# Lecture 4

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**Abstract**

For us, categories are locally small, i.e. the Hom-sets are sets (not proper classes).

# 1 Pre-additive categories

Let $C$ be a categories. We say $C$ is *preadditive* if each morphism set $\mathrm{Mor}_{\mathcal{A}}(x, y)$ is endowed with the structure of an abelian group such that the compositions

$$\mathrm{Mor}(x, y) \times \mathrm{Mor}(y, z) \longrightarrow \mathrm{Mor}(x, z)$$

are bilinear.

In particular for every $x, y$ there exists at least one morphism $x \to y$, namely the zero map.

Eg. The category of modules over a ring.

Eg. Not the category of schemes.

A functor between pre-additive categories is called additive if it is an abelian group hom on each hom-set.

# 2 Limits and colimits

## 2.1 Definitions

### 2.1.1 Cones

Let $S$ be a category (the shape category). Let $C$ be another category.

A *diagram* of shape $S$ is a functor $f : S \to C$.

A *cone over* a diagram $f : S \to C$ is an object $N \in \operatorname{Ob} C$ together with $C$-morphisms $n_s : N \to f(s)$ for all $s \in S$, such that for all $\varphi : s \to t \in S$ we have

$$n_t = f(\varphi) \circ n_s.$$

A *morphism of cones* $(N, (n_s)_S) \to (M, (m_s)_S)$ is a $C$-morphism $\psi : N \to M$ such that for all $s \in S$ we have

$$n_s = m_s \circ \Psi.$$

The *Category of cones over* a diagram $f$ of shape $S$ has cones over $f$ as objects and morphisms of cones as morphisms!

A *cone under* a diagram $f : S \to C$ is an object $N \in \operatorname{Ob} C$ together with $C$-morphisms $n_s : f(s) \to N$ for all $s \in S$, such that for all $\varphi : s \to t \in S$ we have

$$n_s = n_t \circ f(\varphi).$$

We define a morphism of cones under a diagram, and the category of such cones, in an analogous way.

### 2.1.2 Limits and colimits

Let $f$ be a diagram of shape $S$ to a category $C$.

A *limit* of $f$ is a terminal object of the category of cones over $f$.

A *colimit* of $f$ is an initial object of the category of cones under $f$.

Limits and colimits are unique up to unique isomorphism, if they exist.

## 2.2 Examples

### 2.2.1 Products and coproducts

Let $S$ be a category with two objects $s$ and $t$, where the only morphisms are identities.

Let $f : S \to C$ be a diagram. The limit of $f$ is the product of $f(s)$ and $f(t)$. The colimit is the coproduct of $f(s)$ and $f(t)$.

### 2.2.2 Fibred Products

Let $S$ be a category with 3 objects $s, t, u$, with identities and also morphisms $s \to t$ and $u \to t$. A limit over a diagram of shape $S$ is the fibre product of $f(s)$ with $f(u)$ over $f(t)$.

### 2.2.3 Fibred Coproducts

Let $S$ be a category with 3 objects $s, t, u$, with identities and also morphisms $t \to s$ and $t \to u$. A colimit over a diagram of shape $S$ is the cofibred product of $f(s)$ with $f(u)$ under $f(t)$.

### 2.2.4 Equalisers and coequalisers

Let $S$ be a category with 2 objects $s$ and $t$, and two morphisms $a, b : s \to t$. Let $f$ be a diagram of shape $S$. The limit of $f$ is the equaliser of $a$ and $b$. The colimit is the co-equaliser. This generalises kernels and cokernels: suppose now that $C$ is pre-additive. let $f$ be as above, but assume also that $f(a)$ is the zero map. Then the colimit is the cokernel of $f(b)$, and the limit is the kernel of $f(b)$.

