Lecture 2

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Abstract

We discuss fibred products, projective spaces, and the proof that projective space is proper. There is probably too much material here to cover in the lecture - part of the homework is to read and understand these notes. The other part is to do the exercises.

Large parts of this are copied verbatim from the stacks project (though the errors are all mine). In particular, if there is something you do not understand, the stacks project is very likely to be a good reference (they give a lot more details than I have time to include). The only exception to this is the proof that projective space is proper - for this we follow Liu's book, another useful reference.

Contents

1 Fibred products

Definition 1.1. *Write* Sch *for the category of schemes. Let S be a scheme. Write* Sch_S *for the category whose objects are morphisms $X \to S$ in* Sch*, and where*

$$\mathrm{Mor}_{\mathrm{Sch}_S}(X \to S, Y \to S)$$

is the set of commutative diagrams

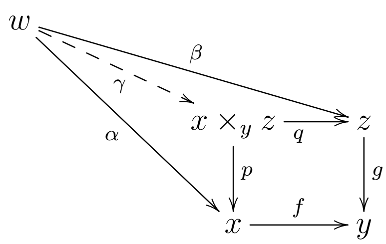
$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow & \downarrow \\ & & S. \end{array}$$

We call objects of Sch_S 'S-schemes'. Note that $\mathrm{Sch}_{\mathbb{Z}}$ is equivalent to Sch.

Definition 1.2. *Let $\mathcal{C}$ be a category. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_{\mathcal{C}}(x, y)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(z, y)$. A fibre product of f and g is an object $x \times_y z \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_{\mathcal{C}}(x \times_y z, x)$ and $q \in \mathrm{Mor}_{\mathcal{C}}(x \times_y z, z)$ making the diagram*

$$\begin{array}{ccc} x \times_y z & \xrightarrow{q} & z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ x & \xrightarrow{f} & y \end{array}$$

commute, and such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(w, x)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(w, z)$ with $f \circ \alpha = g \circ \beta$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(w, x \times_z y)$ making the diagram



commute.

A diagram satisfying this universal property is called a 'cartesian diagram'.

If a fibre product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma.

Definition 1.3. *We say the category $\mathcal{C}$ has fibre products if the fibre product exists for any $f \in \mathrm{Mor}_{\mathcal{C}}(x, z)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(y, z)$.*

Lemma 1.4. *The category of affine schemes has fibre products.*

Proof. Set $\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} C = \operatorname{Spec}(A \otimes_C B)$. □

Lemma 1.5. *Let S be a scheme. Then Sch_S has fibres products.*

Proof. Glue from the affine case. Details omitted. □

Given $X/S \in \mathrm{Sch}_S$ and a morphism of schemes $f : T \to S$, we often write f^*X for the T-scheme $X \times_S T \to T$.

2 Open and closed immersions

Definition 2.1. *Let $(f : X \to Y, \varphi : \mathcal{O}_Y \to \mathcal{O}_X)$ be a morphism of locally ringed spaces. We say that f is a closed (open) immersion if f is a homeomorphism of X onto a closed (open) subset of Y (i.e. a topological closed (open) immersion) and for every $x \in X$, the map*

$$\varphi_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is surjective (is an isomorphism).

Lemma 2.2. *Let $f : Z \to X$ be a morphism of locally ringed spaces. In order for f to be a closed immersion it suffices if there exists an open covering $X = \bigcup U_i$ such that each $f : f^{-1}U_i \to U_i$ is a closed immersion.*

Proof. Omitted. □

Exercise 2.3.

1. *Let R a ring and $I \lhd R$ an ideal. Then the natural map $\operatorname{Spec} R/I \to \operatorname{Spec} R$ is a closed immersion.*

2. *Let R a ring and $f \in R$. Then $\operatorname{Spec} R[1/f] \to \operatorname{Spec} R$ is an open immersion.*

Definition 2.4. *Let X be a scheme.*

1. *A morphism of schemes is called an open immersion if it is an open immersion of locally ringed spaces (see Definition 2.1).*

2. *A morphism of schemes is called a closed immersion if it is a closed immersion of locally ringed spaces (see Definition 2.1).*

3. *A morphism of schemes $f : X \to Y$ is called an immersion, or a locally closed immersion if it can be factored as $j \circ i$ where i is a closed immersion and j is an open immersion.*

Any open (resp. closed) immersion of schemes is isomorphic to the inclusion of an open (resp. closed) subscheme of the target [proof omitted].

