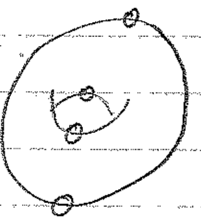
Intro to Morse theory

Let M be a real ~~differentiable~~ C^∞ manifold. We want to investigate topology of M in terms of ~~a~~ C^∞ real-valued fctns on it.

Let $f: M \longrightarrow \mathbb{R}$ be a C^∞ fctn. This has a derivative $df: M \to T^\vee M$. $(df \in \Pi^0(M, T^\vee M))$

~~Defn~~ A point $p \in M$ is called critical if $df(p) = 0 \in T_p M^\vee$. Critical values are images of crit. pts.

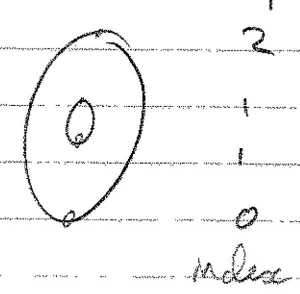
eg ~~of~~ $M =$  $f =$ 'height', 4 critical points circled.

Given a critical point $p \in M$, & local coords $(x_\alpha)_{\alpha \in A}$ we define the Hessian $H_p f$ of ~~f~~ f wrt (x_α) to be the symm. quad form on $T_p M$ which (in terms of local coords near p) is given by

$$H_p f\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} f \Big|_p$$

~~eg: Naively~~ ~~The~~ A critical point p is called 'non-degenerate' if $H_p f$ is non degenerate, & in this case we call the dimension of ~~the~~ a maximal subspace on which $H_p f$ is negative definite the 'index' of p (Indep of choice of local coords [by Sylvester's law of inertia])

eg: Naively, the index measures 'how many directions M slopes down in at p': Hence:



Morse lemma: let p be a non-degenerate critical point of M. Then p is an isolated critical point.

Pf omitted.

There are two main thms. We won't prove them here bcs next week we will need to prove generalisations.

Def: $M^a = f^{-1}(-\infty, a]$.

Thm 1: let M, f as above, $a < b \in \mathbb{R}$ s.t. $f^{-1}[a,b]$ is compact & f has no critical values in $[a,b]$. Then

M^a is diffeo to M^b, & M^b def. retracts onto M^a

(ie. $\exists_h^{cts} \Psi : M^b \times [0,1] \longrightarrow M^b$ s.t. $\forall x \in M_b$ & $y \in M$
$\Psi(x,0) = x$, $\Psi(x,1) \in M^a$,
& $\Psi(y,1) = a$.)

Thm 2 M, f as above, $p \in M$ non-degen crit. pt of index γ, $f(p) = q$.
Suppose $\exists \varepsilon > 0$ s.t.
$f^{-1}[q-\varepsilon, q+\varepsilon]$ is compact & contains no crit. pts apart from p.

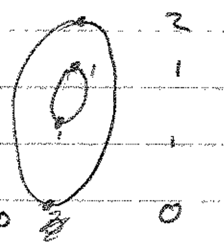
Then
$M^{q+\varepsilon}$ is homotopy equivalent to $M^{q-\varepsilon}$ with a γ-cell attached

Def: $n \in \mathbb{Z}_{\geq 0}$. ~~An~~ The n-cell C_n is the closed ball in $\mathbb{R}^n$, with boundary ∂C_n

Def: let S, T two top spaces, $n \in \mathbb{Z}_{\geq 0}$, ~~We say~~ $\sim$ an $\equiv$ relation on top spaces (eg $\sim$ = 'is homotopy $\equiv$'). We say
'$S \sim T$ with an n-cell attached' iff

$\exists$ cts $g : \partial C_n \longrightarrow T$ s.t. $S \sim (T \sqcup C_n$ glued along $g)$.