### 2.2.5 Stalks

Let $(X, \mathcal{O})$ be a RTS, and $x \in X$. Let $S$ denote the category of open neighbourhoods of $x$, with morphisms the inclusions. Let $f : S \to \mathrm{Rng}$ be the restriction of $\mathcal{O}$ to $S$. The stalk $\mathcal{O}_x$ is the colimit of $f$.

# 3 Adjoint functors

Let $C$ and $D$ be two categories, let $f : C \to D$ and $g : D \to C$ be functors. We say $f$ and $g$ are adjoint ($f$ is right adjoint to $g$, and $g$ is left adjoint to $f$) if there exists a natural isomorphism between the two functors

$$\hom_C(g-, -) : D^{\mathrm{op}} \times C \to \mathrm{Set}$$

and

$$\hom_D(-, f-) : D^{\mathrm{op}} \times C \to \mathrm{Set}\,.$$

In particular, such a natural isomorphism specifies a bijection

$$\hom_C(g(d), c) \cong \hom_D(d, f(c))$$

for all $c \in C$ and $d \in D$. [Possible aid to memory: the functor which is right adjoint appears on the right inside the hom, and conversely].

## 3.1 Examples

### 3.1.1 Free abelian groups

Let $f : \mathrm{Ab} \to \mathrm{Set}$ be the forgetful functor. This has a **left** adjoint $g : \mathrm{Set} \to \mathrm{Ab}$; it is the functor taking a set to the free abelian group generated by that set.

### 3.1.2 Schemes

Let $f : LRS \to \mathrm{Rng}^{\mathrm{op}}$ be the functor sending $(X, \mathcal{O})$ to $\mathcal{O}(X)$. Then $f$ has a **right** adjoint $\mathrm{Spec} : \mathrm{Rng}^{\mathrm{op}} \to LRS$ - see Lecture 1. [Homs from a scheme to an affine scheme are the same as homs from the global sections to the ring. If you get confused, consider the case of $\mathbb{P}^1$ over spec of a field. ]

# 4 Commuting of adjoints and (co)limits

Let $f : C \to D$ be a functor, and $g : D \to C$ be a right adjoint to $f$. Let $S$ be a category, and $\varphi : S \to C$, $\psi : S \to D$ functors. Then

$$f(\operatorname{colim} \varphi) = \operatorname{colim}(f \circ \varphi)$$

and

$$g(\lim \psi) = \operatorname{colim}(g \circ \psi).$$

*Proof.* We do only the first one.

We will show that every cone under $f \circ \varphi$ admits a unique map from $f(\operatorname{colim} \varphi)$, in other words $f(\operatorname{colim} \varphi)$ is initial in the category of cones over $f \circ \varphi$, which is what we wanted.

Consider a cone $N$ under $f \circ \varphi$, so for all $s \in S$ we get maps $f(\varphi(s)) \to N$ (such that a bunch of diagrams commute). Apply adjunction, this is equivalent to a bunch of maps $\varphi(s) \to g(N)$, with more diagrams commuting. This makes $g(N)$ a cone under $\varphi$, so since $\operatorname{colim} \varphi$ is initial we find that the maps above factor as

$$\varphi(s) \to \operatorname{colim} \varphi \to g(N).$$

Applying adjunction again, we get canonical maps $f(\operatorname{colin} \varphi) \to N$. $\square$

**Exercise 4.1** (Good idea if you've not seen these things before)**.** *Do the right adjoints/ limits version.*

Maps from a colimit are just the same as maps from each object in the image of the diagram, plus commuting maps. Maps to a limit have a similarly easy description. Hence:

Table 1: Easy to characterise

| maps out of | maps into |
|---|---|
| colimits<br>tensor products<br>cokernels<br>stalks | limits<br>products<br>kernels |
| left adjoint<br>sheafification | right adjoint |

Moreover, two things from the same side of the diagram tend to commute with one another.

Nb. the things at the bottom are not (co)limits, so it seems a bit silly to put them there. However:
- it is still easy to characterise maps from/to them;
-more generally, it is possible to define colimits as left adjoints to a certain functor, and similarly limits as right adjoints. Hence the columns do fit together.

