Lecture 11: The Chow Ring [26/11/2013]

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Last time, we defined irreducible closed subschemes, & talked about how to define their intersection numbers, in the case where they meet properly. Today we will

- put an equivalence relation on formal sums of irred. subschemes;
- make this into a graded group;
- use the intersection product & a 'moving lemma' to turn this into a graded <u>ring</u>, the Chow ring.

<u>Def</u> Let A a 1-dim integral domain with $\mathrm{Frac}(A)=K$. Given $f\in K^*$, write $f=\frac{a}{b}$ with $a,b\in A$. Define

$$\mathrm{ord}_A f = \mathrm{len}_A\left(A/a.A\right) - \mathrm{len}_A\left(A/b.A\right) \in \mathbb{Z},$$

the 'order of f'. Note that $\forall a,b\in A\setminus\{0\}$, have SES

$$0 \to A/a \to A/ab \to A/b \to 0.$$

<u>Ex</u>: Using this, show

1) $\mathrm{ord}_A f$ is well-defined;
2) $\mathrm{ord}_A : K^* \to \mathbb{Z}$ is a gp. hom

<u>Eg</u> If A is reg. loc. Noeth. 1-dim, then A is a DVR (discrete valuation ring). The ord_A defined above coincides with the (suitably normalised) valuation on A.

Rational Functions

Prop: X an integral scheme, $\underset{\mathrm{Spec}\,A}{U}, \underset{\mathrm{Spec}\,B}{V} \subset X$ non-empty affine opens.
Then
$\mathrm{Frac}(A)$ & $\mathrm{Frac}(B)$ are canonically isomorphic.

Pf: Since X irred, we see $U \cap V$ is non-empty. Let $W \overset{= \mathrm{Spec}\,C}{\subset} U \cap V$ aff-open. We get canonical maps

$$\mathrm{Frac}(A) \longrightarrow \mathrm{Frac}(C)$$
$$\mathrm{Frac}(B) \longrightarrow \mathrm{Frac}(C)$$

They are non-zero & surjective, hence $\simeq$. $\square$

Def:
Given an integral scheme X, we write $FF(X)$ for this field. We call it the 'field of rational functions on X'. [! non-standard notation !]

Exercise: Compute $FF(\mathrm{Spec}\,\mathbb{Z})$, $FF(\mathbb{P}^1_k)$ for k a field.

Def: X integral, $Z \subset X$ closed, integral, codimension 1.

Define $\mathrm{ord}_Z : FF(X)^* \longrightarrow \mathbb{Z}$ by:

- Z corresp. to a pt $x \in X$;
- $\mathrm{Spec}\,A \ni x$ an aff. open nhd;
- $\mathcal{O}_{X,Z}$ = localisation of A at x, a 1-d int. domain with fraction field $FF(X)$;
- $\mathrm{ord}_Z = \mathrm{ord}_{\mathcal{O}_{X,x}} : \mathrm{Frac}\,A \to \mathbb{Z}$

Def:
- A *prime divisor* on a scheme X is a closed integral subscheme of codim 1.
- $\mathrm{Div}(X) :=$ free abelian gp on prime divisors of X.

Condition (†) : X is integral & ~~quasi-projective over a field~~ Noetherian.

Prop. Say X satisfies (†) & $f \in FF(X)^*$. Then there are only finitely many prime divisors $Z \subset X$ s.t. $\mathrm{ord}_Z f \neq 0$.

Pf ~~[Sta, Tag 02RL]~~. Wlog $X = \mathrm{Spec}\, A$ with A integral & Noeth, & $f \in A \setminus \{0\}$ [Exercise: reduce to this case]

Set $V = \mathrm{Spec}\, A/(f) \longrightarrow X$ closed subscheme.

Wts. V contains only fin. many closed subschemes $Z \subset X$ with $\mathrm{codim}_X Z = 1$.

Exercise: Show V has fin. many irred comps. (hint: V Noetherian).

Let $Z \subset V$ a closed integral subschem s.t. $\mathrm{codim}_X Z = 1$. Want to show Z is an irreducible component of V (ie. Z is a maximal integral subscheme).

Let $Z \subset W \subset V$, with W a closed integral subscheme. Want to show $Z = W$.
Let Z correspond to a point p of X (so $\dim \mathcal{O}_{X,p} = 1$).
W ——— " ——— q of X.

