Lecture 10: Intersection Theory (19/11/2013)

David Holmes

Before we start with intersection theory, we need to define a few more properties of schemes, & give some basic definitions.

Def: X a scheme. We say X is:

- **reduced** if $\forall x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced
  i.e. no non-zero nilpotents.
- **irreducible** if it is non-empty & the underlying topological space is irreducible (i.e. cannot be written as union of two proper closed subsets)
- **integral** if it is reduced & irreducible.

Warning: there is also a property of a morphism called 'integral' - it is not the same / analogous.

Prop: X a scheme, $f_1 : Z_1 \to X$ & $f_2 : Z_2 \to X$ closed immersions.

Then $\#\mathrm{Hom}_{Sch_X}(f_1, f_2) \leq 1$ (i.e. any map between closed immersions is unique if it exists.)

Pf: exercise. $\square$

Def (closed subschemes) Let X a scheme. The category of closed subschemes of X has
objects $= \{f : Z \to X \text{ cl. imm}\}/\simeq$ where $f_i : Z_i \to X$ isomorphic iff $\exists$ isom. $Z_1 \to Z_2$ s.t. TFDC:

$$\begin{tikzcd} Z_1 \arrow[rr, "\sim"] \arrow[dr, "f_1"'] & & Z_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

morphisms: $\mathrm{Hom}\left([f_1 : Z_1 \to X], [f_2 : Z_2 \to X]\right) := \mathrm{Hom}_{Sch_X}(f_1, f_2)$.

- We say a closed subscheme $[f : Z \to X]$ is <u>integral</u> if Z is.
- Intersections: $[f_1 : Z_1 \to X] \cap [f_2 : Z_2 \to X] = [Z_1 \times_X Z_2 \xrightarrow{f_1 f_2} X]$
- Inclusion: $[f_1 : Z_1 \to X] \subseteq [f_2 : Z_2 \to X]$ iff $\exists$ factorisation $Z_1 \xrightarrow{f_1} X$, $Z_1 \xrightarrow{g} Z_2 \xrightarrow{f_2} X$.
- <u>Ideal sheaf</u>: Let $[f : Z \to X]$ a closed subscheme. The ideal sheaf of $[f]$ is the kernel of the canonical map
$$f^\flat : \mathcal{O}_X \to f_* \mathcal{O}_Z.$$

<u>Ex</u>: It is independent of the chosen representative of $[f : Z \to X]$.

<u>Remark</u>:
- It is possible to choose distinguished representatives of $[f : Z \to X]$ s.t. the map on sets is inclusion of a subset, etc.
- From now on, we will often talk about 'a closed subscheme $Z \subset X$', to ease notation.

<u>Integral closed subschemes & points</u>

Let X a scheme, & $x \in X$ a point. • Given an affine open $\mathrm{Spec}\, A \subset X$, x corresponds to a prime ideal $\mathfrak{p} \trianglelefteq A$, & hence to a canonical closed immersion
$$\underset{\text{integral}}{\mathrm{Spec}(A/\mathfrak{p})} \to \mathrm{Spec}\, A.$$

- If we make this construction on each chart of an affine cover of X, the resulting closed immersions will glue on overlaps by uniqueness, yielding a closed immersion
$$Z \rightarrowtail X.$$

<u>Exercise</u>: Show that the above construction induces a bijection
$$\{\text{points of } X\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\text{integral closed}\\ \text{subschemes of } X\end{array}\right\}.$$

Dimension

- let A a ring. $\dim A$ = Krull dimension of A $\in \mathbb{Z}_{\geq 0} \cup \{\infty\}$,
  (Not necess. finite/if A noetherian. Is finite if eg. A is fin. type over dedekind domain).
  even
- X a scheme.

$$\dim X = \sup \left\{ n \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{l} \exists \text{ a sequence of points } (p_0, p_1, \dots, p_n) \text{ of } X \\ \text{s.t. } \forall i,\ p_i \in \text{closure of } p_{i+1} \end{array} \right\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

$$= \sup \left\{ n \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{l} \exists \text{ a sequence of integral closed subschemes} \\ (Z_0, \dots, Z_n) \text{ of } X \text{ s.t. } \forall i,\ Z_i \subset Z_{i+1} \end{array} \right\}.$$

