

# Grothendieck groups of coherent sheaves

homsets are ab. gps, composition is bilinear,  
(finite  $\oplus$  ~~is~~ <sup>or limits</sup> ~~coherent~~.)

## 1) K-theory

- Let  $C$  be a full additive subcat. of an abelian cat.  $A$ . <sup>of  $Ob(C)$</sup>
- eg let  $R$  a ring. Then  $Mod R$  is abelian, and the subcategory  $FG Mod R$  of <sup>finite side</sup> finitely generated modules is full & additive (& abelian if  $R$  Noetherian).
- Let  $\underline{Ob(C)}$  denote the class of objects of  $C$  up to isomorphism. We assume from now on that  $\underline{Ob(C)}$  is a set (holds for  $FG Mod R$ ).
- Let  $F(C)$  denote the free abelian gp on  $\underline{Ob(C)}$ .

Given a sequence

$$(E): 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \quad *$$

in  $C$  which is exact in  $A$ , we write  $Q(E)$  for the element  $[M] - [M''] - [M']$  in  $F(C)$ .

- Let  $H(C)$  denote the subgp of  $F(C)$  generated by the  $Q(E)$  as  $E$  runs through sequences  $*$  exact in  $A$ .

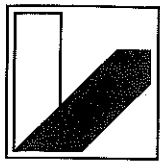
• The Grothendieck group of  $C$  is the quotient  $K(C) = F(C) / H(C)$ .

• Write  $d: C \rightarrow K(C)$

• Note  $K(C)$  depends also on  $A$ .

• Given  $M, N \in C$ , have exact seq.  $0 \rightarrow M \rightarrow M \oplus N \rightarrow N \rightarrow 0$ . ~~It~~

Hence  $d[M \oplus N] = d[M] + d[N]$  in  $K(C)$ .



## Examples

- $R$  a ring,  $N$  an infinite cardinal,  $N\text{-Mod-}R = \text{cat. of } R\text{-modules of cardinality } \leq N$ . For any  $M \in R\text{-mod}$ , we have

$$M \oplus \left( \bigoplus_{n \in N} M \right) \cong \bigoplus_{n \in N} M, \text{ hence}$$

$$d(M) + d\left(\bigoplus_{n \in N} M\right) = d\left(\bigoplus_{n \in N} M\right),$$

so  $d(M) = 0$ . Hence  $K(C) = 0$ .

- $R$  a PID  $A=C = \text{ab. cat. of fin-gen. } R\text{-modules}$ .

Structure theory for f.g. modules over PID, every  $M \in C$  is a direct sum of a free & a torsion module, & a torsion is a direct sum of cyclic modules.

Define rank:  $\text{FGMod}(R) \rightarrow \mathbb{N}$   
 $\pi_1 \rightarrow \text{rank of free part}$

This induces  $\text{rank}: FCC \rightarrow \mathbb{Z}$ , a gp. hom.

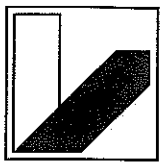
Since rank is additive in exact sequences ( $- \otimes_{\mathbb{Z}} \text{Frac } R$ , then lengths),  
~~flat~~ flat by localization

it induces a gp. hom  $K(C) \rightarrow \mathbb{Z}$ .

Let  $M$  be a cyclic module. We have an exact sequence

$$0 \rightarrow R \rightarrow R \rightarrow M \rightarrow 0 \text{ (because } R \text{ is a PID),}$$

Hence  $d(M) = 0$ , so  $\text{rk}: K(C) \rightarrow \mathbb{Z}$  is an isomorphism Ex: check details



Grothendieck gps of Schemes

Let  $X$  be a scheme.  $\text{Coh}(X) \subset \text{Mod}(X)$  is a full add. subcat of an abelian cat. Define

$$K_0(X) = K(\text{Coh}(X)).$$

Eg. If  $X = \text{Spec } R$  is affine,  $\text{Coh}(X)$  is  $\cong$  to  $\text{FGMod}(R)$ , & so  $K_0(X) = K(\text{FGMod}(R))$ .

