Lecture 12: Differentials, Chern classes & Grothendieck-Riemann-Roch

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1) Differentials [following STA, Tag 08RL]

Def: X a top. space, $\varphi: \mathcal{O}_1 \to \mathcal{O}_2$ a hom. of sheaves of rings. Let $\mathcal{F} \in \mathcal{O}_2$-Mod. An $\mathcal{O}_1$-derivation into $\mathcal{F}$ is a map of sheaves $D: \mathcal{O}_2 \to \mathcal{F}$ which is additive, annihilates the image of $\mathcal{O}_1$ in $\mathcal{O}_2$, & satisfies the Liebnitz rule:

$$D(ab) = aD(b) + D(a).b,$$

for all open $U \subset X$ & $a, b \in \mathcal{O}_2(U)$.

- Write $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ for the set of $\mathcal{O}_1$-derivations into $\mathcal{F}$.

- Given $\alpha: \mathcal{F} \to \mathcal{G}$ a map of $\mathcal{O}_2$-modules, we get an induced map

$$\begin{aligned} \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) &\longrightarrow \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G}) \\ D &\longmapsto \alpha \circ D. \end{aligned}$$

Thus we obtain a functor

$$\begin{aligned} \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, -): \mathrm{Mod}(\mathcal{O}_2) &\longrightarrow Ab \\ \mathcal{F} &\longmapsto \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) \end{aligned}$$

Prop: X a top. space, $\varphi: \mathcal{O}_1 \to \mathcal{O}_2$ hom of sheaves of rings. Then there exists an $\mathcal{O}_2$-module $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ s.t. the functor $\mathrm{Hom}_{\mathrm{Mod}\,\mathcal{O}_2}(\Omega_{\mathcal{O}_2/\mathcal{O}_1}, -): \mathrm{Mod}(\mathcal{O}_2) \to Ab$ coincides with $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, -)$.

[~~This~~ We say $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ represents $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, -)$].

Pf: Omitted / exercise / STA Tag 08RL. ($\square$).

Note that $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \Omega_{\mathcal{O}_2/\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\Omega_{\mathcal{O}_2/\mathcal{O}_1}, \Omega_{\mathcal{O}_2/\mathcal{O}_1})$

has a canonical element (the identity), called the 'universal derivation'.

Def: let $(f, f^{\#}) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morph. of LRTS. The sheaf of differentials of f (called $\Omega_{X/S}$) is the module

$\Omega_{\mathcal{O}_X / f^{-1}\mathcal{O}_S}$ together with its universal derivation

$$d_{X/S} : \mathcal{O}_X \to \Omega_{X/S}.$$

Prop. k a field, $R = k[x_1, \dots, x_n]$, $M = R\langle dx_1, \dots, dx_n\rangle$ (i.e. free R-module on these n formal symbols)

$S = \mathrm{Spec}\, k$, $X = \mathrm{Spec}\, R = \mathbb{A}^n_k$.

$$d : \tilde{R} \to \tilde{M}$$
$$x_i \longmapsto dx_i \quad + \text{ Liebnitz rule}$$

Then $(\tilde{M}, d) = (\Omega_{X/S}, d_{X/S})$.

Pf: Given $\mathcal{F} \in \mathcal{O}_X\text{-mod}$, define a map

$$\mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}) \longrightarrow \mathrm{Der}_k(\mathcal{O}_X, \mathcal{F})$$
$$\varphi \longmapsto \Big(\underset{+\text{Liebnitz}}{x_i \longmapsto \varphi(dx_i)}\Big) \quad \text{on each open.}$$

To show this is an isomorphism, check that an inverse is:

$$(dx_i \longmapsto \psi(x_i)) \longleftarrow\!\shortmid \psi \quad \text{on each open.}$$

$\square$

See exercises for more examples.

2) Chern & Todd classes

ref: Hartshorne, Appendix A, section 3.

