GHRR: 1. varieties over a field, 2. inclusion of a divisor.

0. Recall the thm: $f\colon X \to Y$ proper, X, Y smooth q.proj. $/k$.

Then
$$\begin{tikzcd} K(X) \arrow[r, "ch_X(\cdot)\cdot td(X)"] \arrow[d, "f_!"'] & A(X)_{\mathbb{Q}} \arrow[d, "f_*"] \\ K(Y) \arrow[r, "ch_Y(\cdot)\cdot td(Y)"'] & A(Y)_{\mathbb{Q}} \end{tikzcd}$$

Here $ch_X\colon K(X) \to A(X)_{\mathbb{Q}}$ is a ring morphism, contravariant in X, for $\mathcal{E}$ in $\mathrm{Vect}(X)$ with a filtration $\mathcal{E} = \mathcal{E}^0 \supset \mathcal{E}^1 \supset \cdots \supset \mathcal{E}^r = 0$ s.t. $\forall i \in \{0, \ldots, r-1\}$ $\mathcal{L}_i := \mathcal{E}^i/\mathcal{E}^{i+1}$ is a line bundle;

$$c_t(\mathcal{E}) = \prod_{0\leq i<r} c_t(\mathcal{L}_i) = \prod_{0\leq i<r} (1 + c_1(\mathcal{L}_i)\cdot t)$$
$$\sum_j c_j(\mathcal{E})\cdot t^j, \quad c_j(\mathcal{E}) \in A^j(X), \quad ch_X(\mathcal{E}) = \sum_{0\leq i<r} e^{c_1(\mathcal{L}_i)} \text{ in } A(X)_{\mathbb{Q}}$$

$ch_X(\mathcal{E})$ = a univ. power series in the $c_j(\mathcal{E})$. Note that $A^j(X) = 0$ if $j > \dim(X)$.

$$td(\mathcal{E}) = \prod_{0\leq i<r} \frac{c_1(\mathcal{L}_i)}{1 - e^{-c_1(\mathcal{L}_i)}}, \quad td(X) = td(T_X).$$

1. Take $Y = \{\cdot\}$ ($\mathrm{Spec}(k)$). Then $\mathrm{Coh}(Y)$ = k-vect. sp. fin. dim.

Then $K(Y) \xrightarrow{\dim} \mathbb{Z}$ is an isomorphism.

Also $A(Y) = A^0(Y) = \mathbb{Z}$, $ch_Y = id_{\mathbb{Z}}$, $td(Y) = 1$ (empty product).

For $\mathcal{F} \in \mathrm{Coh}(X)$: $f_![\mathcal{F}] = \sum_{i\geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}) = \chi(\mathcal{F})$.

Let $d = \dim(X)$ (assume X is connected).

Then
$$\begin{tikzcd} A(X) = A^0(X) \oplus \cdots \oplus A^d(X) \arrow[d, "f_*"'] \arrow[dr, "\text{projection}"] & \\ \mathbb{Z} & A^d(X) \arrow[l, "\deg"'] \end{tikzcd}$$

GHRR = HRR: $\chi(\mathcal{F}) = \deg\left(\left[ch_X(\mathcal{F})\cdot td(X)\right]_d\right)$.

Let us write this out for X/k a connected smooth projective curve, and $\mathcal{F} = \mathcal{O}_X(D)$ for D a divisor on X.

Then $\mathrm{ch}_X(\mathcal{O}_X(D)) = e^{[D]} = 1 + [D]$ in $A(X) = \underset{\mathbb{Z}}{\underset{\vee}{A^0(X)}} \oplus \underset{\mathrm{Pic}(X)}{\overset{\shortparallel}{A^1(X)}}$

$$td(X) = td(T_X) = td(\Omega_X^\vee) = 1 - \tfrac{1}{2} c_1(\Omega_X) = 1 - \tfrac{1}{2}[K]$$

$$\left(\frac{x}{1-e^{-x}} = \frac{x}{1-(1-x+\frac{x^2}{2}+\cdots)} = \frac{1}{1-\frac{x}{2}+\cdots} = 1 + \frac{x}{2} + \cdots \right)$$

with $K = \mathrm{div}_{\Omega_X}(\omega)$, $\omega \neq 0$ a rational 1-form.

