LINEAR ALGEBRA AND IMAGE PROCESSING SOLUTIONS
MID-TERM EXAM - MARCH 2014

Time: 1 hour 30 minutes
Fill in your name and student number on all papers you hand in. In this examination you are only allowed to use a pen and examination paper.
In total there are 5 question, and each question is worth the same number of points. In all questions, justify your answer fully and show all your work.
Good luck!

Question 1: Consider the following system of 4 linear equations in 4 variables.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 + \ \ 2x_3 + x_4 &= 3 \\ x_2 + 2x_3 + 3x_4 &= 4 \\ x_1 + x_3 &= 1 \end{aligned}$$

a) Write the general solution of the system in parametric vector form

$$\underline{p} + t\underline{u} \quad (t \in \mathbb{R}).$$

For example, $p = (-1, 0, 2, 0)$**,** $u = (1, -1, -1, 1)$**.**

b) In this case the parametric vector form of the general solution is not unique (this is not unusual). Find vectors $\underline{q} = (0, q_2, q_3, q_4) \in \mathbb{R}^4$ and $\underline{v} = (2, v_2, v_3, v_4) \in \mathbb{R}^4$ such that the general solution of the linear system above is given in parametric vector form by

$$\underline{q} + s\underline{v} \quad (s \in \mathbb{R}).$$

Unique solution $q = (0, -1, 1, 1)$**,** $v = (2, -2, -2, 2)$**.**

Question 2: Consider the 3×3 matrices

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -2 & -1 \\ 0 & 1 & 0 \\ 5 & 5 & 4 \end{pmatrix}.$$

a) Is A invertible? If so, find the inverse of A.
No: not row equivalent to identity.

b) Is B invertible? If so, find the inverse of B.
Yes. Inverse is

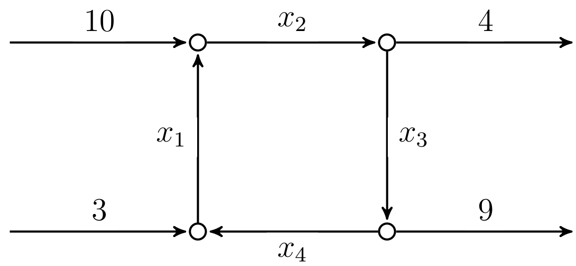
$$B = \begin{pmatrix} 4 & 3 & 1 \\ 0 & 1 & 0 \\ -5 & -5 & -1 \end{pmatrix}.$$

c) Is the product AB^{1000} invertible?
[Hint: you do not need to compute the matrix AB^{1000}.]
No. Suppose AB^{1000} invertible, and let C denote the inverse. Then

$$CAB^{1000} = I = AB^{1000}C,$$

so $B^{-1000} = CA$ and hence $B^{1000}CA = I$, so $B^{1000}C$ is an inverse for A.

Question 3: Consider the following network:



a) Write down a linear system describing the flow in the network.
b) Put the augmented matrix of the linear system from (a) in row reduced echelon form.
Solution:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & -1 & 13 \\ 0 & 0 & 1 & -1 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

c) Can you find a solution where all the flows are positive?
Yes.

Question 4: For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true).

a) If A, B and C are $n \times n$ matrices such that A is not the zero matrix and such that $AB = AC$, then $B = C$.
False, eg. $n = 2$, $A = (1, 0; 0, 0)$, $B = (0, 0; 0, 0)$, $C = (0, 0; 0, 1)$.

b) The function $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ sending (x, y) to $(x + y, x + 1)$ is linear.
False, since $T(0, 0) = (0, 1)$**, and any linear function sends** $\underline{0}$ **to** $\underline{0}$**.**

c) If every column of the augmented matrix of a linear system contains a pivot, then the system is inconsistent.
True: the final column contains a pivot, and hence there is an equation of the form $0 = 1$ **in an equivalent linear system to the one we started with.**

d) If every row of the augmented matrix of a linear system contains a pivot, then the system is inconsistent.
False, eg. the system $x_1 = 1$, $x_2 = 1$.

e) If the function $T \colon \mathbb{R}^5 \to \mathbb{R}^4$ is onto (surjective) and $S \colon \mathbb{R}^4 \to \mathbb{R}^2$ is onto, then the composite $ST \colon \mathbb{R}^5 \to \mathbb{R}^2$ is onto.
True: let $x \in \mathbb{R}^2$**, then choose** $y \in \mathbb{R}^4$ **such that** $S(y) = x$**, then choose** $z \in \mathbb{R}^5$ **such that** $T(z) = y$**, then** $ST(z) = x$**.**

Question 5: Consider the linear transformation $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 - x_2 \\ x_1 - x_3 \\ x_1 + 2x_2 + x_3 \end{pmatrix}.$$

a) Is T onto (surjective)?
Yes, since the standard matrix of T **has a pivot in every row.**

b) Is T 1-to-1 (injective)?
Yes, since the equation $Ax = 0$ **has only the trivial solution** $x = 0$ **(check by row reduction).**

c) Write down the standard matrix for the composite transformation $\mathbb{R}^3 \to \mathbb{R}^3$ sending x to $T(T(x))$.
The matrix is

$$\begin{pmatrix} 8 & -3 & 1 \\ 2 & -3 & -1 \\ 10 & 1 & -1 \end{pmatrix}.$$

-the square of the standard matrix for T**.**