# 5 Additive and abelian categories, exact functors

We defined a pre-additive category above. A category is called additive if it is pre-additive and finite products exist (finite product = limit of a diagram from a shape cat with finitely many objects and all morphisms identities).

We defined kernels and cokernels above using (co)-limits. A *coimage* of $f : X \to Y$ is a cokernel of the canonical map $\ker f \to X$. An *image* of $f$ is a kernel of the canonical map $Y \to \operatorname{coker} f$.

**Lemma 5.1.** *Let $f : x \to y$ be a morphism in a preadditive category such that the kernel, cokernel, image and coimage all exist. Then $f$ can be factored uniquely as $x \to Coim(f) \to Im(f) \to y$.*

**Exercise 5.2** (Basic, reccomended)**.** *Prove the lemma.*

**Definition 5.3.** *A category $\mathcal{A}$ is abelian if it is additive, if all kernels and cokernels exist, and if the natural map $Coim(f) \to Im(f)$ is an isomorphism for all morphisms $f$ of $\mathcal{A}$.*

**Exercise 5.4** (Basic, recommended)**.** *Show that the category of modules over a (commutative) ring is abelian. In particular,* Ab *is abelian.*

Let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be a sequence of maps in an abelian cat. Assume $g \circ f = 0$.

**Exercise:** show that there is a canonical map image $f \to \ker g$.

**Definition 5.5.** *A sequence of maps as above is exact at $B$ if $g \circ f = 0$ and the canonical map* image $f \to \ker g$ *is an isomorphism.*

**Definition 5.6.** *A functor between abelian cats is called exact if it is additive and sends exact sequences to exact sequences. (There is a more general notion of exact functor for cats that are not abelian).*

=== End of category definitions for now ===

=== Beginning of schemes ===

# 6 Kernels and cokernels for $\mathrm{Ab}\, X$, take 1

Let $X$ a top space.

**Exercise 6.1** (First part basic and essential)**.** *Kernels and cokernels exist in* $\mathrm{PAb}\, X$*. Further,* $\mathrm{PAb}\, X$ *is an abelian category.*

Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves of abelian groups on $X$ .

**Lemma 6.2.** *Suppose $\mathcal{F}$ and $\mathcal{G}$ are sheaves. Then the presheaf* $\mathrm{Ker}\, f$ *is a sheaf.*

**Exercise 6.3** (Basic, essential)**.** *Prove the lemma. Note that this does not follow naturally from the formalities in the first part of the lecture.*

However, even if $\mathcal{F}$ and $\mathcal{G}$ are sheaves, the presheaf cokernel is *not* in general a sheaf:

**Example 6.4.** *Let $X = \mathbb{C}$ with the usual topology.*

*Define a sheaf of rings $\mathbb{Z}_X$ on $X$ by $\mathbb{Z}_X(U) = \oplus_{\pi_0(U)}\mathbb{Z}$. [Maybe you will see this sheaf again later on...]. Consider the RTS $(X, \mathbb{Z}_X)$.*

*Let $\mathcal{F}$ denote the sheaf of holomorphic functions $X \to \mathbb{C}$ (a sheaf of $\mathbb{Z}_X$-modules). Define a presheaf $\mathcal{G}$ by*

$$\mathcal{G}(U) = \{\exp(2\pi i) \circ f | f \in \mathcal{F}(U)\}.$$

*We have a natural inclusion $\mathbb{Z}_X \to \mathcal{F}$, and composite with* $\exp(2\pi i)$ *yields a map $\mathcal{F} \to \mathcal{G}$.*

***Exercise:*** *Show that the sequence of presheaves*

$$0 \to \mathbb{Z}_X \to \mathcal{F} \to \mathcal{G} \to 0$$

*is exact.*

*Show that $\mathcal{G}$ is not a sheaf.*

# 7 Sheafification of a presheaf

To construct a cokernel of a map on sheaves (and for many other purposes) it is useful to be able to canonically associate a sheaf to a given presheaf. We do this first for sheaves of sets, and then more generally.

