LINEAR ALGEBRA AND IMAGE PROCESSING
MID-TERM EXAM - MARCH 2014

Time: 1 hour 30 minutes
Fill in your name and student number on all papers you hand in. In this examination you are only allowed to use a pen and examination paper.
In total there are 5 question, and each question is worth the same number of points. In all questions, justify your answer fully and show all your work.
Good luck!

Question 1: Consider the following system of 4 linear equations in 4 variables.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 + \ 2x_3 + x_4 &= 3 \\ x_2 + 2x_3 + 3x_4 &= 4 \\ x_1 + x_3 &= 1 \end{aligned}$$

a) Write the general solution of the system in parametric vector form

$$\underline{p} + t\underline{u} \quad (t \in \mathbb{R}).$$

b) In this case the parametric vector form of the general solution is not unique (this is not unusual). Find vectors $\underline{q} = (0, q_2, q_3, q_4) \in \mathbb{R}^4$ and $\underline{v} = (2, v_2, v_3, v_4) \in \mathbb{R}^4$ such that the general solution of the linear system above is given in parametric vector form by

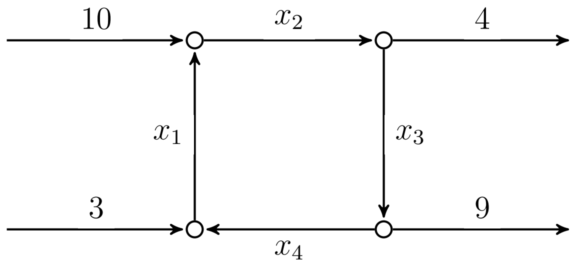
$$\underline{q} + s\underline{v} \quad (s \in \mathbb{R}).$$

Question 2: Consider the 3×3 matrices

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -2 & -1 \\ 0 & 1 & 0 \\ 5 & 5 & 4 \end{pmatrix}.$$

a) Is A invertible? If so, find the inverse of A.
b) Is B invertible? If so, find the inverse of B.
c) Is the product B^{1000} invertible?
[Hint: you do not need to compute the matrix B^{1000}.]
d) Is the product AB^{1000} invertible?
[Hint: you do not need to compute the matrix AB^{1000}.]

Question 3: Consider the following network:



a) Write down a linear system describing the flow in the network.
b) Put the augmented matrix of the linear system from (a) in row reduced echelon form.
c) Can you find a solution where all the flows are positive?

Question 4: For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true).

a) If A, B and C are $n \times n$ matrices with A not the zero matrix, and such that $AB = AC$, then $B = C$.
b) The function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ sending (x, y) to $(x + y, x + 1)$ is linear.
c) If every column of the augmented matrix of a linear system contains a pivot, then the system is inconsistent.
d) If every row of the augmented matrix of a linear system contains a pivot, then the system is inconsistent.
e) If the function $T\colon \mathbb{R}^5 \to \mathbb{R}^4$ is onto (surjective) and $S\colon \mathbb{R}^4 \to \mathbb{R}^2$ is onto, then the composite $ST\colon \mathbb{R}^5 \to \mathbb{R}^2$ is onto.

Question 5: Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 - x_2 \\ x_1 - x_3 \\ x_1 + 2x_2 + x_3 \end{pmatrix}.$$

a) Write down the standard matrix of T.
b) Is T onto (surjective)?
c) Is T 1-to-1 (injective)?
d) Write down the standard matrix for the composite transformation $\mathbb{R}^3 \to \mathbb{R}^3$ sending x to $T(T(x))$.