For more details & examples, see the thesis of Ariyan Javanpeykar on the Algant website. (1)

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Grothendieck groups of coherent sheaves

1) K-theory

(hom sets are ab. gps, composition is bilinear, finite $\oplus$ exist. (or limits?))

- let C be a full additive subcat. of an abelian cat. A ~~s.t. Ob(C) is a set~~

[eg let R a ring. Then Mod-R is abelian, and the subcategory FGMod-R of finitely generated modules is full & additive (& abelian if R Noetherian)

- let $\underset{\simeq}{Ob(C)}$ denote the class of objects of C up to isomorphism. We assume from now on that $\underset{\simeq}{Ob(C)}$ is a set (holds for FGMod-R).

- let $F(C)$ denote the free abelian gp on $\underset{\simeq}{Ob(C)}$.

Given a sequence

$$(E):\ 0 \longrightarrow M' \to M \to M'' \to 0 \qquad *$$

in C which is exact in A, we write $Q(E)$ for the element $[M]-[M'']-[M']$ in $F(C)$.

- let $H(C)$ denote the subgp of $F(C)$ generated by the $Q(E)$ as E runs through sequences $*$ exact in A.

- The Grothendieck group of C is the quotient $K(C) = F(C)/H(C)$.
  - Write $cl: C \to K(C)$
  - Note $K(C)$ depends also on A.

- Given $M, N \in C$, have exact seq. $0 \to M \to M \oplus N \to N \to 0$.

  Hence $cl[M\oplus N] = cl[M] + cl[N]$ in $K(C)$.

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Examples

- R a ring, N an infinite cardinal, N-Mod-R = cat. of ~~...~~ R-modules of cardinality $\leq N$. For any ~~...~~ $M \in R$-mod, we have

$$M \oplus \left(\bigoplus_{n\in N} M\right) \simeq \bigoplus_{n \in N} M \quad \text{, hence}$$

$$d(M) + d\left(\bigoplus_{n\in N} M\right) = d\left(\bigoplus_{n\in N} M\right),$$

so $d(M) = 0$. Hence $K(C) = 0$.

- R a PID C = ab. cat. of fin. gen. ~~...~~ R modules.

Structure theory for f.g. moduls over PID, every $M \in C$ is a direct sum of a free & a torsion ~~...~~ module, & a torsion is a direct sum of cyclic modules.

Define rank: ~~FGRMod~~ $FGMod(R) \longrightarrow \mathbb{N}$

$M \longmapsto$ rank of free part

This induces rank: $F(C) \longrightarrow \mathbb{Z}$, a gp. hom.

Since rank is additive in exact sequences ($- \otimes_R \operatorname{Frac} R$, then lengths),
~~...~~ flat bcs localisation

it induces a gp. hom $K(C) \longrightarrow \mathbb{Z}$.

Let M be a cyclic module. We have an exact sequence

$$0 \to R \to R \to M \to 0 \quad \text{(because R is a PID)},$$

Hence $d(M) = 0$, so rk: $K(C) \longrightarrow \mathbb{Z}$ is an isomorphism [Ex: check details]

Grothendieck gps of Schemes

Def Let X be a scheme. $\mathrm{Coh}(X) \subset \mathrm{Mod}(X)$ is a full add. subcat of an abelian cat. Define

$$K_0(X) = K(\mathrm{Coh}(X)).$$

Eg

- If X is affine, $= \mathrm{Spec}\, R$, $\mathrm{Coh}(X)$ is $\equiv$ to $\mathrm{FGMod}(R)$, & so $K_0(X) = K(\mathrm{FGMod}(R))$.

  Hence, if R is a PID, $K_0(X) \simeq \mathbb{Z}$.

Def. Let X a scheme, & $\mathcal{F} \in \mathrm{Mod}(X)$. We say $\mathcal{F}$ is locally free of finite rank if $\mathcal{F}$ is (q)coherent and there exists a cover $\{U_i\}_{i\in I}$ of X by affine opens s.t. $\forall i \in I$, $\mathcal{F}|_{U_i} \simeq \bigoplus_{j=1}^{n_i} \mathcal{O}_{U_i}$ (some $n_i \in \mathbb{N}$).

Improve

- We often say 'vector bundle' instead of 'locally free sheaf of finite rank'. We write $\mathrm{Vect}(X)$ for the cat. of vector bundles on X. It is a full additive subcat. of $\mathrm{Mod}(X)$.

Def: $K^0(X) = K(\mathrm{Vect}(X))$.

Eg: Let R be a local ring. Then $\mathrm{Vect}(\mathrm{Spec}\, R)$ is $\equiv$ to the category of free R-modules of finite rank. We have a map

$$\mathrm{rank} : \mathrm{FGFreeMod}(R) \longrightarrow \mathbb{N},$$

& by additivity of ranks in exact sequences, we get a surjective gp. hom $K^0(X) \to \mathbb{Z}$. In fact, it is not hard to check that this is an isomorphism.

Ring Structure on $K^0(X)$

Let X be a scheme.

Prop: Tensor product (over $\mathcal{O}_X$) induces a comm. ring structure on $F(\mathrm{Vect}(X))$.