**Lemma 7.1.** *Fix a top space $X$. Write $i : \mathrm{Sh}\, X \to \mathrm{PSh}\, X$ for the inclusion ('view a sheaf as a presheaf'). Then $i$ has a left adjoint, which we call 'sheafification', and write $\mathcal{F} \mapsto \mathcal{F}^{\#}$. In other words, for all $\mathcal{G} \in \mathrm{Sh}\, X$ we have*

$$\mathrm{Mor}_{\mathrm{PSh}\, X}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{\mathrm{Sh}\, X}(\mathcal{F}^{\#}, \mathcal{G}).$$

Once we have proven the Lemma, uniqueness of adjoint functors [Exercise? Useful if you are not familiar with these things. ] allows us to use this as the definition of the sheafification of a presheaf of sets.

The basic construction is the following. Let $\mathcal{F}$ be a presheaf of sets $\mathcal{F}$ on a topological space $X$. For every open $U \subset X$ we define

$$\mathcal{F}^{\#}(U) = \{(s_u) \in \prod_{u \in U} \mathcal{F}_u \text{ such that } (*)\}$$

where $(*)$ is the property:

(∗) For every $u \in U$, there exists an open neighbourhood $u \in V \subset U$, and a section $\sigma \in \mathcal{F}(V)$ such that for all $v \in V$ we have $s_v = (V, \sigma)$ in $\mathcal{F}_v$.

**Proof that $\mathcal{F}^{\#}$ is a sheaf:**

□

**Exercise 7.2** (A little long, but try to do at least the first step)**.** *Given a sheaf $\mathcal{G}$ and map of presheaves $f : \mathcal{F} \to i(\mathcal{G})$, construct a natural map of sheaves $\mathcal{F}^{\#} \to \mathcal{G}$. Complete the proof of the lemma.*

**Lemma 7.3.** *Let $X$ be a topological space. Let $\mathcal{F}$ be an abelian presheaf on $X$. Write $i : \mathrm{Ab}\, X \to \mathrm{PAb}\, X$ for the inclusion ('view an abelian sheaf as an abelian presheaf'). Then $i$ has a left adjoint, which we call 'sheafification', and write $\mathcal{F} \mapsto \mathcal{F}^{\#}$. In other words, for all $\mathcal{G} \in \mathrm{Ab}\, X$ we have*

$$\mathrm{Mor}_{PAb(X)}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{Ab(X)}(\mathcal{F}^{\#}, \mathcal{G}).$$

*Let $f_A : \mathrm{Ab}\, X \to \mathrm{Sh}\, X$ and $f_P : \mathrm{PAb}\, X \to \mathrm{PSh}\, X$ denote the forgetful functors. Then TFDC:*

$$\begin{array}{ccc} \mathrm{PAb}\, X & \longrightarrow & \mathrm{Ab}\, X \\ {\scriptstyle f_P}\downarrow & & \downarrow{\scriptstyle f_A} \\ \mathrm{PSh}\, X & \longrightarrow & \mathrm{Sh}\, X \end{array}$$

*Proof.* One way is to repeat the construction (given above for sets) in the category of abelian groups. Then you also have to repeat the proof that it works, though this will be the same.