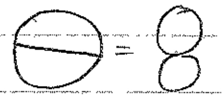
ck to our example:



- outside critical ~~pts~~ values, M^a doesn't change as a varies
- For $a<0$, $M^a = \phi$
- for ~~$-$~~ $0<a<1$, $M^a \overset{htpy}{\simeq}$ C_0 attached to ϕ. point

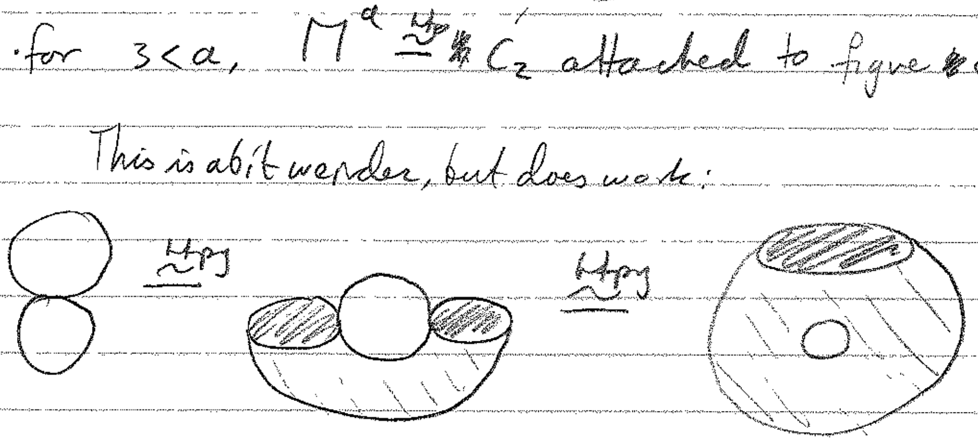
- for $1<a<2$, $M^a \overset{htpy}{\simeq}$ C_1 attached to C_0

- for $2<a<3$, $M^a \overset{htpy}{\simeq}$ C_1 attached to circle



- for $3<a$, $M^a \overset{htpy}{\simeq}$ C_2 attached to figure 8. disc

This is a bit weirder, but does work:



so now you can see where to glue in the disc to get back what we started.

As this example shows, there are often several non-equivalent ways to attach a λ-cell. Morse theory doesn't seem to tell us which. However, can still draw useful consequences, as next lemma will show.

Thm "Most" functions have no ~~non~~ degenerate critical pts. Sketch ~~pf next time.~~

Eg: Flat donut + height has degenerate critical pts (clear since not isolated).

Lemma: Let S, T top spaces & $n \in \mathbb{Z}_{>0}$ be s.t. S is htpy $\simeq$ to T with an n-cell attached.

Then the inclusion $T \hookrightarrow S$ induces isomorphism on π_i for $i < n-1$ & surjection on π_{n-1}.

~~Pf: After htpy may view $T \hookrightarrow S$. We see $\pi_i(S, T) = \pi_i(C_n) = 0 \; \forall i$.~~

~~$\pi_i(S, T, x_0) = 0 \; \forall x_0 \in T$.~~

Pf: Let $x_0 \in T$ be fixed. After htpy, view $T \subset S$.

Injectivity for $i < n-1$: Let $\sigma : C_i \to T$ be cts map, $\partial C_i \mapsto x_0$

representing an elt of $\pi_i(T, x_0)$. Suppose σ is homotopic to identity in S. Then the homotopy can be deformed (*) to live in the boundary of the attached n-cell (since $n > i+1$) (ie puncture & retract the n-cell) so σ is htpic to id in T.

Surjectivity for $i \leq n-1$:

Let $\sigma : C_i \to S$ be cts. $\partial C_i \mapsto x_0$

Let $p \in$ the attached n-cell s.t. $p \notin \sigma(C_i)$ (exists by dimension).

Puncture at c & retract, to obtain

a $\tilde{\sigma} : C_i \to T$ homotopic to σ. $\partial C_i \mapsto x_0$

$\square$

(*) Bcs the homotopy is a space of dim $i+1 < n$, so $\exists p \in C_n$ not in the homotopy. Puncture C_n at p & retract.

Part II: Lefschetz Hyperplane Thm (Homotopy Version).

§ Statement

~~Fix for remainder of ta~~

For remainder of talk, X will be a smooth projective connected variety$/\mathbb{C}$ of (cplx) dim n.

Given a line bundle E on X & $s \in \Gamma(X, E) \setminus \{0\}$ write $V(s) \subset X$ for the zero set of s. We say s is 'nonsingular' if $V(s)$ is smooth$/\mathbb{C}$.

<u>Thm</u> [Bott–Thom]: Let E an ample line bundle on X & $s \in H^0(X, E)$ a nonsing. section. Then X is htpy $\simeq$ to

$$V(s) \text{ with } r \text{ cells } e_1, \dots, e_r \text{ attached, each } \dim_{\mathbb{R}} e_i \geq n.$$

<u>Cor</u>: E, s as above. $j: V(s) \hookrightarrow X$ the inclusion. Then the induced map on htpy gps is surj. in dim $< n$, & injective in dim $< n-1$.