Hence, if  $R$  is a PID,  $K_0(X) \cong \mathbb{Z}$ .

Let  $X$  a scheme, &  $\mathcal{F} \in \text{Mod}(X)$ . We say  $\mathcal{F}$  is locally free of finite rank if  $\mathcal{F}$  is coherent and there exists a cover  $\{U_i\}_{i \in I}$  of  $X$  by affine opens s.t.  $\forall i \in I, \mathcal{F}|_{U_i} \cong \bigoplus_{j=1}^{n_i} \mathcal{O}_{U_i}$  (some  $n_i \in \mathbb{N}$ ). improve

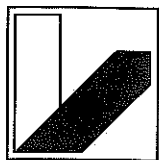
We often say 'vector bundle' instead of 'locally free sheaf of finite rank'. We write  $\text{Vect}(X)$  for the cat. of vector bundles on  $X$ . It is a full additive subcat. of  $\text{Mod}(X)$ .

Def:  $K^0(X) = K(\text{Vect}(X))$ .

Eg: Let  $R$  be a local ring. Then  $\text{Vect}(\text{Spec } R)$  is  $\cong$  to the category of free  $R$ -modules of finite rank. We have a ~~gp hom~~ map

$$\text{rank} : \text{FGFreeMod}(R) \longrightarrow \mathbb{N},$$

& by additivity of ranks in exact sequences, we get a surjective gp. hom  $K^0(X) \rightarrow \mathbb{Z}$ . In fact, it is not hard to check that this is an isomorphism.



### Ring Structure on $K^0(X)$

Let  $X$  be a scheme.

Prop: Tensor product (over  $\mathcal{O}_X$ ) induces a comm. ring structure on  $F(\text{Vect}(X))$ .

Pf:  $\otimes_{\mathcal{O}_X}$  sends vector bundles to vector bundles, & is associative, commutative, ~~distributive over~~  
[ $\mathcal{O}_X$ ] is the multiplicative unit.  $\square$

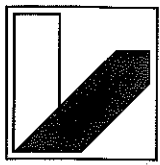
Prop Tensor product (over  $\mathcal{O}_X$ ) induces a comm. ring structure on  $K^0(X)$ .

Pf: We need to show  ~~$H(\text{Vect}(X))$~~   $H(\text{Vect}(X))$  is an ideal in  $F(\text{Vect}(X))$ .

Let  $M \in \text{Vect}(X)$  &  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$  a SES in  $\text{Vect}(X)$ .

Wts  $0 \rightarrow N' \otimes_{\mathcal{O}_X} M \rightarrow N \otimes_{\mathcal{O}_X} M \rightarrow N'' \otimes_{\mathcal{O}_X} M \rightarrow 0$  is exact.

~~flat~~ These modules are  $q$ -coherent, & exactness is local, so we may assume  $X$  is affine and  $M, N, M$  is free. But free modules are flat, so we are done  $\square$



Operations

Let  $f: X \rightarrow Y$  be a morphism of schemes.

Let  $V \in \text{Vect}(Y)$ . Then  $f^*V \in \text{Vect}(X)$ .

PF: wlog  $X$  and  $Y$  are affine;  $X = \text{Spec } A$ ,  $Y = \text{Spec } B$ .

wlog  $V$  is free; it corresponds to a module  $M = \bigoplus_{i=1}^n B$ .

$$\text{Then } f^*V = \left( A \otimes_B \bigoplus_{i=1}^n B \right) = \bigoplus_{i=1}^n A.$$

□

~~Lemma~~

Lemma: This extends to a mg hom  $f^!: K^0(Y) \rightarrow K^0(X)$ .