The Stacks Project takes a different approach (see [3, Tag 0085]) - they show that there is a unique abelian group structure on the sheafification of the underlying presheaf of sets. This is probably a nicer thing to do... □

Note that by the adjunction we have a canonical map of presheaves

$$\mathcal{F} \to i(\mathcal{F}^{\#})$$

since

$$\mathrm{Mor}_{\mathrm{Sh}}(\mathcal{F}^{\#}, \mathcal{F}^{\#}) = \mathrm{Mor}_{\mathrm{PSh}}(\mathcal{F}, i(\mathcal{F}^{\#})).$$

**Lemma 7.4.** *Let $\mathcal{F}$ be a sheaf of sets or abelian groups. Then $\mathcal{F}^{\#} = \mathcal{F}$.*

*Proof.* Adjointness plus Yoneda. □

**Lemma 7.5.** *Let $\mathcal{F}$ be a presheaf of sets or abelian groups. Then for all $x \in X$, the canonical map on stalks is an isomorphism. In other words:*

$$\mathcal{F}_x = (\mathcal{F}^{\#})_x.$$

*Proof.* Sheafification is a left adjoint. Taking stalk is a colimit. Hence this should morally be true, but care is needed to set things up right... □

# 8 Kernels and cokernels in Ab, take 2

Let $X$ a top space. Given a map $f : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Ab}\, X$, we considered the presheaf kernel, and checked that it was in fact a sheaf. This fails for the cokernel.

Write $i : AbX \to PAbX$ for the forgetful functor.

If $\operatorname{coker} f$ exists in $\mathrm{Ab}\, X$ the (since colimits and left adjoints commute) we see that $(\operatorname{coker} i(f))^{\#} = \operatorname{coker} f$.

**Exercise 8.1.** *Prove that cokernels exist in $\mathrm{Ab}\, X$.*

## 8.1 Sheafification in CommRng

You can also sheafifiy a presheaf of commutative ring on a topological space. The definition (left adjoint to 'view a sheaf as a presheaf') is the same, and they can be constructed in the same manner.

**Example 8.2** (Constant presheaf)**.** *Let $X$ be a top space, and let $c$ be an element of a category $C$ which is either set, abelian group or commutative ring. We define the 'constant presheaf' of $c$ to be the presheaf which sends an open of $X$ to the element $c$. We define the 'constant sheaf' of $c$ to be the sheafification of the constant sheaf $c$.*

**Exercise 8.3.** *Let $X$ be the disjoint union of two copies of $\mathbb{R}$ with the usual topology. Describe the constant* sheaf *$\mathbb{Z}$ of rings (or abelian groups). What is $\mathbb{Z}(X)$?*

# 9 Sheaves of modules

Let $(X, \mathcal{O})$ be a ringed topological space (RTS). We wish to define a pre sheaf of $\mathcal{O}$-modules.

Not straightforward; over $U$ we want a sheaf of modules over $\mathcal{O}(U)$, but this depends on $U$. As such, we have to define a sheaf of $\mathcal{O}$-modules as a sheaf of abelian groups *plus some extra data*.

**Definition 9.1.** *Let $X$ be a topological space, and let $\mathcal{O}$ be a presheaf of rings on $X$.*

1. *A presheaf of $\mathcal{O}$-modules is given by an abelian presheaf $\mathcal{F}$ together with a map of presheaves of sets*

$$\mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F}$$

   *such that for every open $U \subset X$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an $\mathcal{O}(U)$-module structure on the abelian group $\mathcal{F}(U)$.*

2. *A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules is a morphism of abelian presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ {\scriptstyle id \times \varphi} \downarrow & & \downarrow {\scriptstyle \varphi} \\ \mathcal{O} \times \mathcal{G} & \longrightarrow & \mathcal{G} \end{array}$$

   *commutes.*

3. *The set of $\mathcal{O}$-module morphisms as above is denoted* $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.