Then GRR says:

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D)) = \overset{\deg}{\vee}\left[(1+[D]) \cdot (1 - \tfrac{1}{2}[K])\right]_1 =$$
$$= \deg(D) - \tfrac{1}{2}\deg(K).$$

Apply with $D = 0$, and use $H^0(X, \mathcal{O}_X) = k$, $\dim H^1(X, \mathcal{O}_X) =: g$, the genus of X, gives: $1 - g = -\tfrac{1}{2}\deg(K)$, $\deg(K) = 2g-2$.

For $X = \mathbb{P}^n_k$, $n \geq 0$: $K(X) = \mathbb{Z}[h]/(h^{n+1})$, $h = [\mathcal{O}_H] = [\mathcal{O}_X] - [\mathcal{O}_X(-1)]$, H a hyperplane.

$A(X) = \mathbb{Z}[\varepsilon]/(\varepsilon^{n+1})$, $\varepsilon = [H]$.

One can verify GRR by a direct calculation.

2. Let X be smooth quasi proj. $/k$, $D \subset X$ closed, smooth $/k$, of codim 1, $i: D \to X$ the inclusion.

Our aim is to prove GRR in this situation for $y \in K(D)$ of the form $y = i^! x$ for some $x \in K(X)$. See Borel–Serre Prop. 14.

The diagram:

$$\begin{array}{ccc} & K(D) & \xrightarrow{ch_D(\cdot).td(D)} & A(D)_{\mathbb{Q}} \\ (\forall j>0\ R^j i_* = 0) & {\scriptstyle i_* = i_!}\downarrow & & \downarrow{\scriptstyle i_*} \\ & K(X) & \xrightarrow{ch_X(\cdot).td(X)} & A(X)_{\mathbb{Q}}. \end{array}$$

We have: $I_D \hookrightarrow \mathcal{O}_X \twoheadrightarrow \mathcal{O}_D$ (0)

$$I_D/I_D^2 \overset{d}{\hookrightarrow} i^*\Omega_X \twoheadrightarrow \Omega_D \quad \text{(Hartshorne II thm 8.17)}$$
$$(\Omega_X = \Omega_{X/k} \text{ etc.})$$

dualise: $T_D \hookrightarrow i^* T_X \twoheadrightarrow N$, $N =$ normal sheaf $= (I_D/I_D^2)^\vee = i^* I_D^\vee = i^* \mathcal{O}_X(-D)^\vee = i^*(\mathcal{O}_X(D))$.

td is multipl: $td(i^*T_X) = td(T_D).td(N)$

Let $y \in K(D)$. Then GRR for y is:

$$i_*(ch_D(y).td(D)) = ch_X(i_* y).td(X)$$

This is equivalent to: $i_*(ch_D(y).td(D)).\,td(X)^{-1} = ch(i_* y)$ ($td(X)$ is a unit)

$$= i_*\left(ch_D(y).td(D).\,i^*td(X)^{-1}\right) = i_*\left(ch_D(y).\,td(T_D).\,td(i^*T_X)^{-1}\right)$$

$$= i_*\left(ch_D(y).\,td(N)^{-1}\right).$$

So we want to prove: $ch(i_* y) = i_*\left(ch_D(y).\,td(N)^{-1}\right)$ (1)

Claim: $i^! i_! y = y \cdot (1 - [I_D/I_D^2])$ in $K(D)$. (2)

Proof. Both sides are $\mathbb{Z}$-linear in y, so it is enough to prove it for $y = [\mathcal{E}]$, $\mathcal{E}$ in $Vect(D)$. Then $i_! y = [i_* \mathcal{E}]$ in $K(X)$. Let $\mathcal{E}_\bullet \to i_*\mathcal{E}$ be a finite res. in ~~Vect~~ with all $\mathcal{E}_j$ in $Vect(X)$.