Throughout this section, $k=k$ is a field & X is a regular, integral, quasi-projective k-scheme (a 'good' scheme)

Recall we have
- $A(X)$ (the Chow ring)
- $K^0(X)$ (Groth. ring of vector bundles)
- $K^0(X) = K_0(X) =: K(X)$.

The Chern & Todd classes relate these rings.

2.1 Divisors & linebundles

linebundles = rank 1 vector bundle
= locally free sheaf of rank 1
= sheaf locally $\cong$ to $\mathcal{O}_X$.

Let X be a good scheme, & D a prime cycle on X of codimension 1. Let $\mathcal{I}_D \subset \mathcal{O}_X$ be the ideal sheaf of D.

Claim: $\mathcal{I}_D$ is locally principle (ie locally generated by 1 element).

Pf: X is regular, so all local rings are UFDs (Atiyah-Mac)
- Image of $\mathcal{I}_D$ in each local ring ~~has~~ height 1 or 0 (def'n of codimension)
- A prime ideal of ht ≤ 1 ~~in~~ in a UFD is principle (Matsumura)

Let $x \in X$. Want an affine open nbd. $\overset{\operatorname{Spec} A}{U} \ni x$ s.t. $\mathcal{I}_D(U)$ is a principle ideal in A. $\mathcal{I}_{D,x}$ is principle in $\mathcal{O}_{X,x}$, say gen. by $f \in \mathcal{O}_{X,x}$.

$\exists \overset{=\operatorname{Spec} A_0}{U_0} \ni x$ open aff. nbd s.t. $f \in \mathcal{I}_D(U_0)$. Moreover, $\mathcal{I}_D(U_0)$ is fin. gen, say by $g_1, \dots, g_n$ (by Noetherianness). In $\mathcal{I}_{D,x}$, $\exists t_1, \dots, t_n$ s.t. $g_j = f \cdot t_i$. $\exists U^{\subset = \operatorname{Spec} A} \ni x$ s.t. $t_i \in \mathcal{O}_X(U)$, so

$$\mathcal{I}_D(U) = (f).$$

$\square$

Ex: $\mathcal{I}_D$ is locally free of rank 1.

<u>Define</u> $\mathcal{O}(D) = \mathcal{H}om_{\mathcal{O}_x\text{-mod}}(\mathcal{I}_D, \mathcal{O}_x)$ [sheaf hom].

<u>Ex</u>: $\mathcal{O}(D)$ is locally ~~principal~~ free of rank 1 (ie invertible)

Given a divisor $D = \sum_i n_i E_i$ a divisor, define $\mathcal{O}(D) = \bigotimes_i \mathcal{O}(E_i)^{\otimes n_i}$ $\left(\begin{array}{l}\text{recall} \\ \mathcal{F}^{\otimes -1} \doteq \mathcal{H}om(\mathcal{F}, \mathcal{O}_x)\end{array}\right)$

Recall that $\mathrm{Pic}(X)$ = isom. classes of invertible sheaves on X.

Then $D \longmapsto \mathcal{O}(D)$ defines a group hom

$$Z^1(X) \longrightarrow \mathrm{Pic}(X)$$

<u>Prop</u> D divisor on X. Then $\mathcal{O}(D) \cong \mathcal{O}_x$ iff $D \in \mathrm{Rat}(X)$.

<u>Pf</u>: Say $D \in \mathrm{Rat}\,X$. Then $\exists f \in FF(X)^*$ s.t. $D = \operatorname{div} f$. Write $D = D_1 - D_2$ where D_i effective (ie $D_i = \sum_j n_j^i E_j^i$ with E_j^i prime & $n_j^i \geq 0$).

Then $\mathcal{O}(D) \cong \mathcal{O}_x \iff \mathcal{O}(-D_1) \cong \mathcal{O}(-D_2) \iff \mathcal{I}_{D_1} \cong \mathcal{I}_{D_2}$,

& we shall show the latter.

<u>Idea</u>: cover X by affine opens U on which $\mathcal{I}_{D_1}(U)$ & $\mathcal{I}_{D_2}(U)$ both principle ideals. On each U, we use f to construct a <u>canonical</u> isomorphism $\mathcal{I}_{D_1}|_U \xrightarrow{\sim} \mathcal{I}_{D_2}|_U$.