<u>Pf of Cor</u>: See last time. $\square$

Our aim for the next ? is to prove the Thm.

§ Non-degenerate Morse Theory

Let M be a C^∞ ~~dff~~ real mfd, "& $\phi: M \to \mathbb{R}$ a C^∞ fctn, with differential $d\phi: M \to TM^\vee$.

Def: critical points are pts where $d\phi = 0$.

(choose local coords.)

Def: Let $m \in M$ a critical pt, $T_m M$ tangent space, Hessian of ϕ is sym. quad ~~form on~~ form on $T_m M$ def'd in local coords by

$$H_m \phi\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \phi \Big|_m$$

~~as last time.~~

Index of a crit. pt is dim of maximal subspace on which $H_m\phi$ is -ve def, & is indep of coords.

So far, same as last time.

Def: A smooth conn. submfd. $V \subset M$ is called a nondegenerate critical manifold of ϕ (NDCM) iff

- $d\phi = 0$ on V ('critical')
- $\forall v \in V$, the nullspace of $H_v \phi$ is exactly the tangent space to V at v.

(So we see that a pt $m \in M$ is a nondegen. crit pt iff it's a ~~nondegen critical submfd.~~ NDCM.)

ϕ is called nondegenerate if it ~~is smooth &~~ its set of crit. pts consists entirely of NDCMs.

Thm II Say M cpct, ϕ nondegen. Let M_* be the set on which ϕ takes its minimum. Let $|\phi|$ be the lowest index which occurs among indices of crit. pts of $M - M_*$. ~~Then M is htpy $\simeq$ to~~ Then

$$M \overset{\text{htpy}}{\simeq} M_* \cup e_1 \cup \cdots \cup e_r,$$

$$\text{each } e_i \text{ a cell with } \dim_{\mathbb{R}} e_i \geq |\phi|.$$

We will deduce Thm I from Thm II. If time allows, we will also prove Thm II.

§ ~~Positive~~ Hermitian forms

X as always, E on X a line bundle.

Def: A hermitian structure on E is a C^∞ fctn h ~~...~~ which assigns to each $x \in X$ a +ve def. hermitian form h_x on the fibre E_x.

Given $U \subset X$ & $s \in H^0(U, E)$, we write ~~... for h~~

$$\cancel{h_x(s,s)}\ \underset{E}{h}(s,s) : X \longrightarrow \mathbb{R}_{\geq 0}$$
$$x \longmapsto h_x(s_x, s_x).$$

Given (E, h), we define its first chern class ~~by~~

$$c(E,h) = \frac{-i}{2\pi} \partial\bar\partial \log(s_u, s_u)$$

on open set $U \subset X$, ~~...~~ where $s_u \in H^0(U,E) \setminus 0$. ~~...~~

In terms of an analytic local coord system $(z_1, \ldots, z_n)$, it is ~~of type~~

$$i g_{\alpha\beta}\, dz_\alpha \wedge d\bar z_\beta, \text{ where } g_{\alpha\beta} \text{ is a Hermitian matrix.}$$

We say h is a positive structure if $g_{\alpha\beta}$ is +ve def at every pt.

E is called positive if it admits a +ve herm. structure.

Prop: Ample line bundles are positive.

Pf: Replacing E with a tensor power, may assume it is v.ample. Then pull back herm. form ~~...~~ on $\mathbb{P}^n$. □

Rk:
[Thm [Kodaira]: positive line bundles are ample].

§ Pf of I from II

Comparing Thm I & Thm II, we see that to deduce I from II we ~~xx~~ need to construct a C^∞ nondegenerate fctn ϕ on X with ~~X_*~~ $= S$ ~~on~~ $X_* = V(s)$ & $|\phi| = \dim_{\mathbb{C}} X$. It turns out that a slight peturbation of (s,s) will do.

X as always, E on X positive line bundle, (,) +ve herm metric on E, $s \in H^0(X,E)\setminus\{0\}$.