Then $i^! i_! y = \sum_j (-1)^j [i^* \mathcal{E}_j] = \sum_j (-1)^j \mathrm{Tor}_j^{\mathcal{O}_X}(\mathcal{O}_D, i_*\mathcal{E})$ now use $I_D \rightarrowtail \mathcal{O}_X \twoheadrightarrow \mathcal{O}_D$

$$= [\mathcal{E}] - [I_D/I_D^2 \otimes_{\mathcal{O}_D} \mathcal{E}]$$
$$= [\mathcal{E}] \cdot (1 - [I_D/I_D^2]). \quad ⊠$$

$$\begin{array}{ccc} I_D \otimes_{\mathcal{O}_X} i_*\mathcal{E} & \longrightarrow & \mathcal{O}_X \otimes_{\mathcal{O}_X} i_*\mathcal{E} \\ \| & & \| \\ I_D/I_D^2 \otimes_{\mathcal{O}_D} \mathcal{E} & \xrightarrow{0} & \mathcal{E} \end{array}$$

Now assume $y = i^! x$ for some $x \in K(X)$.

Then: (LHS of ①)

$$ch(i_! y) = ch(i_! i^! x) = ch(i_!(1 \cdot (i^! x))) = ch((i_! 1) \cdot x) = ch(x \cdot i_! 1) \overset{⓪}{=}$$

$$= ch(x \cdot (1 - [I_D])) = ch(x) \cdot (ch(1) - ch(I_D)) =$$

$$= ch(x) \cdot (1 - e^{-[D]}), \text{ bec. } ch(I_D) = ch(\mathcal{O}_X(-D)),\ c_1(\mathcal{O}_X(-D)) = -[D].$$

But (RHS of ①):

$$i_*(ch_D(y) \cdot td(N)^{-1}) = i_*(ch_D(i^! x) \cdot td(i^* \mathcal{O}_X(D))^{-1}) =$$

$$= i_*(i^*(ch_X(x)) \cdot i^* td(\mathcal{O}_X(D))^{-1})$$

$$= i_*(1 \cdot i^*(ch(x) \cdot td(\mathcal{O}_X(D))^{-1}))$$

$$= i_*(1) \cdot ch(x) \cdot td(\mathcal{O}_X(D))^{-1}$$

$$= [D]\, ch(x) \cdot \left(\frac{[D]}{1 - e^{-[D]}}\right)^{-1}.$$ Done!

Lecture 13.2 10/12/2013

David Holmes

We will treat two more cases of GHRR:

- surface $\to$ pt
- surface $\to$ curve

Fix $k = \bar{k}$

Recall: 'good scheme' = regular integral, quasi-projective scheme$/k$.

Let X be a good projective scheme of dim. 2, & $Y = \operatorname{Spec} k$. Let $f : X \to Y$ be the structure map. Let D be a divisor on X.

We will apply GHRR to $\mathcal{O}(D)$.

Fill in as we go:

$$\begin{tikzcd}
K(X) \arrow[r, "f_!"] \arrow[d, "ch"'] & K(Y) = \mathbb{Z} \arrow[d, "ch"] \\
A(X)_{\mathbb{Q}} \arrow[d, ".td(X)"] & A(Y)_{\mathbb{Q}} = A^0(Y)_{\mathbb{Q}} = \mathbb{Q} \arrow[d, ".td(Y)"] \\
A(X)_{\mathbb{Q}} \arrow[r, "f_*"'] & A(Y)_{\mathbb{Q}}
\end{tikzcd}$$

bcs rel dim 2.