Note $Z \subset W \Leftrightarrow q \subset p$ (& $W \subset Z \Leftrightarrow p \subset q$).

Now $q \cdot \mathcal{O}_{X,p}$ is a prime ideal, so (by dimension) it is zero or the maximal ideal p. If $q \cdot \mathcal{O}_{X,p} = p \cdot \mathcal{O}_{X,p}$

then $p = q$, done. So assume $q \cdot \mathcal{O}_{X,p} = (0)$ [zero ideal].

Then [exercise] $\exists$ open nbd $p \in U$ s.t. $U \subset W$, but $U \not\subset V$, so $W \not\subset V$, contradiction.

$\square$

Ex: Give example of an integral scheme where this fails.

Prop: X integral Noeth fin. dim., $f \in FF(X)^*$. Then there are only finitely many prime divisors $Z \subset X$ s.t. $\mathrm{ord}_Z f \neq 0$.

Pf: May assume $X = \operatorname{Spec} A$ with A Noetherian, & $f \in A \setminus \{0\}$.
~~Codim. 1 prime ideals p s.t. $\mathrm{ord}_p f \neq 0$~~ $\subseteq$ codim
Now $\dim \operatorname{Spec}\left(A/(f)\right) = \dim A - 1$ (Krull's principal ideal thm).

Now $\mathrm{ord}_Z f \neq 0 \Rightarrow Z \subset \operatorname{Spec}\left(A/(f)\right)$, & $\dim Z = \dim\left(A/(f)\right)$.

Hence Z is an irreducible component of $\operatorname{Spec}\left(A/(f)\right)$, & there are only fin. many irred comps since $A/(f)$ is Noetherian. [See STA, Tag 005Z, or exercise]. $\square$

Def: Let X satisfy (†), $f \in FF(X)^*$. Define $\operatorname{div} f \in \operatorname{Div} X$ by

$$\operatorname{div} f = \sum_{\substack{Z \\ \text{irred} \\ \text{codim } 1}} (\mathrm{ord}_Z f)\,[Z] .$$

$[Z]$: class of Z in $\operatorname{Div} X$

We have just seen that the sum is finite.

We don't actually need this, but we can now define the divisor class group of X as

$$\operatorname{Div}(X) \big/ \langle \operatorname{div} f \mid f \in FF(X)^* \rangle .$$

We are interested not only in divisors, but in subschemes of arbitrary codimension. We need a good equivalence relation on these... (see later for why!)

Def: Let X satisfy $\oplus$. $r \in \mathbb{Z}_{\geq 0}$. A prime cycle of codimension r is a closed integral subscheme of X of codim r.

$$Z^r(X) := \text{free abelian gp. gen by prime cycles of codim } r.$$

$$Z(X) = Z^*(X) := \bigoplus_{r \geq 0} Z^r(X), \quad \text{an abelian group.}$$

Note $Z^0(X) = \mathbb{Z}$.

$[\iota : V \to X]$

Def: Let X satisfy $\oplus$. Given a closed integral $V \subset X$ & a rat. fctn $f \in FF(V)$, define a cycle $div(V,f)$ on X as

$$div(V,f) = \iota(div(f)).$$

NB If codim $V = r$, then every component of $div(V,f)$ has codim $r+1$. $(*)$

Def: Let X satisfy $\oplus$. Define the group of rationally trivial cycles by

$$\cancel{A(X)}\ Rat(X) = \left\langle div(V,f) \;\middle|\; V \subset X \text{ prime cycle, } f \in FF(V)^* \right\rangle.$$

Define the Chow group of X by

$$A(X) = A^*(X) = \frac{Z(X)}{Rat(X)}.$$

Note that ~~the~~ rational equivalence respects the grading by codimension (by $*$), so we can also define

$$Rat^r(X) = \langle div(V,f) \mid V \subset X \text{ prime cycle of codim } r,\ f \in FF(V) \rangle,$$

$$A^r(X) = \frac{Z^r(X)}{Rat^{r+1}(X)}, \quad \& \text{ get } A(X) = \bigoplus_{r \geq 0} A^r(X).$$

Moving lemma, Ring Structure

From now on, we mainly restrict our attention to regular, integral, quasi-projective schemes over an algebraically closed field k. We will call such a scheme 'good'.