~~Prop Each connected component of V(s)~~

First, we need to rephrase '$V(s)$ smooth' in terms of local coords:

~~Pro~~ **Lemma**: $V(s)$ is smooth iff s satisfies ~~the~~

'Condition T': $\forall x \in X$ with $s(x)=0$, $\exists$ an open nbhd $\mathcal{U} \ni x$, & (do we need (a))?!

~~a) a holo section s_* of E~~

a) a holo section $s_* \in E(\mathcal{U})$ s.t. $s_*(x) \neq 0$

b) a local ~~xx~~ analytic coord system ~~xx~~ $(z_1, \ldots, z_n)$ ~~xx~~ centred @ x s.t.

$$h(z_1, z_1) = z_1 \bar{z}_1$$

s.t. $s = s_* z_1$ on $\mathcal{U}$.

Pf: omitted

□

Prop. Each conn. comp ~~xx~~ of $V(s)$ is a ~~xx~~ NDCM of $(s,s): X \to \mathbb{R}$.

Pf: Let $p \in V(s)$, & let $(z_1, -, z_n), s^*$ as above, so $s = s_* z_1$. Let $a = (s_*, s_*)$, a C^∞ +ve fctn near p, ~~xx~~ so

$$(s,s) = a\, z_1 \bar{z}_1.$$

Easy to check that ~~x~~ $d(s,s) = 0$ ~~x~~ at p. It remains to check that the nullspace of $H_p(s,s)$ equals the tangent space to $V(s)$ at p.

~~Partial~~ Partial derivatives wrt components of z_i : $i>1$ are clearly 0. Let ~~xx~~ $x_1, x_2 : M \to \mathbb{R}$ s.t. $z_1 = x_1 + i\, x_2$ near p. Then

$$H_p(s,s)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = 2a(p) \cdot \delta_{\alpha\beta} \quad \text{(easy ex)}$$

$\delta_{\alpha\beta}$ = 0 if $\alpha \neq \beta$, 1 if $\alpha = \beta$

so matrix is ~~xx~~ $\underset{\neq 0}{2a(p)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $H_p(s,s)$ is non-degen. exactly along tangent space. □

<u>Prop</u> Let p be a crit. pt of (s,s) on $X - V(s)$. Then the index of p is $\geq \dim_{\mathbb{C}} X = n$.

<u>Pf</u>. On $X - V(s)$, the fctn (s,s) is > 0. Hence $f := \log(s,s)$ is C^∞ near p, & $(s,s) = e^f$. Hence $d(s,s) = 0 \Leftrightarrow df = 0$, so p is a crit. pt of f. Similarly, the index of p wrt f is same as wrt (s,s) ("$d(s,s) = e^f df$", but in more variables).
smooth, nonvanishing.

Let H be the Hessian of f at p (in real coords x_α, y_α ; $z_\alpha = x_\alpha + i y_\alpha$), so

$$H_p f\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} f \Big|_p \quad \text{\& sim for } y.$$

Next, note that by def'n of positivity of h, the differential form

$$\frac{i}{2\pi} \bar\partial \partial f$$

is positive near p. In terms of local analytic coords z_i, this translates into the matrix

$$\left(\frac{\partial^2 f}{\partial z_\alpha \partial \bar z_\beta}\right) \text{ is } \underline{-\text{ve det}}.$$

~~Now~~ Recalling $\dfrac{\partial}{\partial z_\alpha} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_\alpha} - i\dfrac{\partial}{\partial y_\alpha}\right)$, $\dfrac{\partial}{\partial \bar z_\alpha} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_\alpha} + i\dfrac{\partial}{\partial y_\alpha}\right)$

See opposite page.

~~we see that $H_p f$ is $-$ve def on a subspace of dim n.~~ □

~~change coords s.t. $(\ast) = \begin{pmatrix} -1 & 0 \\ 0 & \ddots \end{pmatrix}$.~~

Indeed, on the subsp spanned by $\frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - \frac{i\partial}{\partial y_\alpha}\right)$, $\frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + i\right.$

~~So $\dfrac{\partial^2 f}{\partial z_\alpha \partial \bar z_\alpha} dz_\alpha \wedge d\bar z_\alpha = \dfrac{1}{4}\left(\dfrac{\partial^2 f}{\partial x_\alpha^2} + \dfrac{\partial^2 f}{\partial y_\alpha^2}\right) dx \wedge dy$~~

Don't understand

So the fctn $(s,s) : X \to \mathbb{R}$ nearly satisfies what we need for II: it takes its minimum on $V(s)$, each conn. comp of $V(s)$ is NDCM, & other crit. pts have index $\geq \dim_{\mathbb{C}} X$. All we are missing is that the crit. pts away from $V(s)$ are non-degenerate. To achieve this, we 'perturb' (s,s) a little using partition-of-unity style arguments.