4. *The category of presheaves of $\mathcal{O}$-modules is denoted PMod($\mathcal{O}$).*

**Definition 9.2.** *Let $X$ be a topological space. Let $\mathcal{O}$ be a sheaf of rings on $X$.*

1. *A sheaf of $\mathcal{O}$-modules is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, see Definition 9.1, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.*

2. *A morphism of sheaves of $\mathcal{O}$-modules is a morphism of presheaves of $\mathcal{O}$-modules.*

3. *Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote* $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ *the set of morphism of sheaves of $\mathcal{O}$-modules.*

4. *The category of sheaves of $\mathcal{O}$-modules is denoted Mod($\mathcal{O}$).*

**Example 9.3.** *: Let $k$ be a field, so* $\operatorname{Spec} k = \{*\}$ *is a one-point topological space. Let $V$ be a vector space over $k$. Two presheaves of modules on* $\operatorname{Spec} k$*:*

$$\mathcal{F} : \emptyset \mapsto V, \{*\} \mapsto V$$

*where the map $\mathcal{F}(\emptyset) \to \mathcal{F}(\{*\})$ is zero.*

$$\mathcal{G} : \emptyset \mapsto 0, \{*\} \mapsto V.$$

*where the map $\mathcal{G}(\emptyset) \to \mathcal{G}(\{*\})$ is zero.*

*The map $\mathcal{O} \times \mathcal{G} \to \mathcal{G}$ is the 'usual' scalar multiplication (take care with the zero ring), and the same for $\mathcal{F}$.*

*Which is a sheaf of modules?*

**Exercise 9.4** (Basic)**.** *Let $X$ a top space. Let $\mathbb{Z}$ be the constant sheaf of rings, so we have a RTS $(X, \mathbb{Z})$. Construct an equivalence of categories between* $\operatorname{Ab} X$ *and* $\operatorname{Mod}(\mathbb{Z})$*.*

## 9.1 Sheafification, kernels and cokernels

Let $(X, \mathcal{O})$ be an RTS. Given a morphism in $\operatorname{Mod}(\mathcal{O})$, we wish to form a kernel and cokernel. By the above exercise, we will have the same problem with cokernels as we did when considering abelian groups - the obvious thing (the presheaf cokernel) will fail to be a sheaf. Thus we will need to sheafify presheaves of modules.

Note that a sheaf of modules is not just a sheaf taking values in some cat - there is extra data there too. Hence sheafification will not (quite) follow the model we used before. However, the definition will be similar.

**Lemma 9.5.** *Let $i : \operatorname{Mod}(\mathcal{O}) \to \operatorname{PMod}(\mathcal{O})$ be the forgetful functor. Then $i$ has a left adjoint which we call 'sheafification', and write $\mathcal{F} \mapsto \mathcal{F}^{\#}$. In other words, for all $\mathcal{G} \in \operatorname{Mod} X$ we have*

$$\operatorname{Mor}_{\operatorname{PMod}(\mathcal{O})}(\mathcal{F}, i(\mathcal{G})) = \operatorname{Mor}_{\operatorname{Mod}(\mathcal{O})}(\mathcal{F}^{\#}, \mathcal{G}).$$

*Proof.* Omitted. See [3, Tag 0088]. □

As for abelian groups, the presheaf kernel of a morphism of sheaves of modules is again a sheaf of modules. Cokernels also exist, with analogous proof to the case of abelian groups.

## 10 Sheaves of modules form an abelian category

**Proposition 10.1.** *Let $(X, \mathcal{O})$ be a RTS. The category* $\mathrm{Mod}(\mathcal{O})$ *forms an abelian category. Moreover a complex*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

*is exact at $\mathcal{G}$ if and only if for all $x \in X$ the complex*

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

*is exact at $\mathcal{G}_x$.*

**Exercise 10.2.** *Read [3, Tag 01AF], where this is proven. We have done a lot of the work - most of the relevant objects exist.*

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## 11 Quasi-coherent sheaves

**Definition 11.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map*

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_U$$

*The category of quasi-coherent $\mathcal{O}_X$-modules is denoted QCoh($\mathcal{O}_X$).*

Examples:

The module $\mathcal{O}_X$ is a quasi-coherent module over itself.