$$\begin{tikzcd}
{[\mathcal{O}(D)]} \arrow[r, mapsto] \arrow[d, mapsto] & {[f_*\mathcal{O}(D)] - [R^1f_*\mathcal{O}(D)] + [R^2f_*\mathcal{O}(D)]} \arrow[d, mapsto, "\text{use that } ch(\mathcal{O}_Y^{\oplus r}) = r\,[E_Y]."] \\
1 + D + \frac{1}{2}D^2 \arrow[d, mapsto] & \chi(\mathcal{O}(D)) \arrow[d, mapsto] \\
ch \cdot td \arrow[r, mapsto] & \chi(\mathcal{O}(D))
\end{tikzcd}$$

- $c_t(\mathcal{O}(D)) = 1 + t.[D]$.
- $ch(\mathcal{O}(D)) = 1 + [D] + \frac{1}{2}[D]^2$. (cf Hartshorne, p 432)
- $td(T_X) = 1 + \frac{1}{2}c_1(T_X) + \frac{1}{12}(c_1(T_X)^2 + c_2(T_X))$

  Note $c_i(E) \in A^i(X)$, so $c_i = 0$ $\forall i > \dim X$.

- $ch(Y) = 1$.

- $f_* : A(X)_{\mathbb{Q}} \to A(Y)_{\mathbb{Q}}$ kills $A^0(X)$ & $A^1(X)$. Only interesting bit is deg 2.

$$\left[ch(\mathcal{O}(D)) \cdot td(X)\right]_2 = \left[\left(1 + D + \tfrac{1}{2}D^2\right)\left(1 + \tfrac{1}{2}c_1(T_X) + \tfrac{1}{12}(c_1(T_X)^2 + c_2(T_X))\right)\right]_2$$

$$= \frac{1}{12}(c_1(T_X)^2 + c_2(T_X)) + \frac{1}{2}D.c_1(T_X) + \frac{1}{2}D^2.$$

$$\overset{GHRR}{=} \chi(\mathcal{O}(D)).$$

- $c_1(T_X)$ & $c_2(T_X)$ are two important invariants of X (cf. genus of a curve).
- Setting $D=0$ yields $\chi(\mathcal{O}_X) = \frac{1}{12}(c_1(T_X)^2 + c_2(T_X))$.

Surface $\to$ curve X, C good projective, $\dim X = 2$, $\dim C = 1$, $f: X \to C$ smooth proper.
To simplify formulae, assume $g(C)=1$, so $T_C \cong \mathcal{O}_C$, $td(C)=1$.
- D on X a divisor. Apply GHRR to $\mathcal{O}(D)$.

$$\begin{array}{ccc} K(X) & \xrightarrow{f_!} & K(C) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ A(X)_{\mathbb{Q}} & G & A(Y)_{\mathbb{Q}} \\ \downarrow{\scriptstyle \cdot td} & & \downarrow{\scriptstyle \cdot 1} \\ A(X)_{\mathbb{Q}} & \xrightarrow{f_*} & A(Y)_{\mathbb{Q}} \end{array} \qquad \begin{array}{ccc} [\mathcal{O}(D)] & \longmapsto & [f_*\mathcal{O}(D)] - [R^1 f_*\mathcal{O}(D)] \\ \downarrow & & \\ 1 + D + \frac{1}{2}D^2 & & \\ \downarrow & & \\ ch(\;)td(X) & \longmapsto & \end{array}$$

- $R^i f_* \mathcal{O}(D) = 0 \;\forall i \geq 2$, bcs rel. dim $X/C = 1$.
- $c_t(\mathcal{O}(D)) = 1 + Dt$ as always, so $ch(\mathcal{O}(D)) = 1 + D + \frac{1}{2}D^2$ (Hartshorne P432).
- Have exact seq. of sheaves on X:

$$0 \to T_{X/C} \to T_X \longrightarrow f^* T_C.$$

- Since $g(C)=1$, $T_C = \mathcal{O}_C$, hence $f^* T_C = \mathcal{O}_X$
- Since f is smooth, $T_{X/C}$ is a vector bundle (actually a line bundle, bcs rel. dim $X/C = 1$).