Omitted

~~Lemma~~ Prop: Let p be a crit. pt of (s,s) on $X - V(s)$. Then the index of p is $\geq \dim_{\mathbb{C}} X = n$.

Pf: On $X - V(s)$, the fctn (s,s) is > 0. Hence $f = \log(s,s)$ is C^∞ near p, & $(s,s) = e^f$. Positivity of (E,h) means that the form

$$\frac{i}{2\pi} \bar{\partial}\partial f \quad \text{is positive near } p.$$

In local analytic coords U_α, this means the ~~form~~ matrix

Fix for opposite page:

Same up to here

Set $T_pM_{\mathbb{C}} := T_pM \otimes_{\mathbb{R}} \mathbb{C}$ (real dim $4n$ - weird!)

- H_pf extends uniquely to a Herm. form $H_pf_{\mathbb{C}}$ on $T_pM_{\mathbb{C}}$.
- Index of $H_pf_{\mathbb{C}}$ (as a Herm. form) = index of H_pf (as a real form)
- ~~the subspace~~ Let $V \subset H_pf_{\mathbb{C}}$ be the subspace spanned by the

$$\frac{\partial}{\partial z_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - \frac{i\,\partial}{\partial y_\alpha}\right).$$

Then clearly $H_pf_{\mathbb{C}}$ is -ve def on V, & V has cplex dim n, so index of $H_pf_{\mathbb{C}}$ is $\geq n$. $\square$

§ Proof of Thm II

Now that we have (almost) deduced Thm I from Thm II it is time to prove II.

From now on, M is a cpct C^∞ real mfd, ~~w. a Riemannian structure~~ $f: M \to \mathbb{R}$ a C^∞.

$M^a := f^{-1}(-\infty, a]$.

Thm II follows easily from two more results. First:

Thm III. Let $a<b \in \mathbb{R}$.

If f has no critical value in $[a,b]$ then M^a is a def. retract of M^b.

Pf. Put a Riemannian structure on M^b (apparently we can do this? Yes (bcs M^b is differentiable & paracompact))

Write ∇f for the gradient of f wrt this structure. Given $p \in M^b - M^a$ write L_p for the integral curve of $-\nabla f$ through p with parameter coming from the integral. ("flow p by $-\nabla f$") – well def'd bcs $df \neq 0$ on this set.

Since $\overline{N^b - N^a}$ is cpct, $|\nabla f| > \varepsilon_0 > 0$ on this set.

Hence each L_p intersects $f^{-1}(a)$ at some pt, call it $h(p)$.

The map sending $p \in \overline{N^b - N^a}$, $t \in [0,1]$ to the pt $h_t(p)$ on L_p which divides the segment from p to $h(p)$ in the ratio $1 : 1-t$ is a def. retract of N^b onto N^a. □

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~~Thm III.2~~ Before stating Thm III.2, need ~~a little more notat~~ more def'ns.
Let $a \in \mathbb{R}$. Let a R-structure be given on M.
Let V be a NDCM of f on M^a.

Given $p \in V$, define linear map $\tau_p : T_pM \to T_pM$ by

$$\langle \tau_p X, Y \rangle = H_p f(X, Y) \quad \forall X, Y \in T_pM.$$

inner product on tangent space from Riemannian structure.

By def of NDCM, $\ker \tau_p$ = tangent space ~~of~~ to V.

Hence τ_p induces an automorphism of the normal bundle of V in M.

Def Let ξ_V be the vector bundle on V sub-bundle of this normal bundle which is spanned by the negative eigendirectios of τ_p. Thus the fibre $\xi_{V,p}$ has dimension the index of p, which is independent of $p \in V$.

[Fact: ξ_V is indep. of R-structure].