<u>Because these maps are canonical, they glue to give an isomorphism over whole of X.</u> [EXERCISE]

Construction of the isomorphism: Let $U = \operatorname{Spec} A$ as above. Enough to construct a canonical morphism of A-modules $\mathcal{I}_{D_1}(U) \to \mathcal{I}_{D_2}(U)$ (by quasi-coherence). Let $g_1, g_2 \in A$ s.t.

$$\mathcal{I}_{D_i}(U) = (g_i) \triangleleft A. \qquad \left[\text{note in general } \frac{g_1}{g_2} \neq f \in \mathrm{Frac}(A)\right]$$

Define an isomorphism $(g_1) \longrightarrow (g_2)$.

$$a \longmapsto \frac{a}{f}$$

$\square$

Hence, $D \mapsto \mathcal{O}(D)$ defines a map $A^1(X) \to \mathrm{Pic}(X)$.
In fact, this is an isomorphism – we don't need this for now, but it may be a nice <u>exercise</u>.

2.2 Chern classes

A 'theory of Chern classes' is a way of assigning, to each good scheme X & vector bundle $\mathcal{E}$ on X, an element $c_t(\mathcal{E}) \in A(X)[t]$ satisfying some conditions. More precisely, it is a fctn:

$$c : \left\{ (X, \mathcal{E}) \,\middle|\, \begin{array}{l} X \text{ good scheme} \\ \mathcal{E} \text{ a v.b. on } X \end{array} \right\} \longrightarrow \left\{ (R, r) \,\middle|\, \begin{array}{l} R \text{ comm. ring} \\ r \in R \end{array} \right\}$$

s.t.

- $c(X, \mathcal{E})$ is of the form $(c_t(\mathcal{E}), A(X)[t])$ for some $c_{\mathcal{E}}(t) \in A(X)[t]$;

- $\forall D \in Z^1(X)$, $\quad c_t(\mathcal{O}(D)) = 1 + [D]\cdot t$;

- $\forall$ morphisms $f : X' \to X$ of good schemes, we have

  $$\forall i : \quad \big[\underbrace{c_t(f^*\mathcal{E})}_{\text{pullback on } \mathcal{E}(X')}\big]_i = \underbrace{f^*([c_t(\mathcal{E})]_i)}_{\text{pullback on } A(X)}.$$

  where $[-]_i :=$ coefficient of t^i

- $\forall$ short exact sequences $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of v.b. on X, we have

  $$c_t(\mathcal{E}) = c_t(\mathcal{E}')\, c_t(\mathcal{E}'') \quad \text{in } A(X)[t]$$

<u>Thm</u> There exists a unique theory of Chern classes.

<u>Pf</u>: Omitted (needs quite a bit of work to set up)
$(\square)$

We often write $c_i(\mathcal{E}) = [c_t(\mathcal{E})]_i$.

2.3 Splitting principle (or: how to compute Chern classes):

Def: X good, $\mathcal{E}$ on X a v. bundle. We say $\mathcal{E}$ splits iff it has a filtration
$$\mathcal{E} = \mathcal{E}_0 \supseteq \mathcal{E}_1 \supseteq \cdots \supseteq \mathcal{E}_r = 0$$
s.t. $\forall i$, $\frac{\mathcal{E}_i}{\mathcal{E}_{i+1}}$ is invertible (i.e. a rank 1 vector bundle).

Thm X good, $\mathcal{E}$ on X a v. bundle. Then $\exists$ a morphism of good schemes $f: X' \to X$ s.t. $f^*\mathcal{E}$ splits.

