ may be identified with $\mathbf{R}/\mathbf{Z}\delta$. By Theorem 4.14.7, $\operatorname{Log}\operatorname{Pic}^0(X/S)$ is an extension of $\overline{\mathbf{G}}_m^{\log\dagger}/\mathbf{Z}\delta$ by $\operatorname{Pic}^{[0]}(X/S) \simeq \mathbf{G}_m$.

In order to understand this extension more explicitly, we will use the quotient presentation of Corollary 4.6.3. Recall from Equation (4.7.5) that we may identify $H^1(X, \pi^*\mathbf{G}_m^{\log})^{[0]}$ with $\operatorname{Hom}(H_1(\mathfrak{X}), \mathbf{G}_m^{\log})$. Therefore Corollary 4.6.3 gives us the exact sequence (5.1.3):

$$0 \to H_1(\mathfrak{X}) \to \operatorname{Hom}(H_1(\mathfrak{X}), \mathbf{G}_m^{\log})^\dagger \to \operatorname{Log}\operatorname{Pic}^0(X/S) \to 0 \tag{5.1.3}$$

The pairing $H_1(\mathfrak{X}) \times H_1(\mathfrak{X}) \to \mathbf{G}_m^{\log}$ lifts the intersection pairing on $\mathfrak{X}$, valued in $\overline{\mathbf{G}}_m^{\log}$. Substituting $H_1(\mathfrak{X}) = \mathbf{Z}$, we obtain $\operatorname{Log}\operatorname{Pic}^0(X/S) = \mathbf{G}_m^{\log\dagger}/\mathbf{Z}\tilde{\delta}$ where $\mathbf{G}_m^{\log\dagger}$ denotes the subfunctor of $\mathbf{G}_m^{\log}$ that is bounded by δ, and $\tilde{\delta}$ is a lift of δ to M_S.

The tropicalization sequence from Theorem 4.14.7 now becomes (5.1.4):

$$0 \to \mathbf{G}_m \to \mathbf{G}_m^{\log\dagger}/\mathbf{Z}\tilde{\delta} \to \overline{\mathbf{G}}_m^{\log\dagger}/\mathbf{Z}\delta \to 0 \tag{5.1.4}$$

The element $\tilde{\delta} \in \mathbf{G}_m^{\log}$ can be understood as a 'logarithmic period', in the following sense. The map $M_X^{\mathrm{gp}} \to \Omega_{X/S}^{\log}$ factors through $M_{X/S}^{\mathrm{gp}}$ and therefore gives us a logarithmic differential ϕ on X. We wish to compute $\int_\gamma \phi$ where γ is a basis for $H_1(\mathfrak{X})$, without attempting to introduce any general theory of integration.

Let $\tilde{X}$ be the 'universal cover' of X, whose tropicalization $\tilde{\mathfrak{X}}$ has vertices indexed by the integers, with consecutive vertices connected by an edge. We can recognize $\tilde{X}$ as a subdivision of $\mathbf{G}_m^{\log\dagger}$ and we have $X = \tilde{X}/H_1(\mathfrak{X}) = \tilde{X}/\mathbf{Z}\gamma$.

Locally in X, there is no obstruction to lifting ϕ to $M_{\tilde{X}}^{\mathrm{gp}}$, so there is a global section Φ of $M_{\tilde{X}}^{\mathrm{gp}}$ lifting ϕ. Then $\Phi(\gamma.x) - \Phi(x)$ is a function of $x \in \tilde{X}$ valued in $\pi^* M_S^{\mathrm{gp}}$. It is therefore constant and represents the coboundary of γ in $H^1(X, \pi^* M_S^{\mathrm{gp}}) = M_S^{\mathrm{gp}}$.

5.2. **A curve of genus 2.** Let X consist of 2 rational components joined along 3 nodes. The tropicalization $\mathfrak{X}$ has 2 vertices, v_1 and v_2 and 3 edges, e_1, e_2, and e_3, which we choose to orient from v_1 to v_2, as shown in Figure 5.2. We write δ_i for the length of e_i in $\overline{M}_S$.