PF: Omitted; straightforward w. a little knowledge of flat modules over rings. (□)

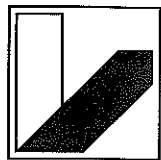
Def  
~~Lemma~~

Let  $S$  be locally Noetherian, &  $f: X \rightarrow Y$  a proper morphism of  $H$ -quasi-proj schemes; let  $\mathcal{F} \in \text{Coh}(X)$ . Then  $\forall q \geq 0$ , have  $R^q f_* \mathcal{F} \in \text{Coh}(Y)$ . Define

$$f_! : K_0(X) \rightarrow K_0(Y) \quad \text{by}$$

$$f_!([\mathcal{F}]) = \sum_{i=0}^{\infty} (-1)^i R^i f_* \mathcal{F}. \quad (\text{a finite sum})$$

This is an additive gp-hom.



## Regular Schemes

Def: A local Noetherian ring  $(R, \mathfrak{m})$  is regular if  $\mathfrak{m}$  can be generated by  $(\dim R)$  elements.

• A Noeth. ring  $R$  is regular if  $\forall p \in \text{Spec } R$ , the local ring  $R_p$  is regular.

• A scheme  $X$  is regular if it admits an ~~affine~~ open cover by the spectra of Noetherian regular rings.

Eg: ~~A~~ A field  $k$  is regular.

• If  $R$  is a regular ring, then  $A^n_R$  is regular, so  $\mathbb{P}^n_R$  is regular.

Eg Not:  
 $R = \frac{\mathbb{C}[x, y]}{(y^2 - x^3)}$

$\dim R = 1$ . Let  $p = (x, y)$ . Then you can't generate  $p$  using fewer than 2 elts, so not regular.

## Comparing $k^\circ(X)$ and $k_0(X)$

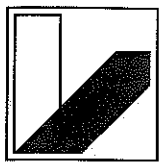
•  $\text{Vect}(X) \subset \text{Coh}(X)$ .

• This induces a map  $k^\circ(X) \rightarrow k_0(X)$ .

Thm: If  $X$  is regular, the above map is an isomorphism.

• Let  $f: X \rightarrow Y$  a proper morphism of <sup>regular</sup>  $\mathbb{A}^1_{\mathbb{H}}$ -quasi-projective schemes over some loc. Noeth. scheme  $S$ . Then

$\forall x \in k_0(X), y \in k_0(Y)$ , we have  $f_!(x \cdot f^!(y)) = f_!(x) \cdot y$ .



Exercises:

1) Which of the following rings is regular?

a)  $\frac{\mathbb{C}[x, y, \frac{1}{x}]}{(y^2 - x^3)}$

b)  $\frac{\mathbb{F}_7(t)[x, y]}{(xy)}$

c)  $\frac{\mathbb{Z}_{(5)}[x, y]}{(y^2 - x^3 - 5^n)}$ ,  $n \in \mathbb{Z}_{\geq 0}$  (answer may depend on  $n$ !)  
( $\mathbb{Z}_{(5)}$  = local ring of  $\mathbb{Z}$  at  $(5)$ ).

~~at~~

2) Let  $X = \text{Spec } k[t]$  ( $k$  a field). Define sheaves ~~in~~  $\text{Mod}(X)$  by:

•  $\mathcal{F}_1 = \tilde{k}$  where we view  $k$  as a  $k[t]$  algebra by ~~the~~  $t \mapsto 0$ .

•  $\mathcal{F}_2 = \widetilde{k(t)}$  where  $k(t)$  is a  $k[t]$  module in the obvious way.

•  $\mathcal{F}_3(U) = \begin{cases} \mathcal{O}_X(U) & \text{if } (t) \notin U \\ 0 & \text{if } (t) \in U \end{cases}$

Fill in the following table with True/False, giving a proof in each case!

	quasi-coherent	coherent	vector bundle
$\mathcal{F}_1$			
$\mathcal{F}_2$			
$\mathcal{F}_3$			

$\mathcal{O}(n)$  on  $\mathbb{P}^m$ ,  
 $n \in \mathbb{Z}$ ,  $m \geq 0$ .