Pf. Omitted. $\square$

~~[illegible]~~

Prop: Let X good, $\mathcal{E}$ on X a vector bundle which splits, & $\mathcal{L}_1, \dots, \mathcal{L}_r$ the quotient line bundles. Then
$$c_t(\mathcal{E}) = \prod_{i=1}^{r} c_t(\mathcal{L}_i)$$

Pf: Exercise. $\square$

2.4 Chern character & Todd class.

Let X good, $\mathcal{E}$ on X a v. bundle of rank r. Introduce formal symbols $a_1, \dots, a_r$, & write
$$c_t(\mathcal{E}) = \prod_{i=1}^{r} (1 + a_i t).$$
[note $\deg c_t(\mathcal{E}) \leq r$ (exercise)]

Define the exponential Chern character of $\mathcal{E}$ as
$$ch(\mathcal{E}) = \sum_{i=1}^{r} e^{a_i},$$
where $e^x = 1 + x + \frac{x^2}{2} + \cdots$

& the <u>Todd class of $\mathcal{E}$</u> as

$$td(\mathcal{E}) = \prod_{i=1}^{r} \frac{a_i}{1-e^{-a_i}} \qquad \left(\frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \frac{x^2}{12} + \cdots \right)$$

At the moment, these are just power series in formal symbols. However, note they are symmetric in the a_i, & hence can be written as power series in the $c_i(\mathcal{E})$. Moreover, bcs X has finite dimension (say d), we know $A^n(X) = 0 \ \forall n > d$. Hence

$ch(\mathcal{E})$ & $td(\mathcal{E})$ actually make sense in $A(X) \otimes_{\mathbb{Z}} \mathbb{Q} = A(X)_{\mathbb{Q}}$.

We have maps ch & $td : K(X) \to A(X)_{\mathbb{Q}}$.

<u>Prop</u>:
- $\forall$ morphisms $f : X \to Y$ of good schemes, have $f^* \circ ch = ch \circ f^*$
- $ch : K(X) \to A(X)_{\mathbb{Q}}$ is a ring homomorphism
- $\forall D \in \mathrm{Div}(X)$, have $ch([\mathcal{O}(D)]) = \sum_{i \geq 0} \frac{1}{i!} [D]^i$.

<u>Pf</u>: omitted. □

<u>Def</u>: X good. Define the Todd class of X by $td(X) = td(\underbrace{T_{X/\mathrm{Spec}\, k}}_{\text{tangent sheaf}})$.

<u>Thm</u> [Grothendieck - Riemann - Roch]: $f: X \to Y$ proper morphism of good schemes. Then the following diagram commutes:

$$\begin{tikzcd}
K(X) \arrow[r, "f_!"] \arrow[d, "ch"] & K(Y) \arrow[d, "ch"] \\
A(X)_{\mathbb{Q}} \arrow[d, "- \mapsto - \cdot td(X)"'] & A(Y)_{\mathbb{Q}} \arrow[d, "- \mapsto - \cdot td(Y)"] \\
A(X)_{\mathbb{Q}} \arrow[r, "f_*"] & A(Y)_{\mathbb{Q}}
\end{tikzcd}$$

<u>Pf</u>: Omitted. $(\square)$

<u>Next time: applications.</u>

<u>Exercises</u>:

1) Let $f: U \to X$ be an open immersion over S. Show that

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[dr, "\pi_U"'] & X \arrow[d, "\pi_X"] \\
& S
\end{tikzcd}$$

$$\Omega_{U/S} = f^* \Omega_{X/S}.$$

2) Compute $\Gamma(\mathbb{P}^n_k, \Omega_{\mathbb{P}^n_k / \operatorname{Spec} k})$.

3) Let X good, $\mathcal{E}, \mathcal{F}$ on X vector bundles of ranks r, s resp.

- show $\deg c_t(\mathcal{E}) \leq r$
- write $c_t(\mathcal{E}) = \prod_{i=1}^{r}(1+a_i t)$, $c_t(\mathcal{F}) = \prod_{i=1}^{s}(1+b_i t)$ formally

compute $c_t(\mathcal{E} \otimes \mathcal{F})$, $c_t(\mathcal{E}^\vee)$ & $c_t(\Lambda^n \mathcal{E})$.