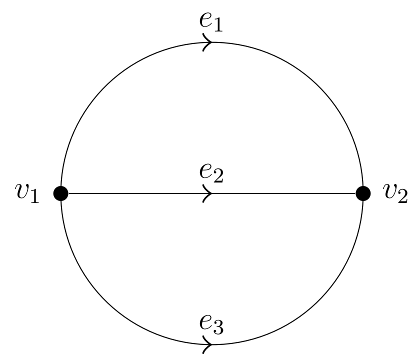

FIGURE 2. A tropical curve of genus 2.

The differences $e_1 - e_2$ and $e_2 - e_3$ form a basis for $H_1(\mathfrak{X})$. In this basis, the matrix of the intersection pairing is (5.2.1):

$$A = \begin{pmatrix} \delta_1 + \delta_2 & -\delta_2 \\ -\delta_2 & \delta_2 + \delta_3 \end{pmatrix} \tag{5.2.1}$$

The presentation $\operatorname{Tro Jac}(X/S) = \operatorname{Hom}(H_1(\mathfrak{X}), \overline{\mathbf{G}}_m^{\log})^\dagger / H_1(\mathfrak{X})$ becomes (5.2.2):

$$\operatorname{Tro Jac}(X/S) = (\overline{\mathbf{G}}_m^{\log} \times \overline{\mathbf{G}}_m^{\log})^\dagger / A\mathbf{Z}^2 \tag{5.2.2}$$

In particular, the real points are $\mathbf{R}^2 / A\mathbf{Z}^2 \simeq S^1 \times S^1$.

The commutative diagram in (3.4.1) gives a morphism (5.2.3):

$$H^0(\mathfrak{X}, \mathscr{V}) \to \operatorname{Tro Pic}(\mathfrak{X}) \subset H^1(\mathfrak{X}, \mathfrak{L}) \tag{5.2.3}$$

In concrete terms, this sends an integer linear combination of vertices D on $\mathfrak{X}$ to the torsor of piecewise linear functions on $\mathfrak{X}$ that are linear along the edges of $\mathfrak{X}$ and whose failure of linearity at each vertex v of $\mathfrak{X}$ is $D(v)$. We denote this torsor $\mathfrak{L}(D)$.

The exact sequence in the first row of (3.4.1) shows that lifts of $\mathfrak{L}(D)$ to $H^1(\mathfrak{X}, \overline{M}_S^{\mathrm{gp}}) = \operatorname{Hom}(H_1(\mathfrak{X}), \overline{M}_S^{\mathrm{gp}})$ correspond to the trivializations of the induced $\mathscr{H}$-torsor $\mathscr{H}(D)$. This torsor is the sheaf of assignments of integers to the vertices of $\mathfrak{X}$ such that the sum of outgoing slopes at each vertex v is $D(v)$.

The same reasoning applies equally well to any subdivision $\mathfrak{Y}$ of $\mathfrak{X}$. Since $\operatorname{Tro Pic}(\mathfrak{Y}) = \operatorname{Tro Pic}(\mathfrak{X})$ and $\operatorname{Hom}(H_1(\mathfrak{Y}), \overline{M}_S^{\mathrm{gp}}) = \operatorname{Hom}(H_1(\mathfrak{X}), \overline{M}_S^{\mathrm{gp}})$, giving $D \in H^0(\mathfrak{Y}, \mathscr{V})$ and a trivialization of $\mathscr{H}(D)$ will also produce points in $\operatorname{Tro Pic}(\mathfrak{X})$ and $\operatorname{Hom}(H_1(\mathfrak{X}), \overline{M}_S^{\mathrm{gp}})$. Figure 5.2 shows a piece of $\operatorname{Hom}(H_1(\mathfrak{X}), \mathbf{R})$ with horizontal coordinate $e_1 - e_2$ and vertical coordinate $e_2 - e_3$. For $D \in H^0(\mathfrak{X}, \mathscr{V})$ and trivialization of $\mathscr{H}(D)$ chosen according to the following rules, we have plotted a picture of those data at the corresponding position in $\operatorname{Hom}(H_1(\mathfrak{X}), \mathbf{R})$:

(1) D is supported on a quasistable model $\mathfrak{Y}$ of $\mathfrak{X}$, meaning that each edge of $\mathfrak{X}$ is subdivided at most once;
(2) if $v \in \mathfrak{Y}$ is a point of subdivision of $\mathfrak{X}$ then $D(v) = 1$;
(3) we have $0 \leq D(v_1) \leq 2$ and $-2 \leq D(v_2) \leq 0$.