Pf: $\otimes_{\mathcal{O}_X}$ sends vector bundles to vector bundles, & is associative, commutative, ~~distributive over $\oplus$~~. $[\mathcal{O}_X]$ is the multiplicative unit. $\square$

Prop Tensor product (over $\mathcal{O}_X$) induces a comm. ring structure on $K^0(X)$.

Pf: We need to show $H(\mathrm{Vect}(X))$ is an ideal in $F(\mathrm{Vect}(X))$.

Let $M \in \mathrm{Vect}(X)$ & $0 \to N' \to N \to N'' \to 0$ a SES in $\mathrm{Vect}(X)$.

Wts $0 \to N' \otimes_{\mathcal{O}_X} M \to N \otimes_{\mathcal{O}_X} M \to N'' \otimes_{\mathcal{O}_X} M \to 0$ is exact.

These modules are q-coherent, & exactness is local, so we may assume X is affine and M is free. But free modules are flat, so we are done $\square$

Operations

Let $f: X \to Y$ be a morphism of schemes.

~~Lemma~~ Let $\mathcal{V} \in Vect(Y)$. Then $f^*\mathcal{V} \in Vect(X)$.

Pf: Wlog X and Y are affine; $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$.
Wlog $\mathcal{V}$ is free; it corresponds to a module $M = \bigoplus_{i=1}^{n} B$.
Then $f^*\mathcal{V} = \left(A \otimes_B \bigoplus_{i=1}^{n} B\right) = \left(\bigoplus_{i=1}^{n} A\right)$.

$\square$

~~Lemma~~

Lemma: This extends to a ring hom $f^!: K^0(Y) \to K^0(X)$.

Pf: Omitted; straightforward w. a little knowledge of flat modules over rings. $(\square)$

Def ~~Lemma~~: Let S be locally Noetherian, & $f: X \to Y$ a proper morphism of H-quasi-proj schemes$/_S$; let $\mathcal{F} \in Coh(X)$. Then $\forall q \geq 0$, have $R^q f_* \mathcal{F} \in Coh(Y)$. Define

$$f_!: K_0(X) \to K_0(Y) \quad \text{by}$$

$$f_!([\mathcal{F}]) = \sum_{i=0}^{\infty} (-1)^i R^i f_* \mathcal{F}. \quad (\text{a finite sum}).$$

This is an additive gp-hom.

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Regular Schemes

Def: • A local Noetherian ring $(R, \mathfrak{m})$ is regular if $\mathfrak{m}$ can be generated by $(\dim R)$ elements.

• A Noeth. ring R is regular if $\forall\, p \in \operatorname{Spec} R$, the local ring R_p is regular.

• A scheme X is regular if it admits an ~~affine~~ open cover by the spectra of Noetherian regular rings.

Eg: • A field k is regular.

• If R is a regular ring, then $\mathbb{A}^n_R$ is regular, so $\mathbb{P}^n_R$ is regular.

Eg Not: $R = \dfrac{\mathbb{C}[x,y]}{(y^2 - x^3)}$. $\operatorname{Dim} R = 1$. Let $p = (x,y)$. Then you can't generate p using fewer than 2 elts, so not regular.

Comparing $K^0(X)$ and $K_0(X)$

• $\operatorname{Vect}(X) \subset \operatorname{Coh}(X)$.

• This induces a map $K^0(X) \to K_0(X)$.

Thm: • If X is regular, the above map is an isomorphism.

• Let $f: X \to Y$ a proper morphism of regular quasi-projective schemes over some ~~loc.~~ Noeth. scheme S. Then $\forall\, x \in K_0(X)$, $y \in K_0(Y)$, we have $f_!(x \cdot f^!(y)) = f_!(x) \cdot y$.

Exercises:

1) Which of the following rings is regular?

a) $\dfrac{\mathbb{C}[x, y, \frac{1}{x}]}{(y^2 - x^3)}$.

b) $\dfrac{\mathbb{F}_7(t)[x, y]}{(x\,y)}$.

c) $\dfrac{\mathbb{Z}_{(5)}[x, y]}{(y^2 - x^3 - 5^n)}$, $n \in \mathbb{Z}_{\geqslant 0}$ (answer may depend on n!).
($\mathbb{Z}_{(5)}$ = local ring of $\mathbb{Z}$ at (5)).

2) Let $X = \operatorname{Spec} k[t]$ (k a field). Define sheaves in $\operatorname{Mod}(X)$ by:

- $\mathcal{F}_1 = \tilde{k}$ where we view k as a $k[t]$ algebra by $t \mapsto 0$.
- $\mathcal{F}_2 = \widetilde{k(t)}$ where $k(t)$ is a $k[t]$ module in the obvious way.
- $\mathcal{F}_3(U) = \begin{cases} \mathcal{O}_X(U) & \text{if } (t) \notin U \\ 0 & \text{if } (t) \in U. \end{cases}$

Fill in the following table with True/False, giving a proof in each case:

	quasi-coherent	coherent	vector bundle
$\mathcal{F}_1$			
$\mathcal{F}_2$			
$\mathcal{F}_3$			
$\mathcal{O}(n)$ on $\mathbb{P}^m$, $n \in \mathbb{Z}$, $m \geqslant 0$.			