- $\dim \operatorname{Spec} A = \dim A$.
- dimension is better behaved for integral schemes (think about $\dim(\mathbb{A}^1_k \sqcup \operatorname{Spec} k)$).
- X Noetherian $\not\Rightarrow$ $\dim X$ finite. X fin. type over dedekind ring $\Rightarrow$ $\dim X$ finite.
- Given X int. scheme of fin. dim. & $Z \subset X$ closed integral, we define

$$\operatorname{codim} Z = \dim \mathcal{O}_{X,x}$$ where $x \in X$ is pt. corresp. to Z.

~~Prop: X integral fin. dim, $x \in X$ a point, corresponding closed integral $Z \subset X$. Then~~

$$\cancel{\dim \mathcal{O}_{X,x} = \operatorname{codim} Z.}$$ ~~(often write $\mathcal{O}_{X,Z}$ for $\mathcal{O}_{X,x}$).~~

~~Pf: Reduce to X affine, then comm. algebra~~ □

Def: X integral fin. dim, $V, W \subset X$ closed integral. We say V & W meet properly iff $\forall$ irreducible components $Z \subset V \cap W$, we have

$$\operatorname{codim} Z = \operatorname{codim} V + \operatorname{codim} W.$$

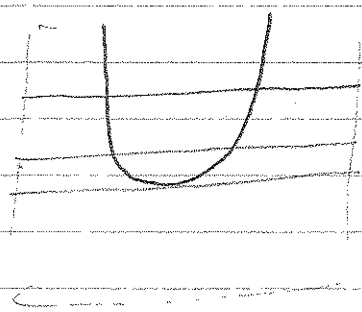
[irreducible component = maximal irreducible closed subscheme].

Intersection numbers

For the remainder of this section:
- X is an integral regular scheme (s.o.f.n.dim);
- V, W are closed integral subschemes of X which meet properly;
- Z is an irreducible component of $V \cap W$.

We want to define the 'multiplicity of the intersection of V and W along Z' $(z_Z(V,W))$.

Eg: k a field, $X = \mathbb{A}^2_k$, $V: y - x^2 = 0$.
$W: y - a = 0$, some $a \in k$.



$$V \cap W = \mathrm{Spec}\left(\frac{k[x,y]}{(y-a,\ y-x^2)}\right) \quad \text{(why?)}$$

$$\simeq \mathrm{Spec}\left(\frac{k[x]}{(x^2-a)}\right).$$

Case 1: $a \in (k^*)^2$. Then $V \cap W = \mathrm{Spec}\left(\frac{k[x]}{x-\sqrt{a}}\right) \sqcup \mathrm{Spec}\left(\frac{k[x]}{x+\sqrt{a}}\right) \simeq (\mathrm{Spec}\, k) \sqcup (\mathrm{Spec}\, k)$
$= \mathrm{Spec}\, k \times k$. not integral

Case 2: $a \notin k^2$. Then $V \cap W \simeq \mathrm{Spec}\left(\frac{k[x]}{x^2-a}\right)$, integral. (not irred)

Case 3: $a = 0$; $V \cap W = \mathrm{Spec}\,\frac{k[x]}{x^2}$, irreducible, not reduced.

Case 1: two irred comps, $z_Z(V,W) = 1$ for both.

Case 2: one —— " ——————.

Case 3: —" ——————, $z_Z(V,W) = 2$.

In this situation, you can probably guess a reasonable def'n.

N.B. $M \in \mathrm{Mod}\, A$. $\mathrm{length}_A(M) = \sup \left\{ n \in \mathbb{Z}_{\geq 0} \,\middle|\, \exists\, N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n \text{ chain of sub } A\text{-modules of } M \right\}$. Additive in exact seq.

However, when the codimensions get larger, things get more complicated.