Let $(\mathbb{R}, C^0)$ denote the LRTS whose top. space is $\mathbb{R}$ with the usual topology, and with structure sheaf the continuous functions. Let $p = 0 \in \mathbb{R}$. Define a sheaf $\mathcal{F} \in \mathrm{Mod}\,\mathcal{O}$ by

$$\mathcal{F}(U) = C^0(U) \text{ if } p \notin U$$
$$\mathcal{F}(U) = 0 \text{ if } p \in U$$

**Exercise 11.2.** *The presheaf $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, and is not quasi-coherent.*

**Note** An essential point is that the pullback of a quasi-coherent sheaf is again quasi-coherent. Bas will define the pullback in the next lecture.

# 12 Quasi-coherent sheaves on affine schemes

Quasi-coherent sheaves are great - they correspond to modules over rings:

**Definition 12.1.** *Let $R$ be a ring, and $M \in \operatorname{Mod} R$. Given $f \in R$, define*

$$\tilde{M}(D(f)) = M \otimes_R R[1/f].$$

*Since the open sets $D(f)$ form a basis for the Zariski topology on $\operatorname{Spec} R$, we know (see Lecture 1) that the given data uniquely defines a sheaf of abelian groups on $X$. A similar argument shows that this is in fact a sheaf of $\mathcal{O}$-modules.*

*Given a morphism $f : M \to N$ in $\operatorname{Mod} R$, there is a natural map $\tilde{f} : \tilde{M} \to \tilde{N}$.*

*GIven a prime ideal $p \in \operatorname{Spec} R$, we have $M_p = \tilde{M}_p$.*

**Lemma 12.2.** *Let $R$ a ring. Let $(X, \mathcal{O})$ be the associated affine scheme. The functors $M \mapsto \tilde{M}$ and $\mathcal{F} \mapsto \mathcal{F}(X)$ define quasi-inverse equivalences of categories between $\operatorname{QCoh} \mathcal{O}$ and $\operatorname{Mod} R$.*

*Outline of proof .* Exercise: read [3, Tag 01IA] to see the details.

First, we show $\tilde{M}$ is quasi-coherent.

- Write a presentation of $M$:

$$\bigoplus_{j \in J} R \to \bigoplus_{i \in I} R \to M \to 0.$$

- By definition, we have $\tilde{R} = \mathcal{O}$.
- We have induced maps

$$\bigoplus_{j \in J} \mathcal{O} \to \bigoplus_{i \in I} \mathcal{O} \to \tilde{M} \to 0.$$

- it remains to show that the above sequence of sheaves is exact. Check this on stalks.

Certainly $\mathcal{F}(X) \in \operatorname{Mod} R$. It remains to check that $\tilde{M}(X) = M$ and $\tilde{\mathcal{F}(X)} \cong \mathcal{F}$. The first is easy. For the second:

- Let $f_1, \ldots f_n \in R$ such that $(f_1, \ldots, f_n) = R$ and such that over each $D(f_r)$ we have a presentation

$$\bigoplus_{j \in J_r} \mathcal{O} \to \bigoplus_{i \in I_r} \mathcal{O} \to \mathcal{F}(D(f_r)) \to 0.$$

- For each $f_r$, use the above presentation to construct a module $M_r$ over $R[1/f_r]$
- On overlaps $D(f_r f_s)$ we get isomorphisms $\varphi_{r,s} : M_r \otimes_{R[1/f_r]} R[1/(f_r f_s)] \to M_s \otimes_{R[1/f_s]} R[1/(f_r f_s)]$.
- these morphisms satisfy the cocycle condition $\varphi_{r,t} = \varphi_{r,s} \circ \varphi_{s,t}$.
- Exercise: glue these modules to a module $M$ over $R$ (see [3, Tag 00EQ]).
- check that $\tilde{M} = \mathcal{F}$ by constructing a map and checking locally that it is an isomorphism.

□

# 13 Example: Jordan canonical form

# 14 Push forward and inverse image

If time allows

# References


[1] Robin Hartshorne *Algebraic Geometry*

[2] Qing Liu *Arithmetic Geometry and Algebraic Curves*

[3] de Jong et al. *Stacks Project*