Def: Let A, B be top spaces, $\alpha : A \to B$ a map, & $\mathcal{F}$ a vector bundle on A with a hermitian metric.
Let $D = \left\{ (a, f) \in A \times \mathcal{F} \;\middle|\; \begin{array}{l} \cdot f \text{ lies over } a \\ \cdot |f| \le 1 \end{array} \right\}$, ∂D be the same thing but $|f| = 1$,
$\pi : \partial D \to A$ projection.
Then 'B with $\mathcal{F}$ attached' is constructed by gluing

D to B along $\alpha \circ \pi$.

Finally, Thm III.2: ~~...~~ R-structure on M, f as before, $a<b\in\mathbb{R}$
If f has a single NDCM V in $f^{-1}[a,b]$, then M^b is homotopic to M^a with the negative bundle of f along V attached.

can be generalised to case with several NDCM

Pf Wlog $f(V)=0$, & f has no critical pt in $[-\varepsilon_0,0)\cup(0,\varepsilon_0]$.
It suffices (by III.1) to prove $\exists\, 0<\varepsilon<\varepsilon_0$ s.t. $M^\varepsilon \underset{htp}{\simeq} M^{-\varepsilon}\cup \xi_{M}$.

-ve bundle of f along V.

Let ξ^+ be -ve bundle of $-f$ along V, & η the normal bundle to V in M. Easy to see

$$\eta = \xi^+\oplus\xi .$$

Let $\pi:\eta\longrightarrow\xi$ be the projection. Length of $x\in\eta$ is written $|x|$,

set $\varphi(x)=|x|^2$.

Let $\rho:\eta\longrightarrow M$ be the exponential map: it is a homeo in the vicinity of V. Thus ρ induces an R-structure $(\cdot,\cdot)$ on η near zero section.
Set $f_*=f\circ\rho:\eta\longrightarrow\mathbb{R}$.

Since f NDCM, we see restriction of f_* to any fibre of η has a non-degen crit. pt @ 0. More precisely:

the fctn f_* restricted to any fibre of ξ^+ (resp ξ) has a non-degen critical ~~max (~~ minimum (resp. maximum) at 0.

1) Hence, if $\varepsilon>0$ is suff. small then the set $f_*\le\varepsilon$ on a fibre of ξ^+ is contractible.

[He works a little more to get linearly contractible: do we need it?]
If so, is a calculation ...

~~Also~~ (2) If $\mu > 0$ is suff. small, then the gradient of $-f_*$ points out of the set $\varphi(x) \leq \mu$ at all pts with $\varphi(x) = \mu$ on any fibre of ξ.

Given $\varepsilon, \mu > 0$, define

$$X_\mu^\varepsilon = \{ x \in \eta \mid f_* x \leq \varepsilon \ \& \ \varphi \circ \pi(x) \leq \mu \}.$$

Then we can find $\varepsilon, \mu > 0$ suff. small s.t:

a) $\varepsilon < \varepsilon_0$
b) ρ is a homeo. on X_μ^ε
c) If $A_\mu^\varepsilon \subset X_\mu^\varepsilon$ is the subset on which $\varphi \circ \pi(x) = \mu$, then the pair

$(X_\mu^\varepsilon \cap \xi, A_\mu^\varepsilon \cap \xi)$ is a def. retract of $(X_\mu^\varepsilon, A_\mu^\varepsilon)$ (by (1))
(maybe here we use loc. contractible?)

d) The gradient of $-f$ points out of the set $\rho(X_\mu^\varepsilon)$ at pts of $\rho(A_\mu^\varepsilon)$ (by (2))

From now on, <u>assume ε, μ chosen s.t. the above hold.</u>
Let $Y_\mu^\varepsilon = N^\varepsilon - \rho(X_\mu^\varepsilon)$.

From (b), we see that $N^\varepsilon \overset{htp}{=} Y_\mu^\varepsilon \cup_\alpha X_\mu^\varepsilon$, where $\alpha = \rho|_{A_\mu^\varepsilon}$

From (c), we see that $N^\varepsilon \overset{htp}{=} Y_\mu^\varepsilon \cup \xi$. (*)

From (d), we see that the gradient $-\nabla f$ points <u>inward</u> at boundary pts of Y_μ^ε.

~~Further,~~ From (a), there are no pts with $-\varepsilon \leq f$ & $\nabla f = 0$.

Then argue as in pf of III.1 to see that Y_μ^ε is a def. retract of $N^{-\varepsilon}$.

Thus by (*), we see $N^\varepsilon \overset{htp}{=} N^{-\varepsilon} \cup \xi$, as desired.

$\square$