Multiplicativity:
$$c_t(T_X) = c_t(T_{X/C}) \cdot c_t(f^* T_C) = c_t(T_{X/C})$$

Write $T_{X/C} = \mathcal{O}_X(-K_{X/C})$ (it is a line bundle, so such a divisor $K_{X/C}$ exists).

Then $td(X) = 1 - \frac{K_{X/C}}{2} + \frac{1}{12} K_{X/C}^2$.

Hence $ch(\mathcal{O}(D)) . td(X) = \underbrace{(1)}_{\deg 0} \underbrace{\left(-\frac{K_{X/C}}{2} + D\right)}_{\deg 1} + \underbrace{\left(\frac{1}{12}K_{X/C}^2 - \frac{D.K_{X/C}}{2} + \frac{1}{2}D^2\right)}_{\deg 2}$.

(higher order vanishes bc $\dim X = 2$)

Ok, so we understand the LHS of the picture quite well. What abt. the RHS?

Need to calculate $ch[f_* \mathcal{O}(D)] - ch[R^1 f_* \mathcal{O}(D)]$.

Claim: $f_* \mathcal{O}(D)$ is a vector bundle

Pf: Wlog base is affine, $U = \operatorname{Spec} A$ where A a dedekind domain. Cover $f^{-1}U$ by affine opens V_i, then

$$f_* \mathcal{O}(D) \hookrightarrow \prod_i \mathcal{O}_X(V_i).$$

(If you like, think about the zero part of the Čech complex.)

Now $\mathcal{O}_X(V_i)$ is torsion free, hence so is $f_* \mathcal{O}(D)$. Then because A is dedekind, this implies $f_* \mathcal{O}(D)$ locally free by structure of fin.-gen. modules over a Dedekind domain. □

[Rk: If char $k = 0$, then $R^1 f_* \mathcal{O}_X$ is also loc. free (uses f smooth) – see [du Bois, Complexe de de Rham filtré d'une variété singulière]. We will not use this].

For $R^1 f_* \mathcal{O}(D)$, write

$$0 \to \underbrace{(R^1 f_* \mathcal{O}(D))_{tors}}_{\text{torsion}} \rightarrowtail R^1 f_* \mathcal{O}(D) \twoheadrightarrow \underbrace{\overline{R^1 f_* \mathcal{O}(D)}}_{\text{vector bundle}} \to 0$$

Then $ch(f_![\mathcal{F}]) = ch(f_*\mathcal{F}) - ch(R^1f_*\mathcal{F})$

$$= rk_\eta(f_*\mathcal{F}) + c_1(f_*\mathcal{F})$$

$$- rk_\eta(\overline{R^1f_*\mathcal{F}}) - c_1(\overline{R^1f_*\mathcal{F}})$$

$$- \sum_{\substack{p\in C^0 \\ \text{closed pts}}} \Big[\dim_k \underbrace{(R^1f_*\mathcal{F})_{tors,p}}_{\text{stalk at } c}\Big]\cdot \underbrace{[p]}_{\text{class of } p \text{ in } A^1(C)}.$$

Recalling $td(C)=1$, GHRR says:
Let η = generic pt of C.

In $A^0(C)_{\mathbb{Q}}$: $\deg_\eta\left(-\frac{K_{X/C}}{2}+D\right) = rk_\eta(f_*\mathcal{F}) - rk_\eta(R^1f_*\mathcal{F})$.

In $A^1(C)_{\mathbb{Q}} = \mathrm{Pic}\, C \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$f_*\left(\tfrac{1}{12}K_{X/C}^2 - \frac{D.K_{X/C}}{2} + \tfrac{1}{2}D^2\right) = c_1(f_*\mathcal{F}) - c_1(R^1f_*\mathcal{F})$$

$$= c_1(f_*\mathcal{F}) - c_1(\overline{R^1f_*\mathcal{F}}) - \sum_{p\in C^0} \dim_k (R^1f_*\mathcal{F})_{tors,p}\cdot[p].$$

Note this is in $(\mathrm{Pic}\, C)\otimes_{\mathbb{Z}}\mathbb{Q}$, not just an equality of degrees.