Exercise 2.5. *A composite of open (closed) immersions is an open (closed) immersion.*

3 Separated morphisms

A topological space X is Hausdorff if and only if the diagonal $\Delta \subset X \times X$ is a closed subset. The analogue in algebraic geometry is, given a scheme X over a base scheme S, to consider the diagonal morphism

$$\Delta_{X/S} : X \longrightarrow X \times_S X.$$

This is the unique morphism of schemes such that $\mathrm{pr}_1 \circ \Delta_{X/S} = \mathrm{id}_X$ and $\mathrm{pr}_2 \circ \Delta_{X/S} = \mathrm{id}_X$ (it exists in any category with fibre products).

Exercise 3.1. *The diagonal morphism of a morphism between affines is a closed immersion. Hint: use Exercise 2.3.*

Definition 3.2. *Let $f : X \to S$ be a morphism of schemes.*

1. *We say f is separated if the diagonal morphism $\Delta_{X/S}$ is a closed immersion.*

2. *We say a scheme Y is separated if the morphism $Y \to \mathrm{Spec}(\mathbb{Z})$ is separated.*

Example 3.3 (Affine line (with origin doubled))**.** *Let $\mathbb{A}^1_{\mathbb{Z}} := \mathrm{Spec}\,\mathbb{Z}[x]$, the affine line over $\mathbb{Z}$. Let S be a scheme with structure morphism $f : S \to \mathrm{Spec}\,\mathbb{Z}$, and let $\mathbb{A}^1_S = f^*\mathbb{A}^1_{\mathbb{Z}}$, the affine line over S.*

Take two copies $\mathrm{Spec}\,\mathbb{Z}[x]$ and $\mathrm{Spec}\,\mathbb{Z}[y]$ of the affine line over $\mathbb{Z}$. Let $U_x = \mathrm{Spec}\,\mathbb{Z}[x, 1/x]$ and $U_y = \mathrm{Spec}\,\mathbb{Z}[y, 1/y]$. Let $f : U_x \to U_y$ be the isomorphism sending x to y. Consider the scheme obtained by glueing the two copies of $\mathbb{A}^1_{\mathbb{Z}}$ along f (see the exercises for a definition of this). This is the 'affine line with origin doubled', a non-separated scheme. Base change yields an example of a non-separated scheme over any S.

Note that if you glued along the isomorphism $U_x \to U_y$ sending x to $1/y$ instead, you would obtain $\mathbb{P}^1_{\mathbb{Z}}$ - see below.

4 Proj of a graded ring

A *graded ring* is ring S endowed with a direct sum decomposition $S = \bigoplus_{d \geq 0} S_d$ such that $S_d \cdot S_e \subset S_{d+e}$. Note that we do not allow nonzero elements in negative degrees. The *irrelevant ideal* is the ideal $S_+ = \bigoplus_{d>0} S_d$.

Let S be a graded ring. An element $s \in S$ is called homogeneous if there exists $d \geq 0$ such that $s \in S_d$. A *homogeneous ideal* is an ideal generated by homogeneous elements.

Definition 4.1. *Let S be a graded ring. We define Proj(S) to be the set of homogeneous, prime ideals $\mathfrak{p}$ of S such that $S_+ \not\subset \mathfrak{p}$. As Proj($S$) is a subset of* $\operatorname{Spec}(S)$ *and we endow it with the induced topology. The topological space Proj(S) is called the homogeneous spectrum of the graded ring S.*

Note that by construction there is a continuous map

$$\operatorname{Proj}(S) \longrightarrow \operatorname{Spec}(S_0)$$

Let $f \in S$ homogeneous of degree > 0.