In the picture, each vertex v is labelled by $D(v)$ unless $D(v) = 0$ and each edge is labelled by the slope it has been assigned in a choice of trivialization of $\mathscr{H}(D)$. The shaded parallelogram is the fundamental domain (5.2.4) for the quotient by $\partial H_1(\mathfrak{X})$.

$$\left\{ x\partial(e_1 - e_2) + y\partial(e_2 - e_3) \,\middle|\, 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \right\} \tag{5.2.4}$$

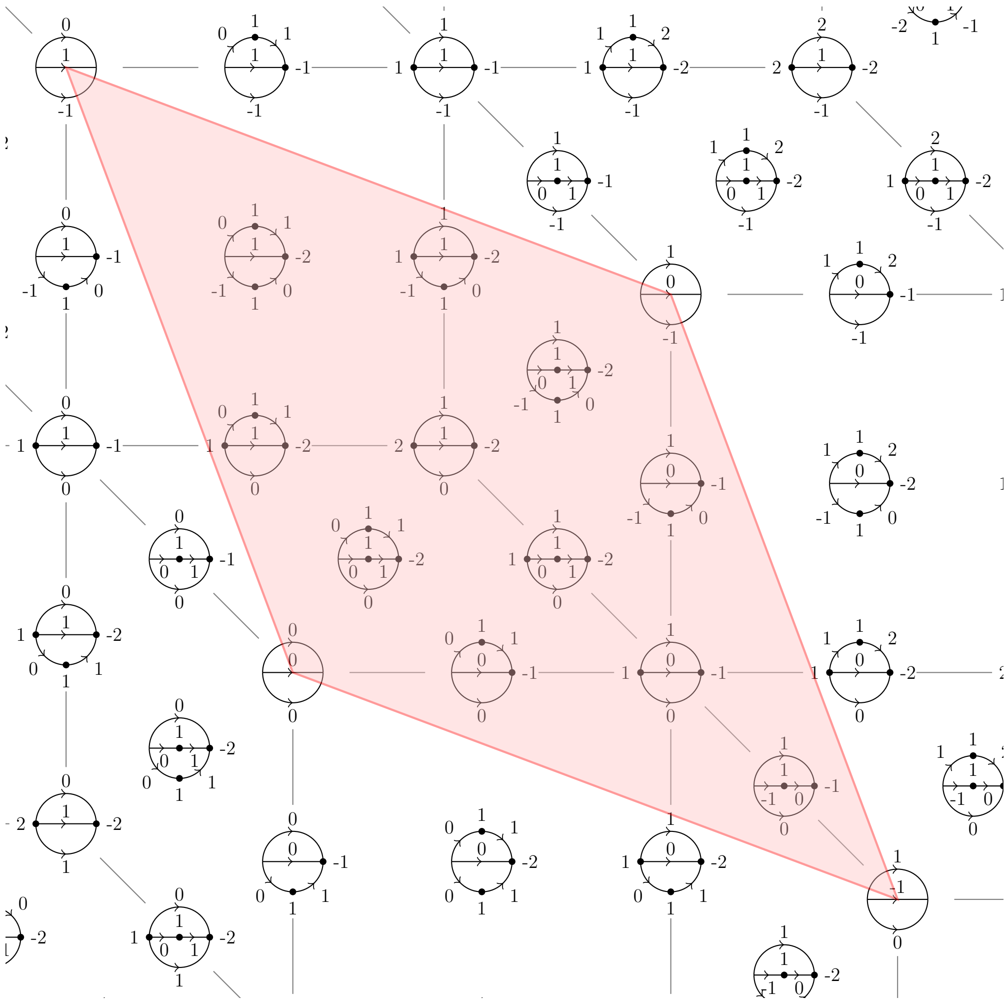

FIGURE 3. A fundamental domain for the quotient $\operatorname{Hom}(H_1(\mathfrak{X}), \mathbf{R})/\partial H_1(\mathfrak{X})$ and the subdivision, under an isomorphism to $\operatorname{Tro}\operatorname{Pic}^2(\mathfrak{X})$, into regions parameterizing balanced tropical divisors on quasistable models of $\mathfrak{X}$.

This subdivision is suggested by Caporaso's compactification of $\operatorname{Pic}^2(X)$. We originally computed it with the help of Margarida Melo, Martin Ulirsch, and Filippo Viviani. The same example also appears in [ABKS14, Figure 1] and [AP18, Figure 4].