<u>Def</u>: X, V, W, Z as above. $\mathcal{I}$ ($\mathcal{J}$) = ideal sheaf of V (W).

$$I = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,Z}, \quad J = \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,Z}.$$

$$i_Z(V,W) := \sum_{i \geq 0} (-1)^i \, \mathrm{length}_{\mathcal{O}_{X,Z}} \mathrm{Tor}_i^{\mathcal{O}_{X,Z}} \left(\frac{\mathcal{O}_{X,Z}}{I}, \frac{\mathcal{O}_{X,Z}}{J} \right).$$

Since X regular, this sum is finite (see Cor. 3 of lecture 9).

~~Def: X, V, W as above.~~

The following def'n & result will make it easier to work with examples.

$\mathcal{I}_V$ = ideal sheaf of V, $\mathcal{I}_W = \cdots$ of W.

<u>Def</u> X, V, W as above. We say V, W are in <u>general position</u> if $\forall x \in V \cap W$ $\exists$ aff. open nbhd $x \in \mathrm{Spec}\, A \subset X$ & elements $f_1, \ldots, f_r, g_1, \ldots, g_s \in A$ s.t.

- $(f_1, \ldots, f_r) = \mathcal{I}_V(\mathrm{Spec}\, A) =: I$, reg. seq.
- $(g_1, \ldots, g_r) = \mathcal{I}_W(\mathrm{Spec}\, A) =: J$, reg. seq.
- $(f_1, \ldots, f_r, g_1, \ldots, g_r)$ is a regular sequence in A.

Note: V, W in general position $\underset{\not\Leftarrow}{\Rightarrow}$ intersect properly (ex).

<u>Prop</u>: X, V, W, Z as usual, V, W in gen. pos. ~~$x \in X$ pt corresp. to Z.~~ ~~$x \in \mathrm{Spec}\, A \subset X$, f_i, g_i as in def. ab~~

Then
$$i_Z(V,W) = \mathrm{length}_{\mathcal{O}_{X,Z}} \left(\frac{\mathcal{O}_{X,Z}}{I + J} \right).$$

Pf: Let ~~x~~ $x \in X$ corresp. to z, & let A, f_i, g_i as in def'n above.

~~Pf~~: Enough to check that $\mathrm{Tor}_i^A\left(A/I, A/J\right) = 0 \ \forall i > 0$.

By assumption, ~~enough to show~~ we see that $(f_1 + J, \ldots, f_r + J)$ is a regular sequence in A/J.

Consider the Koszul complexes:

$$K_\bullet^A(f_1, \ldots, f_r) \quad \& \quad K_\bullet^{A/J}(f_1 + J, \ldots, f_r + J).$$

We see quite easily that

$$K_\bullet^A(f_1, \ldots, f_r) \otimes_A A/J \simeq K_\bullet^{A/J}(f_1 + J, \ldots, f_r + J).$$

Now $K_\bullet^A(f_1, \ldots, f_r)$ is a resolution of A/I, ~~so~~ (Thm 2, lecture 9),

so $\mathrm{Tor}_i^A\left(A/I, A/J\right) = H_i\left(K_\bullet^A(f_1, \ldots, f_r) \otimes_A A/J\right)$.

Applying (Thm 2, lecture 9) again we see that $K_\bullet^{A/J}(f_1 + J, \ldots, f_r + J)$ is a resolution of A/J, hence (def'n of resolution)

$$H_i\left(K_\bullet^{A/J}(f_1 + J, \ldots, f_r + J)\right) = \begin{cases} \dfrac{A}{I+J} & i = 0 \\ 0 & i > 0. \end{cases}$$

$\square$

Using this prop'n, it is straightforward to verify the example on page 4. [Exercise]

Exercises

1) Let k be a field. Compute $\dim \mathbb{P}^n_k$.

2) Fill in with T/F:

X	X is irreducible	X is reduced
$\mathbb{P}^n_{\mathbb{F}_p}$		
Affine line with double origin		
any X s.t. X admits an open immersion to an integral scheme.		

3) X, V, W, Z given.
In each case, say whether V & W meet properly, & if they do compute $i_Z(V,W)$.

a) X integral Noeth. regular, fin. dim, $V=W=X=Z$.

b) k field, $X=\mathbb{A}^2_k$, $V: x=0$, $W: x=1, y=0$, $Z=W$.

c) k field, $X=\mathbb{A}^2_k$, $V: y=x^2$, $W: y=x^3$, $Z: x=y=0$.