- We define $S_{(f)}$ to be the subring of S_f consisting of elements of the form r/f^n with r homogeneous and $\deg(r) = nd$.
- We define *standard open sets*

$$D_+(f) = \{\mathfrak{p} \in \operatorname{Proj}(S) \mid f \notin \mathfrak{p}\}.$$

For a homogeneous ideal $I \subset S$ we define

$$V_+(I) = \{\mathfrak{p} \in \operatorname{Proj}(S) \mid I \subset \mathfrak{p}\}.$$

Lemma 4.2 (Topology on Proj)**.** *Let $S = \oplus_{d \geq 0} S_d$ be a graded ring.*

1. *The sets $D_+(f)$ are open in* $\operatorname{Proj}(S)$.
2. *We have $D_+(ff') = D_+(f) \cap D_+(f')$.*
3. *The open sets $D_+(f)$ form a basis for the topology of* $\operatorname{Proj}(S)$.
4. *Let $f \in S$ be homogeneous of positive degree. The ring S_f has a natural $\mathbf{Z}$-grading. The ring maps $S \to S_f \leftarrow S_{(f)}$ induce homeomorphisms*

$$D_+(f) \leftarrow \{\mathbf{Z}\textit{-graded primes of } S_f\} \to \operatorname{Spec}(S_{(f)}).$$

5. *The sets $V_+(I)$ are closed.*

6. *Any closed subset $T \subset \operatorname{Proj}(S)$ is of the form $V_+(I)$ for some homogeneous ideal $I \subset S$.*

7. *For any graded ideal $I \subset S$ we have $V_+(I) = \emptyset$ if and only if $S_+ \subset \sqrt{I}$.*

Proof. Since $D_+(f) = \operatorname{Proj}(S) \cap D(f)$, these sets are open. For the rest see [3, Tag 00JP]. □

By (3) and (4) of Lemma 4.2, the open sets $D^+(f)$ give an open cover of the topological space $\operatorname{Proj} S$ by spaces homeomorphic to the underlying topological spaces of affine schemes. Hopefully this makes it reasonable that we can glue together these affine schemes to put a scheme structure on $\operatorname{Proj} S$. Read [3, Tag 01M3] for details.

4.1 Base change for projective schemes

Let A be a ring, and B a graded A-algebra - that is, B is a graded ring and multiplication by elements of A preserves the graded pieces of B. In particular, B_0 is naturally an A-algebra, and so we have natural maps $\operatorname{Proj} B \to \operatorname{Spec} B_0 \to \operatorname{Spec} A$.

Given another A-algebra C, we have

$$B \times_A C = \bigoplus_{d \geq 0} (B_d \otimes_A C),$$

so that $B \times_A C$ has a natural grading.

Lemma 4.3. *Let A, B, C as above. Then we have a canonical isomorphism*

$$\operatorname{Proj}(B \otimes_A C) = \operatorname{Proj} B \otimes_{\operatorname{Spec} A} \operatorname{Spec} C.$$

Proof. [2, 3.1.9]. □

5 Projective space

Fix $n \geq 0$ an integer. Let $R = \mathbb{Z}[x_0, \ldots, x_n]$, graded by degree of the polynomial (setting $\deg x_i = 1$ for all i). We define n-dimensional projective space over $\mathbb{Z}$ by $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} R$.

Let $S \in \mathrm{Sch}$. Define $\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} S$.

6 Universally closed morphisms

Definition 6.1. *Let $f : X \to Y$ be a morphism of schemes. We say f is* universally closed *if for all morphisms $T \to Y$, the map $X \times_Y T \to T$ induced by the fibre product is a closed morphism of the underlying topological spaces.*

The base change of a universally closed morphism is closed. A composite of universally closed morphisms is universally closed. Being universally closed is local on the target (not the source!)

7 Projective space is universally closed

Lemma 7.1. *Let S a scheme and $n \in \mathbb{Z}_{\geq 0}$. The S-morphism $\mathbb{P}^n_S \to S$ is universally closed.*

Proof. (Following [2, 3.3.30]). From the definition and because being universally closed is local on the target, it is enough to show that for any affine scheme $Y = \operatorname{Spec} A$, the morphism $\pi : \mathbb{P}^n_Y \to Y$ is closed.