5.3. **Nonmaximal degeneracy.** Let us finally look at an example which is not maximally degenerate. Suppose X is the union of two curves Y_1 and Y_2, glued along two points p_1, p_2; p_i in the first copy is glued to p_i in the second copy. The dual graph $\mathfrak{X}$ of X is again topologically a circle, with two vertices, v_1 and v_2, and two edges, e_1 and e_2, with lengths δ_1 and δ_2. As in Section 5.1, we find that $\operatorname{Tro}\operatorname{Jac}(\mathfrak{X}/S) = \overline{\mathbf{G}}_m^{\log\dagger}/\mathbf{Z}(\delta_1+\delta_2)$ and $\operatorname{Log}\operatorname{Pic}^0(X/S)$ is an extension of this torus by the algebraic Jacobian.

To compute $\operatorname{Log}\operatorname{Pic}^0(X/S)$, we use the quotient presentation from Corollary 4.6.3. Equation (4.7.5) presents $H^1(X, \pi^*\mathbf{G}_m^{\log})$ as an extension of $H^1(X^\nu, \mathbf{G}_m) = \operatorname{Pic}(Y_1) \times \operatorname{Pic}(Y_2)$ by $\operatorname{Hom}(H_1(\mathfrak{X}), \mathbf{G}_m^{\log})$:

$$0 \to \mathbf{G}_m^{\log} \to H^1(X, \pi^*\mathbf{G}_m^{\log}) \to \operatorname{Pic}^0(Y_1) \times \operatorname{Pic}^0(Y_2) \to 0 \tag{5.3.1}$$

Then Corollary 4.6.3 says that $\operatorname{Log}\operatorname{Pic}^0(X/S)$ is the quotient of $H^1(X, \pi^*\mathbf{G}_m^{\log})^\dagger$ by $H_1(\mathfrak{X})$.

In general, the composition (5.3.2) is nonzero:

$$H_1(\mathfrak{X}) \to H^1(X, \pi^*\mathbf{G}_m^{\log}) \to H^1(X^\nu, \mathbf{G}_m) = \mathrm{Pic}^0(Y_1) \times \mathrm{Pic}^0(Y_2) \tag{5.3.2}$$

Indeed, recall that the map $H_1(\mathfrak{X}) \to H^1(X, \pi^*\mathbf{G}_m^{\log})$ is induced from the composition (5.3.3),

$$H_1(\mathfrak{X}) \subset H^0(X, \overline{M}_{X/S}^{\mathrm{gp}}) \to H^1(X, \pi^*\overline{M}_S^{\mathrm{gp}}) \tag{5.3.3}$$

which was itself induced from the short exact sequence (4.6.2.1). Identifying $H^0(X, \overline{M}_{X/S}^{\mathrm{gp}}) = \mathbf{Z}^E$, where E is the set of edges of $\mathfrak{X}$, the basis element e corresponding to the node p is sent to $(\mathcal{O}_{Y_1}(p), \mathcal{O}_{Y_2}(-p))$. Therefore the basis $e_1 - e_2$ of $H_1(\mathfrak{X})$ is sent to $(\mathcal{O}_{Y_1}(p_1 - p_2), \mathcal{O}_{Y_2}(-p_1 + p_2))$.

If Y_1 or Y_2 has positive genus, the map $H_1(\mathfrak{X}) \to \mathrm{Pic}^{[0]}(X)$ is therefore nonzero, and will even be injective if $\mathcal{O}_{Y_i}(p_1 - p_2)$ is not a torsion point of the Jacobians of both curves. This shows that the surjection $H^1(X, \pi^*\mathbf{G}_m^{\log})^{[0]\dagger} \to \mathrm{Pic}^{[0]}(X^\nu/S)$ does not factor through $\mathrm{Log\,Pic}^0(X/S)$, even though its restriction to $\mathrm{Pic}^{[0]}(X/S) \subset H^1(X, \pi^*\mathbf{G}_m^{\log})$ does factor through its image in $\mathrm{Log\,Pic}^0(X/S)$. Indeed, the map $\mathrm{Pic}^{[0]}(X/S) \to \mathrm{Log\,Pic}^0(X/S)$ is injective by Theorem 4.14.7.

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