Hertentamen – Linear Algebra and Image Processing

8 August 2012

Time: 3 hours.
Fill in your name and student number on all papers you hand in.

In all questions, justify your answer fully and show all your work. Responses that give only the numerical answer will not receive full credit. Only pen and paper are permitted during the exam.

There are 2 pages and 10 questions. Each question is worth an equal number of points.

1. Using reduction to upper-triangular form, determine whether the following linear system is consistent or inconsistent. State why you know this is true.

$$\begin{aligned} x_2 &= 8 + 4x_3, \\ 2x_3 - 1 &= 3x_2 - 2x_1, \\ 5x_1 + 7x_3 &= 8x_2 + 1. \end{aligned}$$

2. Give the general solution to the following linear system, shown here as an augmented matrix that has been row-reduced to upper-triangular form:

$$\left[\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right].$$

3. Compute the inverse of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 6 & -2 & 2 \\ -4 & 1 & -3 \end{bmatrix}.$$

4. (a) For each of the following geometric 2D transformations, write down the corresponding 3×3 matrix for homogeneous coordinates.
   i. Scaling (dilating) each coordinate by a factor 3.
   ii. Translation by $(2, 4)$.
   iii. Rotation by $90°$ counter-clockwise.

   (b) For the transformations in part (a),
   i. is scaling followed by translation the same as translation followed by scaling?
   ii. is scaling followed by rotation the same as rotation followed by scaling?
   iii. is translation followed by rotation the same as rotation followed by translation?

5. Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

Compute the determinant $\det A$ by cofactor expansion along

(a) the middle column;

(b) the lowest row.

6. Determine whether S is a basis for $\mathbb{R}^3$. If S is not a basis for $\mathbb{R}^3$, determine whether S is linearly independent and whether S spans $\mathbb{R}^3$.

$$S = \left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}.$$

7. In this question A represents an $n \times n$ matrix.

(a) State 3 conditions that are equivalent to A being invertible. In other words, state 3 conditions that complete the sentence "A is invertible if and only if [condition]".

(b) If the equation $Ax = b$ is inconsistent for some b in $\mathbb{R}^n$, what can you say about the equation $Ax = 0$? Justify your answer.

8. For the matrix

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix},$$

find eigenvalues and eigenvectors for A, after first determining the characteristic equation and factorising it.

9. Let A be the matrix

$$A = \begin{bmatrix} -1 & 6 \\ 3 & -8 \\ 1 & -2 \\ 1 & -4 \end{bmatrix}.$$

(a) Find an orthonormal basis for $\operatorname{Col} A$, the column space of A.

(b) Find a QR factorization $A = QR$ for the matrix A.

10. As preparation for computing a singular value decomposition $A = U\Sigma V^T$ for the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix},$$

(a) compute the eigenvalues of A^TA and the corresponding normalised eigenvectors; and

(b) give the matrix Σ of singular values.

Succes!