A Néron model of the universal Jacobian

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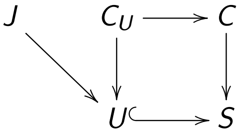
Basic setup:



with:

- S regular separated (eg. $\mathbb{A}^2_{\mathbb{C}}$);
- $C \to S$ a family of nodal curves;
- $U \hookrightarrow S$ dense open;
- $C_U \to U$ smooth;

We write $J = \mathrm{Jac}(C_U/U)$.

Question

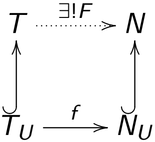
Can we find a 'nice' model for J over S?

Definition

A *Néron model* for J consists of:

- a smooth separated algebraic space N/S
- an isomorphism $N_U \to J$

satisfying the *Néron mapping property*:
for every smooth morphism $T \to S$, and for every U-morphism $f\colon T_U \to N_U$, there is a unique $F\colon T \to N$ extending f.



$$\begin{array}{ccc} J & & C \\ \downarrow & & \downarrow \\ U & \hookrightarrow & S \end{array}$$

Can we find a model of J satisfying the *Néron mapping property*:
for every smooth morphism $T \to S$, and for every U-morphism $f\colon T_U \to N_U$, there is a unique $F\colon T \to N$ extending f.

Examples

- Let $T = S$. The restriction map $N(S) \to N_U(U) = J(U)$ is bijection.
- Let $T = N \times_S N$. Then $T_U = J \times_U J$ (via given isomorphism). Multiplication $m\colon J \times_U J \to J \hookrightarrow N$ extends uniquely to $m_N\colon T = N \times_S N \to N$. The Néron model inherits a group structure.

Theorem (Néron, 1965)

If $\dim S = 1$ *then a Néron model exists.*

$$\begin{array}{ccc} & & C \\ & & \downarrow \\ U & \hookrightarrow & S \end{array}$$

Can we find a model of J satisfying the *Néron mapping property*:
for every smooth morphism $T \to S$, and for every U-morphism $f\colon T_U \to N_U$, there is a unique $F\colon T \to N$ extending f.

Example

$S \subseteq \mathbb{A}^1_{\mathbb{C},u}$, $U = S \setminus \{u = 0\}$

$$C : y^2 = ((x-1)^2 - u)((x+1)^2 - u) \subseteq \mathbb{P}_S(1,1,2)$$

- Elliptic curve, so $C_U = J$ (choose section at infinity).
- Not smooth: $C_{u=0} : y^2 = (x-1)^2(x+1)^2$ [picture]
- Néron model $N = C^{\text{sm}}$

What happens when $\dim S > 1$?

Example

$S \subseteq \mathbb{A}^2_{\mathbb{C},u,v}$, $U = S \setminus \{uv = 0\}$

$$C : y^2 = ((x-1)^2 - u)((x+1)^2 - v)$$

Is C^{sm} a Néron model of $J = C_U$? [Picture]

- If so, has group structure compatible with that on $J = C_U$.
- Let $\lambda \in \mathbb{C}^*$, define a line $L_\lambda : u = \lambda v \subset S$. Then C_{L_λ} is over a 1-dimensional base.The Néron model is $C^{\mathrm{sm}}_{L_\lambda}$.
- This gives a group structure on $C^{\mathrm{sm}}_{L_\lambda}|_{u=v=0} = C^{\mathrm{sm}}|_{u=v=0}$.
- Problem: this group structure depends continuously on λ!
- No Néron model exists.

Question (Qing Liu, 2010)

Does there exist an alteration $S' \to S$ such that a Néron model exists after pullback to S'?

What happens when $\dim S > 1$?

Example

$C : y^2 = ((x-1)^2 - u)((x+1)^2 - v)$ - no Néron model exists.

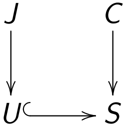
Question (Qing Liu, 2010)

Does there exist an alteration $S' \to S$ such that a Néron mdoel exists after pullback to S'?

Theorem (H., 2014)

No.

- In fact, give a general classification of when Néron models exist
- Proof via study of failure of flatness of $\mathrm{Pic}_{C/S}$ - closure of unit section 'blows up'



- No Néron model if $\dim S > 1$
- blowing up or altering the base doesn't help

On the other hand, there are many morphisms $f\colon S' \to S$ such that a Néron model *does* exist after pullback to S'

Examples

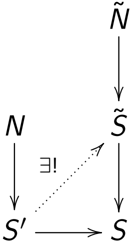
- $S' = U$
- S' a curve (Dedekind scheme)

(should require $f^{-1}U$ dense and S' regular separated).

We call such $f\colon S' \to S$ a *Néron model admitting (NMA) morphism*.

Question

Does there exist a universal Néron model admitting morphism $\tilde{S} \to S$?



$$\begin{array}{ccc} J & & C \\ \downarrow & & \downarrow \\ U & \hookrightarrow & S \end{array}$$

- $f\colon S' \to S$ is *NMA* if f^*J has a Néron model over S';
- Does there exist a universal NMA morphism?

Theorem (H. 2014)

A universal Néron-model admitting morphism does exist if $S = \bar{\mathscr{M}}_{g,n}$ and C is the universal curve.

The universal Néron-model admitting morphism is

- birational
- separated
- locally of finite presentation
- *not* quasi-compact

[Picture]

Summary

- When $\dim S = 1$, everything is nice
- if $\dim S > 1$, no Néron model in general
- blowing up/altering S doesn't help
- in the universal case, there is a canonical 'infinite chain of blowups' after which a Néron model *does* exist.