~~Thm [Chow's moving lemma]: Let X a good scheme,~~

Def: Let X good, & $\alpha, \beta \in Z(X)$. We say α & β meet properly if for all irred subschemes $V, W \subset X$ s.t.
V has non-zero coefficient in α
& W ——— " ——— β,
we have that V & W meet properly.

ie $\operatorname{codim} V \cap W = \operatorname{codim} V + \operatorname{codim} W$.

Thm [Chow's moving Lemma]. Let X a good scheme, $\alpha, \beta \in Z(X)$. Then $\exists \gamma \in Z(X)$ s.t. $\alpha - \gamma \in Rat(X)$ & β meets γ properly.

Pf: Hard, omitted. Cf. 'Fulton, intersection theory, 11.4, for discussion & references.'

($\square$)

NB: • Can weaken assumptions by only requiring k infinite, but can't expect it to work for all finite fields (as far as I know).
• More modern treatments of intersection theory don't use the moving lemma - cf. Fulton, loc. cit.

Def: Let X good, $\alpha, \beta \in Z(X)$ meeting properly. Write $\alpha = \sum_{v \in V} n_v [V]$, $\beta = \sum_{w \in W} n_w [W]$. Define $\alpha \cdot \beta \in Z(X)$ by

$$\alpha \cdot \beta = \sum_{\substack{w \in W \\ v \in V}} \sum_{\substack{Z \subset X \text{ cl. integral} \\ \text{s.t. } Z \subset V \cap W}} i_Z(V, W) \cdot [Z].$$

Prop: Given $\alpha \in Z(X)$, $\beta \in Rat(X)$, meeting properly, X good. we have $\alpha \cdot \beta \in Rat(X)$.

Pf: Omitted, less hard. $(\square)$

This ~~+~~ moving lemma $\Rightarrow$ there is a well defined intersection product on $A(X)$.

Prop: This product makes $A(X)$ into a commutative graded ring.

Pf omitted. $(\square)$.

Functoriality

- Let $f: X \to Y$ a proper morphism of good schemes. Let $V \subset X$ closed integral. Define $f_* V \in Z(Y)$ by

$$f_* V = \begin{cases} 0 & \text{if } \dim f(V) < \dim V \\ [FF(V) : FF(f(V))]\,[V] & \text{else.} \end{cases}$$

Thm This induces a ~~graded ring~~ group hom $f_* : A(X) \to A(Y)$.

- Let $f: X \to Y$ a flat morphism of good schemes. Given closed integral $V \subset Y$, define
$$f^*[V] = [f^{-1}V] = [V \times_Y X].$$ This defines $f^*: A(Y) \to A(X)$.

- Let $f: X \to Y$ a morphism of good schemes. Let $\alpha \in Z(Y)$. Let $\pi: X \times Y \to Y$ the (flat) projection, & $\Gamma: X \to X \times Y$ the graph of f. Let $\beta \in Z(X \times Y)$ be a cycle s.t.
  - $\beta - \pi^* \alpha \in Rat(X \times Y)$;
  - β meets $\Gamma(X)$ properly.

  Define $f^*[\alpha] = [\beta \cdot \Gamma(X)]$, a cycle on X.

Thm. This extends a gp. hom $f^*: A(Y) \to A(X)$.

Thm [Projection formula]: $f: X \to Y$ proper map of good varieties, $\alpha \in A(X)$, $\beta \in A(Y)$. Then $f_*(\alpha \cdot f^* \beta) = (f_* \alpha) \cdot \beta$.

Exercises

A) $k=\bar{k}$, alg. closed field, $X=\operatorname{spec} k$, $Y=\mathbb{A}^1_k$, $Z=\mathbb{P}^1_k$

1) Compute $A(X)$, $A(Y)$, $A(Z)$ as graded rings.

Consider the following maps:
- structure maps $Y \to X$, $Z \to X$
- inclusion of origin $X \to Y$
- inclusion of $(0:1)$: $X \to Z$
- inclusion of coord chart $Y \to X$

2) Which of these maps are flat? Compute pullback on $A(-)$

3) ——— " ——— proper? Compute push forward on $A(-)$

B) $k=\bar{k}$, ~~__~~ $X=\mathbb{P}^2_{k\ x:y:z}$, ~~______~~ $H=$ hyperplane $(x=0)$.

Compute / a representative of $[H]^2$ in $A(X)$.