Write $B = A[x_0, \ldots, x_n]$. Let $V^+(I)$ be a closed subset of $\mathbb{P}^n_Y$. We need to show that $Y \setminus \pi(V^+(I))$ is open. Let $y \in Y$. By Lemma 4.3 we have

$$V^+(I) \cap \pi^{-1}(y) = V^+(I \otimes_A k(y)),$$

(where $k(y)$ is the residue field at y). Hence $y \in Y \setminus \pi(V^+(I))$ if and only if $(B \otimes_A k(y))^+ \subset \sqrt{I \otimes_A k(y)}$ (by Lemma 4.2). Since B is finitely generated as an A-algebra (by the x_i, this is equivalent to

$$B_m \otimes_A k(y) \subset I \otimes_A k(y)$$

for some integer m. This is in turn equivalent to

$$(B/I)_m \otimes_A k(y) = 0.$$

Note that we have a surjective A-module map $B_m \to (B/I)_m$. Easy to see B_m is fin bern as A-module, so the same holds for $(B/I)_m$.

Let us now take $y \in Y \setminus \pi(V^+(I))$. Let $m \geq 1$ such that $((B/I) \otimes_A k(y))_m = 0$. Since $(B/I)_m$ is finitely generated as an A-module [exercise: why?], it follows from Nakayama's lemma that

$$(B/I)_m \otimes_A \mathcal{O}_{Y,y} = 0.$$

Hence there exists $f \in A$ such that $y \in D(f)$ and $f \cdot (B/I)_m = 0$, and hence $(B/I)_m \otimes_A A_f = 0$. From this we see

$$y \in D(f) \subset Y \setminus \pi(V^+(I)),$$

and we are done. □

8 Proper morphisms

Definition 8.1. *Let $f : X \to Y$ be a morphism of schemes. We say f is*

- quasi-compact *if the inverse image of any affine open of Y is compact (as a topological space).*
- locally of finite type *if for every affine open $V \subset Y$ and every affine open $U \subset f^{-1}V$, the canonical map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ makes $\mathcal{O}_X(U)$ into a finitely generated $\mathcal{O}_Y(V)$-algebra.*
- of finite type *if it is quasi-compact and locally of finite type.*
- proper *if is is separated, of finite type and universally closed.*

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In the old version of these notes, there was a proof that projective space was proper. The proof of separateness was incorrect, and was done correctly in Bas' lecture.

9 Exercises

A few exercises are distributed in the text. Some more are below.

Exercise 9.1. *Let A be a ring. Show that* Spec A *is compact as a topological space.*

Exercise 9.2. *Show that every closed immersion of schemes is proper.*

Exercise 9.3 (Hartshorne II.4.3)**.** *Let S be an affine scheme and let $X \to S$ be separated. Let U and V be affine open subsets of X. Show that $U \cap V$ is also affine. Give an example to show that this fails without the assumption that X/S is separated.*

Exercise 9.4. *Let X a scheme, and U, $V \subset X$ affine opens. Then there exists a cover $\mathcal{Y} = \{Y_i\}$ of $U \cap V$ such that every Y_i is a principal open subset in U and is a principal open subset in V.*

Hints:

1 Fix $p \in U \cap V$.

2 Show there exists $f \in \mathcal{O}(U)$ with $p \in D_U^+(f) \subset U \cap V$.

3 Show there exists $g \in \mathcal{O}(V)$ such that $p \in D_V^+(g) \subset D_U^+(f)$.

4 Show there exists $n \in \mathbb{Z}_{\geq 0}$ such that $f^n g \in \mathcal{O}(U)$.

5 Show $D_V^+(g) = D_U^+(f^{n+1}g)$.

References

[1] Robin Hartshorne *Algebraic Geometry*

[2] Qing Liu *Arithmetic Geometry and Algebraic Curves*

[3] de Jong et al. *Stacks Project*